On surjective ring homomorphisms between semi-simple commutative Banach algebras

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Abstract. Let $A$ and $B$ be semi-simple commutative Banach algebras. We give a representation of surjective ring homomorphisms from $A$ onto $B$ in terms of complex ring homomorphisms and injective, continuous and closed mapping between the maximal ideal spaces. As a corollary, we prove that neither the disc algebra $A(\bar{D})$ nor the commutative Banach algebra of all bounded holomorphic functions $H^\infty(D)$ are ring homomorphic image of any semi-simple commutative regular Banach algebras. Under additional assumptions on the maximal ideal spaces, we also prove automatic linearity of ring homomorphisms.

1. Introduction and results

Let $A$ and $B$ be algebras over the complex number field $\mathbb{C}$. We say that a mapping $\rho : A \rightarrow B$ is a ring homomorphism provided that

$$\rho(f + g) = \rho(f) + \rho(g)$$
$$\rho(fg) = \rho(f)\rho(g)$$

for every $f, g \in A$. By definition, ring homomorphisms need not be linear nor continuous. If, in addition, $\rho$ is homogeneous, that is, $\rho(\lambda f) = \lambda \rho(f)$ for every $\lambda \in \mathbb{C}$ and $f \in A$, then $\rho$ is a usual homomorphism.

One might expect that ring homomorphisms are quite similar to homomorphisms. In fact, under some additional assumptions, it is known to be true. For

Mathematics Subject Classification: 46J10.
Key words and phrases: automatic linearity, commutative Banach algebras, maximal ideal spaces, ring homomorphisms.
example, Arnold [1] proved that a ring isomorphism between the Banach algebras of all bounded operators from an infinite dimensional Banach space to another is automatically linear, or conjugate-linear. Unfortunately, ring homomorphisms need not be linear nor conjugate-linear in general. For example, let us consider a ring homomorphism $\tau$ from $\mathbb{C}$ to $\mathbb{C}$. For simplicity, we shall call $\tau$ a ring homomorphism on $\mathbb{C}$. It is obvious that the zero mapping $\tau(z) = 0$ ($z \in \mathbb{C}$), the identity $\tau(z) = z$ ($z \in \mathbb{C}$) and the complex conjugate $\tau(z) = \overline{z}$ ($z \in \mathbb{C}$) are ring homomorphisms on $\mathbb{C}$. We call them trivial ring homomorphisms on $\mathbb{C}$. In fact, Kestelman [5] proved that there exists a non-trivial ring homomorphism on $\mathbb{C}$. It follows from a result of Charnow [2] that the cardinal number of the set of all non-trivial ring automorphisms on $\mathbb{C}$ is $2^c$, where $c$ denotes the cardinality of continuum. Ring homomorphisms have more surprising feature. Let $\Omega \subset \mathbb{C}$ be a region and let $H(\Omega)$ be the algebra of all holomorphic functions on $\Omega$. In [8] it is proven that there exists an injective ring homomorphism from $H(\Omega)$ to $\mathbb{C}$. Thus we may regard $H(\Omega)$ as a subring of $\mathbb{C}$. Thus the study of ring homomorphic image is complicated and interesting.

Let $\overline{D}$ be the closure of the open unit disc $D$, and let $T = \overline{D} \setminus D$. Molnár [9] considered ring homomorphic image of commutative $C^*$-algebras. More explicitly, he proved that the group algebras $L^1(\mathbb{R})$, $L^1(\mathbb{T})$ and the disc algebra $A(\overline{D})$ are not ring homomorphic images of any commutative $C^*$-algebras. Let $1 \leq p < \infty$, $n$ a positive integer and let $G$ be a compact abelian group. Takahasi and Hatori [10] proved that $L^1(\mathbb{R}^n)$, $A(\overline{D})$ and $C^n([a, b])$, the commutative Banach algebra of all $n$-times continuously differentiable functions on $[a, b]$, are not ring homomorphic image of the $L^p$-space $L^p(G)$.

The purpose of this paper is to generalize and unify the above results concerning ring homomorphic images. To do this, we will study surjective ring homomorphisms between semi-simple commutative Banach algebras. Kaplansky [4] studied ring isomorphisms between semi-simple Banach algebras. Although a part of Theorem 1.1 below can be deduced from [6, Corollary 2.8], just for the sake of completeness we give a direct proof. In fact, we shall prove that surjective ring homomorphisms are represented by continuous, injective and closed mapping between the maximal ideal spaces.

**Theorem 1.1.** Let $A$ and $B$ be semi-simple commutative Banach algebras with maximal ideal spaces $M_A$ and $M_B$, respectively. If $\rho : A \to B$ is a surjective ring homomorphism, then there exist a mapping $\Phi : M_B \to M_A$ and a partitioning $\{M_{-1}, M_1, M_d\}$ of $M_B$ satisfying the following conditions:

(a) $\Phi$ is an injective, continuous and closed mapping,
(b) both $M_{-1}$ and $M_1$ are clopen, and $M_d$ is at most finite, and
(c) for each $\varphi \in M_d$, there exists a non-trivial ring automorphism $\tau_{\varphi}$ on $\mathbb{C}$ such that
\[
\hat{\rho}(\hat{f})(\varphi) = \begin{cases} 
\hat{f}(\Phi(\varphi)) & \varphi \in M_{-1} \\
\hat{f}(\Phi(\varphi)) & \varphi \in M_1 \\
\tau_{\varphi}(\hat{f}(\Phi(\varphi))) & \varphi \in M_d
\end{cases}
\] (1.1)
for every $f \in A$, where $\hat{\cdot}$ denotes the Gelfand transform.

As a corollary from Theorem 1.1, we can prove the following two results, which generalize some results in [9, Corollary] and [10, Corollary 4]. Corollary 1.3 (b) is also a generalization of [3, Corollary 3.1]. In fact, HATORI, ISHI, the first and second authors of this paper considered the case where $A$ and $B$ have units.

Corollary 1.2. Let $A$ be a semi-simple regular commutative Banach algebra and let $B$ be a semi-simple commutative Banach algebra. If there exists a surjective ring homomorphism $\rho : A \to B$, then $B$ is regular.

Corollary 1.3. Let $A$ and $B$ be semi-simple commutative Banach algebras. Suppose that the maximal ideal space $M_B$ of $B$ is infinite and connected.
(a) If the maximal ideal space $M_A$ of $A$ is discrete, then there is no surjective ring homomorphism from $A$ onto $B$.
(b) If there exists a surjective ring homomorphism $\rho : A \to B$, then $\rho$ is linear or conjugate-linear.

2. Construction of the mapping $\Phi$

Before proving lemmas, we need a characterization of trivial ring homomorphisms on $\mathbb{C}$. The following result is well-known, so we omit a proof (For a proof, see, for example, [7, Proposition 2.1]).

Proposition 2.1. Let $\tau$ be a ring homomorphism on $\mathbb{C}$. Then each of the following three conditions implies the other two.
(a) $\tau$ is trivial.
(b) There exist $\alpha_0, \beta_0 > 0$ such that $|z| < \alpha_0$ implies $|\tau(z)| \leq \beta_0$.
(c) $\tau$ is continuous at 0.

Remark 2.1. By Proposition 2.1, we see that a ring homomorphism $\tau$ on $\mathbb{C}$ is non-trivial if and only if the following conditions are satisfied:
for each $\alpha, \beta > 0$, there exists $z \in \mathbb{C}$ with $|z| < \alpha$ but $|\tau(z)| > \beta$.

We shall use this fact several times.

Until the end of this section, $A$ and $B$ denote semi-simple commutative Banach algebras with maximal ideal spaces $M_A$ and $M_B$, respectively. We also denote by $\rho$ a surjective ring homomorphism from $A$ onto $B$.

**Definition 1.** For each $\varphi$ of $M_B$, we define the induced mapping $\rho_\varphi$ from $A$ into $\mathbb{C}$ by

$$\rho_\varphi(f) = \hat{\rho(\varphi)} (f \in A),$$

where $\hat{\cdot}$ is the Gelfand transform. Since $\rho$ is surjective, $\rho_\varphi$ is a surjective ring homomorphism for every $\varphi \in M_B$.

**Notation.** Let $A_e$ be the commutative Banach algebra obtained by adjunction of a unit element $e$ to $A$. Here we notice that $A_e$ is well-defined even for unital $A$.

The maximal ideal space $M_{A_e}$ of $A_e$ is the one-point compactification $M_A \cup \{x_\infty\}$ of $M_A$.

**Lemma 2.2.** For each $\varphi \in M_B$, there exists a unique ring homomorphism $\tilde{\rho}_\varphi$ from $A_e$ onto $\mathbb{C}$ with $\tilde{\rho}_\varphi|_A = \rho_\varphi$.

**Proof.** Take $\varphi \in M_B$. Since $\rho_\varphi$ is surjective, there exists $a \in A$ with $\rho_\varphi(a) = 1$. Define the mapping $\tilde{\rho}_\varphi$ from $A_e$ to $\mathbb{C}$ by

$$\tilde{\rho}_\varphi(f + \lambda e) = \rho_\varphi(f) + \rho_\varphi(\lambda a) \quad (f + \lambda e \in A_e).$$

By definition, $\tilde{\rho}_\varphi|_A = \rho_\varphi$, and so $\tilde{\rho}_\varphi$ is surjective since so is $\rho_\varphi$. By the definition of $\tilde{\rho}_\varphi$, it is obvious that $\tilde{\rho}_\varphi$ is additive. We shall prove that $\tilde{\rho}_\varphi$ is multiplicative. Take $f + \lambda e, g + \mu e \in A_e$. Since $\rho_\varphi(a) = 1$, we have

$$\rho_\varphi(\lambda ma) = \rho_\varphi(\lambda ma)\rho_\varphi(a) = \rho_\varphi(\lambda a)\rho_\varphi(\mu a). \quad (2.1)$$

Note also that

$$\rho_\varphi(\mu f) = \rho_\varphi(\mu f)\rho_\varphi(a) = \rho_\varphi(f)\rho_\varphi(\mu a) \quad (2.2)$$

since $\rho_\varphi$ is multiplicative. By the same reasoning, we have $\rho_\varphi(\lambda g) = \rho_\varphi(g)\rho_\varphi(\lambda a)$.

It follows that

$$\tilde{\rho}_\varphi((f + \lambda e)(g + \mu e)) = \tilde{\rho}_\varphi(fg + \mu f + \lambda g + \lambda \mu e)$$

$$= \rho_\varphi(fg + \mu f + \lambda g) + \rho_\varphi(\lambda ma)$$

$$= \rho_\varphi(f)\rho_\varphi(g) + \rho_\varphi(f)\rho_\varphi(\mu a) + \rho_\varphi(g)\rho_\varphi(\lambda a)$$

$$+ \rho_\varphi(\lambda a)\rho_\varphi(\mu a) \quad \text{(by (2.1) and (2.2))}$$

$$= \{\rho_\varphi(f) + \rho_\varphi(\lambda a)\} \{\rho_\varphi(g) + \rho_\varphi(\mu a)\}$$

$$= \tilde{\rho}_\varphi(f + \lambda e) \tilde{\rho}_\varphi(g + \mu e).$$
This proves that \( \rho_\varphi \) is multiplicative. We thus conclude that \( \tilde{\rho}_\varphi \) is a surjective ring homomorphism from \( A_e \) onto \( C \) with \( \tilde{\rho}_\varphi|_A = \rho_\varphi \).

Finally, we prove the uniqueness of \( \tilde{\rho}_\varphi \). Let \( \rho_\varphi^* : A_e \to C \) be another ring homomorphism with \( \rho_\varphi^*|_A = \rho_\varphi \). Note, for each \( \lambda \in C \), that

\[
\rho_\varphi^*(\lambda e) = \rho_\varphi^*(\lambda e) \rho_\varphi(a) = \rho_\varphi^*(\lambda a) = \rho_\varphi(\lambda a)
\]

since \( \rho_\varphi(a) = 1 \). For each \( f + \lambda e \in A_e \), we have

\[
\rho_\varphi^*(f + \lambda e) = \rho_\varphi^*(f) + \rho_\varphi^*(\lambda e) = \rho_\varphi(f) + \rho_\varphi(\lambda a) = \tilde{\rho}_\varphi(f + \lambda e),
\]

which proves the uniqueness. This completes the proof. \( \square \)

**Lemma 2.3.** Let \( \tilde{\rho}_\varphi \) be from Lemma 2.2 for each \( \varphi \in M_B \). There exists unique \( \psi \in M_{A_e} \setminus \{x_{\infty}\} \) with \( \ker \tilde{\rho}_\varphi = \ker \psi \). For such \( \psi \), we have \( \ker \rho_\varphi = \ker(\psi|_A) \).

**Proof.** Take \( \varphi \in M_B \). By Lemma 2.2, there is a unique ring homomorphism \( \tilde{\rho}_\varphi \) from \( A_e \) onto \( C \) with \( \tilde{\rho}_\varphi|_A = \rho_\varphi \). We show that the kernel \( \ker \tilde{\rho}_\varphi \) is an algebra ideal. Since \( \rho_\varphi \) preserve both additions and multiplications, it is enough to show that \( \lambda f \in \ker \tilde{\rho}_\varphi \) whenever \( \lambda \in C \) and \( f \in \ker \tilde{\rho}_\varphi \). Take \( \lambda \in C \) and \( f \in \ker \tilde{\rho}_\varphi \). Since \( \tilde{\rho}_\varphi(f) = 0 \), for \( a \in A \) with \( \tilde{\rho}_\varphi(a) \neq 0 \), we have

\[
\tilde{\rho}_\varphi(\lambda f)\tilde{\rho}_\varphi(a) = \tilde{\rho}_\varphi(f)\tilde{\rho}_\varphi(\lambda a) = 0.
\]

It follows that \( \tilde{\rho}_\varphi(\lambda f) = 0 \) since \( \tilde{\rho}_\varphi(a) \neq 0 \). Thus \( \lambda f \in \ker \tilde{\rho}_\varphi \), and so ker \( \tilde{\rho}_\varphi \) is an algebra ideal of \( A \).

Note that ker \( \tilde{\rho}_\varphi \) is a proper algebra ideal since \( \tilde{\rho}_\varphi|_A = \rho_\varphi \) is non-zero. There exists \( \psi \in M_{A_e} \setminus \{x_{\infty}\} \) with ker \( \tilde{\rho}_\varphi \subset \ker \psi \). We shall prove that ker \( \tilde{\rho}_\varphi = \ker \psi \). Take \( u_0 \in A_e \) with \( u_0 \notin \ker \tilde{\rho}_\varphi \). Since \( \tilde{\rho}_\varphi \) is surjective, there is \( v_0 \in A_e \) such that \( \tilde{\rho}_\varphi(v_0) = 1/\tilde{\rho}_\varphi(u_0) \). Then

\[
\tilde{\rho}_\varphi(u_0v_0 - e) = \tilde{\rho}_\varphi(u_0)\tilde{\rho}_\varphi(v_0) - \tilde{\rho}_\varphi(e) = 0,
\]

and so \( u_0v_0 - e \in \ker \tilde{\rho}_\varphi \subset \ker \psi \). Thus we have \( \psi(u_0)\psi(v_0) = 1 \), which implies \( u_0 \notin \ker \psi \). This proves ker \( \psi \subset \ker \tilde{\rho}_\varphi \), and so ker \( \tilde{\rho}_\varphi = \ker \psi \).

Since \( \tilde{\rho}_\varphi|_A = \rho_\varphi \), we have

\[
\ker \rho_\varphi = \ker(\tilde{\rho}_\varphi|_A) = (\ker \tilde{\rho}_\varphi) \cap A = (\ker \psi) \cap A = \ker(\psi|_A).
\]

In particular, \( \psi|_A \) is non-zero. Thus \( \psi \in M_{A_e} \setminus \{x_{\infty}\} \). \( \square \)
Definition 2. By Lemma 2.3, for each $\varphi \in M_B$, there exists a unique element $\Phi(\varphi) \in M_A \setminus \{x_\infty\}$ with $\ker \tilde{\rho}_\varphi = \ker \Phi(\varphi)$. We may regard $\Phi$ as a mapping from $M_B$ to $M_A \setminus \{x_\infty\}$.

Definition 3. For each $\varphi \in M_B$, we consider the mapping $\tau_\varphi : C \to C$ defined by

$$\tau_\varphi(\lambda) = \tilde{\rho}_\varphi(\lambda e) \quad (\lambda \in C),$$

where $\tilde{\rho}_\varphi$ is from Lemma 2.2.

Lemma 2.4. For each $\varphi \in M_B$, let $\tau_\varphi$ be from Definition 3. Then $\tau_\varphi$ is a ring automorphism on $C$ with

$$\rho_\varphi(f) = \tau_\varphi(\hat{f}(\Phi(\varphi)))$$  \hfill (2.3)

for every $f \in A$. If, in addition, $\rho_\varphi(f) \neq 0$, then

$$\tau_\varphi(\lambda) = \frac{\rho_\varphi(\lambda f)}{\rho_\varphi(f)}$$  \hfill (2.4)

for every $\lambda \in C$.

Proof. Take $\varphi \in M_B$. By the definition of $\Phi$, we have, for each $f \in A$,

$$f - \hat{f}(\Phi(\varphi))e \in \ker \Phi(\varphi) = \ker \tilde{\rho}_\varphi,$$

and so

$$0 = \tilde{\rho}_\varphi(f) - \tilde{\rho}_\varphi(\hat{f}(\Phi(\varphi)))e = \rho_\varphi(f) - \tau_\varphi(\hat{f}(\Phi(\varphi))).$$

This proves $\rho_\varphi(f) = \tau_\varphi(\hat{f}(\Phi(\varphi)))$ for every $f \in A$.

Next, we show that $\tau_\varphi$ is a ring automorphism. By the definition of $\tau_\varphi$, it is obvious that $\tau_\varphi$ is a non-zero ring homomorphism. We see that $\tau_\varphi$ is injective: for if there were $\lambda_1, \lambda_2 \in C$ with $\lambda_1 \neq \lambda_2$ and $\tau_\varphi(\lambda_1) = \tau_\varphi(\lambda_2)$, then we would have

$$\tau_\varphi(\lambda) = \tau_\varphi(\lambda_1 - \lambda_2) = \frac{\lambda}{\lambda_1 - \lambda_2} = 0$$

for all $\lambda \in C$, since $\tau_\varphi(\lambda_1 - \lambda_2) = \tau_\varphi(\lambda_1) - \tau_\varphi(\lambda_2) = 0$. This is a contradiction since $\tau_\varphi$ is non-zero. We need to prove the surjectivity of $\tau_\varphi$. Since $\rho_\varphi$ is surjective, for each $\lambda \in C$, there is $a \in A$ with $\rho_\varphi(a) = \lambda$. By (2.3), we have $\tau_\varphi(\hat{a}(\Phi(\varphi))) = \lambda$, and so $\tau_\varphi$ is surjective. Thus $\tau_\varphi$ is a ring automorphism.

Finally, for each $\lambda \in C$ and $f \in A$ with $\rho_\varphi(f) \neq 0$, we have

$$\rho_\varphi(\lambda f) = \tilde{\rho}_\varphi(\lambda f) = \tilde{\rho}_\varphi(\lambda e) \tilde{\rho}_\varphi(f) = \tau_\varphi(\lambda) \rho_\varphi(f).$$

This proves (2.4), and so the proof is complete. \qed
Lemma 2.5. Let \( \Phi \) be the mapping from Definition 2. Then \( \Phi \) is injective.

Proof. Take \( \varphi_0, \varphi_1 \in M_B \) with \( \varphi_0 \neq \varphi_1 \). There is \( b \in B \) with \( \hat{b}(\varphi_0) = 0 \) and \( \hat{b}(\varphi_1) = 1 \) since \( B \) is semi-simple. Choose \( a \in A \) so that \( \rho(a) = b \); this is possible since \( \rho \) is surjective. Then \( \rho_{\varphi_0}(a) = \hat{b}(\varphi_0) = 0 \) and \( \rho_{\varphi_1}(a) = \hat{b}(\varphi_1) = 1 \). By Lemma 2.4, we have

\[
\tau_{\varphi_0}(\hat{a}(\Phi(\varphi_0))) = 0 \quad \text{and} \quad \tau_{\varphi_1}(\hat{a}(\Phi(\varphi_1))) = 1,
\]

where \( \tau_{\varphi} (\varphi \in M_B) \) is the mapping from Definition 3. Note that \( \tau_{\varphi}(0) = 0 \) and \( \tau_{\varphi}(1) = 1 \) for every non-trivial ring homomorphism. Since \( \tau_{\varphi} \) is injective by Lemma 2.4, we have \( \hat{a}_0(\Phi(\varphi_0)) = 0 \) and \( \hat{a}_0(\Phi(\varphi_1)) = 1 \). We thus conclude \( \Phi(\varphi_0) \neq \Phi(\varphi_1) \), and so \( \Phi \) is injective. \( \square \)

Definition 4. We define the subsets \( M_{-1}, M_1 \) and \( M_d \) of \( M_B \) by

\[
M_{-1} = \{ \varphi \in M_B : \tau_{\varphi}(\lambda) = \bar{\lambda} (\lambda \in \mathbb{C}) \},
\]
\[
M_1 = \{ \varphi \in M_B : \tau_{\varphi}(\lambda) = \lambda (\lambda \in \mathbb{C}) \} \quad \text{and}
\]
\[
M_d = \{ \varphi \in M_B : \tau_{\varphi} \text{ is non-trivial} \}.
\]

By definition, \( \{M_{-1}, M_1, M_d\} \) is a partitioning of \( M_B \), that is, \( M_{-1}, M_1 \) and \( M_d \) are mutually disjoint subsets of \( M_B \) with \( M_{-1} \cup M_1 \cup M_d = M_B \).

From Lemma 2.6 to 2.8, \( \{M_{-1}, M_1, M_d\} \) will denote the partitioning of \( M_B \) from Definition 4.

Lemma 2.6. Both \( M_{-1} \) and \( M_1 \) are closed subsets of \( M_B \).

Proof. We show that \( \text{cl}(M_k) \subset M_k \) for \( k = \pm 1 \), where \( \text{cl}(M_k) \) denotes the closure of \( M_k \) in \( M_B \). Take \( \varphi \in \text{cl}(M_k) \) and let \( \{\varphi_\alpha\} \) be a net in \( M_k \) converging to \( \varphi \). Choose \( \alpha \in A \) so that \( \rho(\alpha)(\varphi) = \rho_{\varphi}(\alpha) \neq 0 \). Since \( \rho(\alpha) \) is continuous on \( M_B \), \( \rho_{\varphi}(a) = \rho(\alpha)(\varphi_\alpha) \) converges to \( \rho_{\varphi}(a) \neq 0 \). So, without loss of generality we may assume \( \rho_{\varphi}(a) \neq 0 \) for every \( \alpha \). It follows from (2.4) that

\[
\tau_{\varphi}(\lambda) = \frac{\rho_{\varphi}(\lambda a)}{\rho_{\varphi}(a)} \to \frac{\rho_{\varphi}(\lambda a)}{\rho_{\varphi}(a)} = \tau_{\varphi}(\lambda).
\]

Since \( \varphi_\alpha \in M_k \), (2.5) implies that \( \tau_{\varphi}(\lambda) = \bar{\lambda} \) if \( k = -1 \), and \( \tau_{\varphi}(\lambda) = \lambda \) if \( k = 1 \). Thus \( \varphi \in M_k \) for \( k = \pm 1 \), and the proof is complete. \( \square \)

Lemma 2.7. \( M_d \) is an open and at most finite subset of \( M_B \).
By Lemma 2.6, $M_d = M_B \setminus (M_{-1} \cup M_1)$ is open. Assume to the contrary that $M_d$ contains a countable subset $\{\varphi_n\}_{n=1}^\infty$ with $\varphi_i \neq \varphi_j$ ($i \neq j$). Set, for each $n \in \mathbb{N}$, the set of all natural numbers, $\psi_n = \Phi(\varphi_n)$. By Lemma 2.5, $\Phi$ is injective, and so $\psi_i \neq \psi_j$ ($i \neq j$). Since $\varphi_n \in M_d$, the ring homomorphism $\tau_{\varphi_n}$ from Definition 3 is non-trivial. For simplicity, we will write $\tau_n$ instead of $\tau_{\varphi_n}$.

By (2.3), we have

$$\rho_{\varphi_n}(f) = \tau_n(\hat{f}(\varphi_n)) = \tau_n(\hat{f}(\psi_n))$$

(2.6)

for each $f \in A$.

Take $a_1 \in A$ with $\hat{a}_1(\psi_1) = 1$. Since $\tau_1$ is non-trivial, there exists $\lambda_1 \in \mathbb{C}$ with $|\lambda_1| < (2 \|a_1\|)^{-1}$ and $|\tau_1(\lambda_1)| > 2$ (cf. Remark 2.1). Set $f_1 = \lambda_1 a_1 \in A$. Then

$$\|f_1\| < 2^{-1} \quad \text{and} \quad |\tau_1(\hat{f}_1(\psi_1))| > 2.$$

By induction, we shall prove that, for each $n \in \mathbb{N}$ with $n \geq 2$, there exists $f_n \in A$ such that

$$\|f_n\| < 2^{-n}, \quad |\tau_n(\hat{f}_n(\psi_n))| > 2^n + \left| \tau_n \left( \sum_{k=1}^{n-1} \hat{f}_k(\psi_n) \right) \right|$$

and that

$$\hat{f}_n(\psi_1) = \hat{f}_n(\psi_2) = \cdots = \hat{f}_n(\psi_{n-1}) = 0.$$

Take $a_2 \in A$ with $\hat{a}_2(\psi_1) = 0$ and $\hat{a}_2(\psi_2) = 1$. Since $\tau_2$ is non-trivial, there exists $\lambda_2 \in \mathbb{C}$ such that

$$|\lambda_2| < \frac{1}{2^2 \|a_2\|} \quad \text{and} \quad |\tau_2(\lambda_2)| > 2^2 + |\tau_1(\hat{f}_1(\psi_1))|.$$

Set $f_2 = \lambda_2 a_2 \in A$. Then $\hat{f}_2(\psi_1) = 0$ and $\hat{f}_2(\psi_2) = \lambda_2$. It follows that

$$\|f_2\| < 2^{-2}, \quad \hat{f}_2(\psi_1) = 0 \quad \text{and} \quad |\tau_2(\hat{f}_2(\psi_2))| > 2^2 + |\tau_1(\hat{f}_1(\psi_1))|.$$

Suppose that there are $f_k \in A$ ($k = 2, \cdots, n-1$) with

$$\|f_k\| < 2^{-k}, \quad \hat{f}_k(\psi_1) = \cdots = \hat{f}_k(\psi_{k-1}) = 0 \quad \text{and} \quad |\tau_k(\hat{f}_k(\psi_k))| > 2^k + \left| \tau_k \left( \sum_{j=1}^{k-1} \hat{f}_j(\psi_k) \right) \right|.$$ 

Choose $a_n \in A$ so that $\hat{a}_n(\psi_n) = 1$ and

$$\hat{a}_n(\psi_1) = \cdots = \hat{a}_n(\psi_{n-1}) = 0.$$
In fact, take \( b_i \in A \), for each \( i \) (\( 1 \leq i \leq n - 1 \)), with \( \hat{b}_i(\psi_i) = 0 \) and \( \hat{b}_i(\psi_n) = 1 \). Then \( \Pi_{i=1}^{n-1} b_i \in A \) is the desired element. Since \( \tau_n \) is non-trivial, there is \( \lambda_n \in \mathbb{C} \) with
\[
|\lambda_n| < \frac{1}{2^n \|a_n\|} \quad \text{and} \quad |\tau_n(\lambda_n)| > 2^n + \left| \tau_n \left( \sum_{j=1}^{n-1} \hat{f}_j(\psi_n) \right) \right|.
\]
Set \( f_n = \lambda_n a_n \in A \). Then
\[
\|f_n\| < 2^{-n}, \quad \hat{f}_n(\psi_1) = \cdots = \hat{f}_n(\psi_{n-1}) = 0.
\]
Since \( a_n(\psi_n) = 1 \), we have \( \hat{f}_n(\psi_n) = \lambda_n \), and so
\[
|\tau_n(\hat{f}_n(\psi_n))| > 2^n + \left| \tau_n \left( \sum_{j=1}^{n-1} \hat{f}_j(\psi_n) \right) \right|
\]
as desired.

Since \( \|f_n\| < 2^{-n} \), the series \( \sum_{n=1}^{\infty} f_n \) converges to an element, say \( f_0 \in A \). We have, for each \( n \in \mathbb{N} \), \( \hat{f}_0(\psi_n) = \sum_{k=1}^{n} \hat{f}_k(\psi_n) \) since \( \hat{f}_k(\psi_n) = 0 \) for each \( k = n + 1, n + 2, \cdots \). By (2.6), we have, for each \( n \in \mathbb{N} \),
\[
|\rho_{\varphi_n}(f_0)| = |\tau_n(\hat{f}_0(\psi_n))| = \left| \tau_n \left( \sum_{k=1}^{n} \hat{f}_k(\psi_n) \right) \right|
\]
\[
= \left| \sum_{k=1}^{n} \tau_n \left( \hat{f}_k(\psi_n) \right) \right| \geq |\tau_n(\hat{f}_n(\psi_n))| - \left| \tau_n \left( \sum_{k=1}^{n-1} \hat{f}_k(\psi_n) \right) \right|,
\]
and so, by (2.7),
\[
|\rho(f_0)(\varphi_n)| = |\rho_{\varphi_n}(f_0)| > 2^n.
\]
Since \( \rho(f_0) \) is bounded on \( M_B \), we now reach a contradiction. We thus proved that \( M_d \) is at most finite subset of \( M_B \). \( \square \)

**Lemma 2.8.** The mapping \( \Phi : M_B \to M_{A_c} \setminus \{x_\infty\} \) is continuous.

**Proof.** Let \( \varphi_0 \in M_B \) and let \( \{\varphi_\alpha\} \subset M_B \) be a net converging to \( \varphi_0 \). We prove that \( \Phi(\varphi_\alpha) \) converges to \( \Phi(\varphi_0) \). If \( \varphi_0 \in M_d \), then \( \{\varphi_0\} \) is open by Lemma 2.6 and 2.7. So, we may assume that \( \varphi_\alpha = \varphi_0 \) for each \( \alpha \). Thus \( \Phi(\varphi_\alpha) = \Phi(\varphi_0) \), and so \( \Phi(\varphi_\alpha) \) converges to \( \Phi(\varphi_0) \).

Next, we consider the case where \( \varphi_0 \in M_k \) for \( k = \pm 1 \). By Lemma 2.6 and 2.7, \( M_k \) is clopen of \( M_B \). Thus we may assume \( \{\varphi_\alpha\} \subset M_k \) for each \( \alpha \).
By the definition of \( M_k \) for \( k = \pm 1 \), \( \tau_{\varphi_\alpha} \) is the complex conjugate for each \( \alpha \) when \( k = -1 \), and \( \tau_{\varphi_\alpha} \) is the identity for each \( \alpha \) when \( k = 1 \). Note that \( \rho_{\varphi_\alpha}(f) \) converges to \( \rho_{\varphi_\alpha}(f) \) for each \( f \in A \) since \( \hat{\rho}(f) \) is continuous on \( M_B \). It follows from (2.3) that \( \hat{f}(\Phi(\varphi_\alpha)) \) converges to \( \hat{f}(\Phi(\varphi_0)) \) for every \( f \in A \). Thus \( \hat{u}(\varphi_\alpha) \to \hat{u}(\varphi_0) \) for each \( u \in A_e \). By the definition of the Gelfand topology, we conclude that \( \Phi(\varphi_\alpha) \) converges to \( \Phi(\varphi_0) \). □

3. Proofs and application

Proof of Theorem 1.1. Let \( \Phi \) and \( \{M_{-1}, M_1, M_d\} \) be from Definitions 2 and 4, respectively. Then \( \Phi \) is an injective and continuous mapping by Lemmas 2.5 and 2.8. It follows from Lemmas 2.6 and 2.7 that \( M_{-1} \) and \( M_1 \) are clopen, and \( M_d \) is at most finite. Let \( \tau_{\varphi} \) be from Definition 3 for each \( \varphi \in M_d \). By (2.3) and Definition 4, \( \rho \) is of the form (1.1).

It remains to be proved that \( \Phi \) is a closed mapping. We define a mapping \( \tilde{\Phi} : M_{B_e} \to M_{A_e} \) by

\[
\tilde{\Phi}(\varphi) = \begin{cases} 
\Phi(\varphi) & \varphi \in M_B \\
x_\infty & \varphi = y_\infty
\end{cases}
\]

where \( \{x_\infty\} = M_{A_e} \setminus M_A \) and \( \{y_\infty\} = M_{B_e} \setminus M_B \). Here we notice that for each \( f \in A \subset A_e \), \( \hat{f} \), as a function on \( M_{A_e} \), is 0 at \( x_\infty \). The same remark holds for \( b \in B \subset B_e \) and \( y_\infty \). We observe that \( \tilde{\Phi} \) is continuous: by definition, it is enough to prove the continuity of \( \tilde{\Phi} \) at \( y_\infty \). Let \( \{\varphi_\alpha\} \subset M_{B_e} \) be a net converging to \( y_\infty \). By Lemma 2.7, \( M_{B_e} \setminus M_d \) is an open neighborhood of \( y_\infty \), and so we may assume \( \{\varphi_\alpha\} \subset M_{B_e} \setminus M_d \). Take \( f \in A \). By the definition of \( \tilde{\Phi} \), we have

\[
\hat{f}(\tilde{\Phi}(\varphi_\alpha)) = \begin{cases} 
\hat{f}(\Phi(\varphi_\alpha)) & \varphi_\alpha \in M_B \setminus M_d \\
0 & \varphi_\alpha = y_\infty
\end{cases}
\]

(3.1)

On the other hand, since \( \varphi_\alpha \notin M_d \), it follows from (2.3) that

\[
|\hat{\rho}(f)(\varphi_\alpha)| = |\rho_{\varphi_\alpha}(f)| = \begin{cases} 
|\hat{\rho}(\Phi(\varphi_\alpha))| & \varphi_\alpha \in M_B \setminus M_d \\
0 & \varphi_\alpha = y_\infty
\end{cases}
\]

By (3.1), we have, for each \( \alpha \),

\[
|\hat{f}(\tilde{\Phi}(\varphi_\alpha))| = |\hat{\rho}(\Phi(\varphi_\alpha))|.
\]

(3.2)
Let pair $F$ of algebra $A$, a commutative Banach algebra of all bounded holomorphic functions on valued continuous functions on $\bar{D}$. Let $\Phi(\varphi_0)$ be continuous on $\Phi(\varphi_0)$ of the Gelfand topology, $\tilde{\Phi}(\varphi_0)$ converges to $\tilde{\Phi}(y_\infty)$. We thus conclude that $\tilde{\Phi} : M_{B_n} \to M_A$ is continuous.

Take a closed subset $F$ of $M_B$. Then $F \cup \{y_\infty\} \subset M_{B_n}$ is compact. Since $\tilde{\Phi}$ is continuous on $M_{B_n}$, $\tilde{\Phi}(F \cup \{y_\infty\}) = \Phi(F) \cup \{x_\infty\}$ is compact in $M_A$, and so $\Phi(F) \subset M_A \setminus \{x_\infty\}$ is closed in $M_A$. This proves that $\Phi$ is a closed mapping. □

Recall that a commutative Banach algebra $A$ is regular if and only if for each pair $F$, $\psi_0$ of closed subset $F \subset M_A$ and $\psi_0 \in M_A \setminus F$, there exists $f \in A$ with $\tilde{f}(\psi_0) = 1$ and $\tilde{f}(\psi) = 0$ for every $\psi \in F$.

**Proof of Corollary 1.2.** Take $\varphi_0 \in M_B$ and closed $F \subset M_B$ with $\varphi_0 \notin F$. Let $\Phi$ be an injective and closed mapping from Theorem 1.1. Then $\Phi(F) \subset M_A \setminus \{x_\infty\}$ is closed with $\Phi(\varphi_0) \notin \Phi(F)$. Since $A$ is regular, there exists $f_0 \in A$ with $\tilde{f}_0(\Phi(\varphi_0)) = 1$ and $\tilde{f}_0(\Phi(\varphi)) = 0$ for every $\varphi \in F$. Recall that if $\tau_\varphi$ is a non-trivial ring homomorphism, then $\tau_\varphi(r) = r$ for every $r \in \mathbb{Q}$ and $\varphi \in M_B$. By (2.3), we have $\rho(\tilde{f}_0)(\varphi_0) = \rho_{\varphi_0}(f_0) = 1$ and $\rho(\tilde{f}_0)(\varphi) = \rho_{\varphi}(f_0) = 0$ for every $\varphi \in F$, and so $B$ is regular. □

**Proof of Corollary 1.3.** (a) Assume to the contrary that there is a surjective ring homomorphism $\rho : A \to B$. Let $\Phi$ be from Theorem 1.1. Then $M_B$ is homeomorphic to $\Phi(M_B) \subset M_A$. By hypothesis, $M_A$ is discrete, and so is $M_B$. Now we reach a contradiction since $M_B$ is infinite and connected.

(b) Let $\{M_{-1}, M_1, M_2\}$ be from Theorem 1.1. Then $M_{-1}, M_1, M_2$ are clopen, and $M_2$ is at most finite. Since $M_2$ is assumed to be infinite and connected, it follows that $M_B = M_{-1}$, or $M_B = M_1$. So, by Theorem 1.1, there exists an injective, continuous and closed mapping $\Phi : M_B \to M_A$ with $\rho(\tilde{f}(\varphi)) = \tilde{f}(\Phi(\varphi))$ for every $f \in A$ and $\varphi \in M_B$, or $\rho(\tilde{f}(\varphi)) = \tilde{f}(\Phi(\varphi))$ for every $f \in A$ and $\varphi \in M_B$. Since $B$ is semi-simple, we have that $\rho$ is conjugate-linear, or linear, respectively. □

**Example 1.** Let $\mathbb{D}$ and $\bar{\mathbb{D}}$ be the open unit disc and the closure of $\mathbb{D}$, respectively. Let $A(\mathbb{D})$ be the disc algebra, that is, the uniform algebra of all complex-valued continuous functions on $\mathbb{D}$, which are holomorphic in $\mathbb{D}$. Let $H^\infty(\mathbb{D})$ be the commutative Banach algebra of all bounded holomorphic functions on $\mathbb{D}$. Neither $A(\bar{\mathbb{D}})$ nor $H^\infty(\mathbb{D})$ are regular. By Corollary 1.2, both $A(\mathbb{D})$ and $H^\infty(\mathbb{D})$ can not be the ring homomorphic images of any semi-simple regular commutative Banach algebra $A$. The case where $A = C_0(X)$ was proved by MOLNÁR [9, Corollary].
Example 2. Let $n \in \mathbb{N}$ and let $C^n([a, b])$ be the set of all $n$-times continuously differentiable complex-valued functions on a closed interval $[a, b]$. Then $C^n([a, b])$ is a semi-simple commutative Banach algebra with respect to the pointwise operations and the norm $\|f\|_n = \sum_{k=0}^{n} \|f^{(k)}\|_{\infty} / k!$ for $f \in C^n([a, b])$.

If $\rho$ is a surjective ring homomorphism from $C^n([a, b])$ onto itself, then $\rho$ is of the form
\[
\rho(f)(x) = \overline{f(\Phi(x))} \quad (f \in C^n([a, b]), \ x \in [a, b]),
\]
(3.3)
or
\[
\rho(f)(x) = f(\Phi(x)) \quad (f \in C^n([a, b]), \ x \in [a, b]).
\]
(3.4)
Here, $\Phi \in C^n([a, b])$ is injective and closed. For if $\rho$ is a surjective ring homomorphism from $C^n([a, b])$ onto itself, then by the Proof of Corollary 1.3 (b), there exists an injective, continuous and closed mapping $\Phi$ from $[a, b]$ into itself such that $\rho$ is of the form (3.3), or (3.4). If we take $f = \text{Id}$, the identity function, then we have $\Phi \in C^n([a, b])$.

Example 3. Let $1 \leq p \leq \infty$ and let $G$ be a compact abelian group. Then the $L^p$-space $L^p(G)$ is a commutative Banach algebra with respect to convolution as a multiplication. The maximal ideal space of $L^p(G)$ is the dual group $\hat{G}$ of $G$ for each $1 \leq p \leq \infty$. Let $B$ be a semi-simple commutative Banach algebra with infinite and connected maximal ideal space. By Corollary 1.3 (a), $B$ can not be the ring homomorphic image of $L^p(G)$ since $\hat{G}$ is discrete. The case where $B = L^1(\mathbb{R}^n), A(\mathbb{D}), C^n([a, b])$ was obtained by [10, Corollary 4].

References

On surjective ring homomorphisms.


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(Received July 1, 2007; revised October 10, 2007;)