Iterative Pexider equation

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Abstract. We consider the Pexider equation $F_{st} = H_s \circ G_t$ for $(s, t)$ belonging to the domain of a binary operation on a groupoid $K$, where $\{F_t : t \in K\} \subset Z^X$, $\{G_t : t \in K\} \subset Y^X$, $\{H_t : t \in K\} \subset Z^Y$ are unknown families of functions. It is shown that, in the case when there exists a unit element $e$ in $K$ and $H_e$ is an injection and $G_e$ is a surjection, the equation can be reduced to the Cauchy equation. Using the above result we solve the following problem: when does it follow from the equality $F_{st} = H_s \circ G_t$, for $(s, t)$ belonging to a set $L \subset R^2_+$, that $F_{st} = H_s \circ G_t$ for $(s, t) \in R^2_+$? Finally, some other conditions are established under which the equation may be reduced to the Cauchy equation.

Let $K$ be a non-empty set endowed with a binary operation (i.e. a mapping of a subset $D(K)$ of $K \times K$ into $K$). The set $K$ with the binary operation is called a groupoid (cp. [2]). If $(s, t) \in D(K)$ then we say that $st$ is defined.

The binary operation is said to be associative in case the following implication holds: if in the equation $s(tp) = (st)p$, $s, t, p \in K$, one of its sides or both the products $tp$ and $st$ are defined then both sides of the equation are defined and the equality holds.

An element $e \in K$ will be called a unit if for every $t \in K$ the products $te$ and $et$ are defined and $te = et = t$.

For $t \in K$ we denote by $K_t(t)$ the set of all elements $e \in K$ such that $et$ ($te$) is defined and $et = t$ ($te = t$).

Let $K$ be a groupoid and $X, Y, Z$ arbitrary non-empty sets. We shall consider the Pexider functional equation

$$F_{st} = H_s \circ G_t \quad \text{for} \quad (s, t) \in D(K),$$

where $\{F_t : t \in K\} \subset Z^X$, $\{G_t : t \in K\} \subset Y^X$, $\{H_t : t \in K\} \subset Z^Y$ are unknown families of functions. We understand (1) in such a way that if $st$ is defined, then the composition $H_s \circ G_t$ is defined (i.e. $\text{Ran} \ G_t \subset \text{Dom} \ H_t$).

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Let us denote by $\text{In}(X, Y)$ (Sur$(X, Y)$) the set of all injections (surjections) of $X$ into (onto) $Y$.

**Theorem 1.** Let $K$ be a groupoid such that there exists a unit element $e$ in $K$. 
(i) If $\{F_t : t \in K\} \subset Z^X$, $\{G_t : t \in K\} \subset Y^X$, $\{H_t : t \in K\} \subset Z^Y$ satisfy (1) and $H_e \in \text{In}(Y, Z)$, $G_e \in \text{Sur}(X, Y)$ then there exist functions $A \in \text{In}(Y, Z)$, $B \in \text{Sur}(X, Y)$ and a family of functions $\{T_t : t \in K\} \subset Y^Y$ such that

\[(2) \quad T_{st} = T_s \circ T_t \quad \text{for } (s, t) \in D(K)\]

and

\[(3) \quad \begin{cases} 
F_t = A \circ T_t \circ B, \\
G_t = T_t \circ B, \\
H_t = A \circ T_t, \quad t \in K.
\end{cases}\]

(ii) If $A \in Z^Y$, $B \in Y^X$ are arbitrary functions and $\{T_t : t \in K\} \subset Y^Y$ fulfils condition (2) then the functions $F_t$, $G_t$, $H_t$ given by (3) satisfy equation (1).

**Proof.** 

Put $F(t) := F_t$, $G(t) := G_t$, $H(t) := H_t$. Setting in (1) $t = e$ and then $s = e$ we get

\[(4) \quad F(s) = H(s) \circ G(e), \quad s \in K,\]
\[(5) \quad F(t) = H(e) \circ G(t), \quad t \in K.\]

Comparing the right hand sides of (4) and (5) for $s = t$ we obtain

\[(6) \quad H(t) \circ G(e) = H(e) \circ G(t), \quad t \in K.\]

By (6) and the relation $G(e) \in \text{Sur}(X, Y)$ we infer that

\[(7) \quad \text{Ran } H(t) \subset \text{Ran } H(e) \quad \text{for } t \in K.\]

Note that in view of (4) and the fact that $G(e) \in \text{Sur}(X, Y)$

\[(8) \quad \text{Ran } H(t) = \text{Ran } F(t), \quad t \in K.\]

From (1) we have

\[(9) \quad F(st) = F(e(st)) = H(e) \circ G(st) \quad \text{for } (s, t) \in D(K).\]

Hence

\[(10) \quad \text{Ran } F(st) \subset \text{Ran } H(e) \quad \text{for } (s, t) \in D(K).\]
Thus from (7), (8), (10) we have the following relations
\[
\begin{align*}
\text{Dom } H(e)^{-1} & \supset \text{Ran } H(t) = \text{Ran } F(t), \quad t \in K, \\
\text{Dom } H(e)^{-1} & \supset \text{Ran } F(st), \quad (s, t) \in D(K).
\end{align*}
\]

We introduce on \(X\) an equivalence relation \(\sim\) putting
\[
x \sim y \quad \text{iff} \quad G(e)(x) = G(e)(y).
\]
Denote \(\tilde{X} := X/\sim\) and let \(g\) be an invertible mapping such that \(g([x]) \in [x]\). Thus the function \(G(e) \circ g : \tilde{X} \to Y\) is a bijection. From (4) we obtain
\[
F(t) \circ g = H(t) \circ G(e) \circ g \quad \text{whence}
\]
(11) \[
H(t) = F(t) \circ g \circ (G(e) \circ g)^{-1} \quad \text{for } t \in K.
\]
Hence (1) may be written as follows:
(12) \[
F(st) = F(s) \circ g \circ (G(e) \circ g)^{-1} \circ G(t) \quad \text{for } (s, t) \in D(K).
\]
(5) yields
(13) \[
G(t) = H(e)^{-1} \circ F(t), \quad t \in K.
\]
Next (6) implies
(14) \[
G(t) = H(e)^{-1} \circ H(t) \circ G(e), \quad t \in K.
\]
Putting (13) into (12) we obtain
(15) \[
F(st) = F(s) \circ g \circ (G(e) \circ g)^{-1} \circ H(e)^{-1} \circ F(t), \quad (s, t) \in D(K).
\]
Define \(T(t) := H(e)^{-1} \circ F(t) \circ g \circ (G(e) \circ g)^{-1}, \quad t \in K\). Hence by (12) and (13) we can write
\[
T(st) = H(e)^{-1} \circ F(st) \circ g \circ (G(e) \circ g)^{-1} =
\]
\[
= H(e)^{-1} \circ F(s) \circ g \circ (G(e) \circ g)^{-1} \circ G(t) \circ g \circ (G(e) \circ g)^{-1} =
\]
\[
= H(e)^{-1} \circ F(s) \circ g \circ (G(e) \circ g)^{-1} \circ H(e)^{-1} \circ F(t) \circ g \circ (G(e) \circ g)^{-1} =
\]
\[
= T(s) \circ T(t).
\]
Then (2) holds, where \(T_t := T(t), \quad t \in K\). By (11) we have
(16) \[
H(t) = H(e) \circ T(t), \quad t \in K,
\]
and from (16), (14)
(17) \[
G(t) = T(t) \circ G(e), \quad t \in K.
\]
Setting $s = e$ in (15) and (12) we get
\begin{equation}
F(t) = F(e) \circ g \circ (G(e) \circ g)^{-1} \circ H(e)^{-1} \circ F(t), \quad t \in K,
\end{equation}
and
\begin{equation}
F(t) = F(e) \circ g \circ (G(e) \circ g)^{-1} \circ G(t), \quad t \in K
\end{equation}
respectively. Hence by (17), the definition of $T$, and (18) we obtain
\begin{equation}
F(t) = H(e) \circ T(t) \circ G(e), \quad t \in K.
\end{equation}
Putting $A := H(e)$, $B := G(e)$ we get from (19), (17) and (16) the formulas (3).

The proof of (ii) is easy.

Now we present an application of the above result.

Let $D(R_+) := \{(s,t) \in R_+: s \cdot t = 0 \text{ or } s = c \cdot t\}$ where $R_+$ denotes the set of all non-negative real numbers and $c \in R_+$.

Assume that \{\text{$F_t : t \in R_+$,}$\}$, \{\text{$G_t : t \in R_+$,}$\}$, \{\text{$H_t : t \in R_+$,}$\}$ are one-parameter families of functions mapping a real interval $I := (a,b)$ into itself. We consider the following problem: when does the equality $F_{t+s} = H_s \circ G_t$ for $(s,t) \in D(R_+)$ imply that $F_{t+s} = H_s \circ G_t$ for $(s,t) \in R_+^2$? The analogous problem for the Cauchy equation
\begin{equation}
F_{t+s} = F_t \circ F_s, \quad (s,t) \in D(R_+)
\end{equation}
has been considered by M.C. Zdun in [8] and M. Sablik in [3]. In the latter paper there has been proved the following

**Theorem 2.** If the limit $\lim_{t \to 0} \frac{F_t(x) - x}{t} =: d(x) \neq 0$ exists in $(a,b)$, $d$ is a continuous function, $F(x,t) = F_t(x)$ is continuous (as a function of two variables) and (20) holds, then \{\text{$F_t : t \in R_+$,}$\}$ is an iteration semigroup (i.e. $F_{t+s} = F_t \circ F_s$ for $(s,t) \in R_+^2$, cp. [4]).

Using Theorems 1(i) and 2 we shall prove the following

**Proposition 1.** Suppose that $H_0 \in \text{In}(I)$, $G_0 \in \text{Sur}(I)$, the functions $H(x,t) := H_t(x)$, $H_0$ are continuous and $F_{t+s} = H_s \circ G_t$ for $(s,t) \in D(R_+)$. If the limit $\lim_{t \to 0} \frac{(H_0^{-1} \circ H_t)(x) - x}{t} =: d(x) \neq 0$ exists in $(a,b)$ and $d$ is continuous then $F_{t+s} = H_s \circ G_t$ for $(s,t) \in R_+^2$.

**Proof.** By the proof of Theorem 1(i) we have the formulas
\begin{equation}
\begin{cases}
F_t = H_0 \circ T_t \circ G_0, \\
G_t = T_t \circ G_0, \\
H_t = H_0 \circ T_t, \quad t \in R_+,
\end{cases}
\end{equation}
where \( \{ T_t : t \in R_+ \} \subset I^I \) is a family of functions such that \( T_{t+s} = T_t \circ T_s \) for \( (s, t) \in D(R_+) \). On account of (21) we have
\[
\lim_{t \to 0} \frac{(H^{-1} \circ H_t)(x) - x}{t} = \lim_{t \to 0} \frac{T_t(x) - x}{t}.
\]
Hence, by Theorem 2, we infer that \( T_{t+s} = T_t \circ T_s \) for \( (s, t) \in R^2_+ \) and consequently, from formulas (21), we get the Proposition.

In the associative case we have the following general Lemma which will be used in the proof of the next Theorem:

Lemma. (i) Let \( K \) be a groupoid such that the binary operation is associative and \( K_l(t) \neq \emptyset, K_r(t) \neq \emptyset \) for every \( t \in K \). Suppose that \( l \in \times_{t \in K} K_l(t), r \in \times_{t \in K} K_r(t) \) and
\[
\{ F_t : t \in K \} \subset Z^X, \{ G_t : t \in K \} \subset Y^X, \{ H_t : t \in K \} \subset Z^Y \text{ satisfy (1) for} (s, t) \in D(K), \text{and}
\]
\[
(C) \quad H_l(t) \in \text{In}(Y, Z), \ G_r(t) \in \text{Sur}(X, Y) \quad \text{for} \ t \in K
\]
then there exist \( \{ M_t \} t \in K \subset \text{In}(Y, Z), \{ N_t \} t \in K \subset \text{Sur}(X, Y) \) and \( \{ T_t \} t \in K \subset Y^Y \) such that
\[
(H) \quad T_{st} = T_s \circ T_t, \quad M_{st} \circ T_{st} = M_s \circ T_{st}, \quad T_{st} \circ N_{st} = T_{st} \circ N_t
\]
for \( (s, t) \in D(K) \), and
\[
(22) \quad \left\{
\begin{array}{l}
F_t = M_t \circ T_t \circ N_t, \\
G_t = T_t \circ N_t, \\
H_t = M_t \circ T_t,
\end{array}
\right. \quad t \in K
\]

(ii) Conversely, if \( \{ M_t \} t \in K \subset Z^Y, \{ N_t \} t \in K \subset Y^X, \{ T_t \} t \in K \subset Y^Y \) satisfy (H) then the functions \( F_t, G_t, H_t \) given by (22) fulfil equation (1).

Proof. Suppose that \( \{ F_t \} t \in K, \{ G_t \} t \in K, \{ H_t \} t \in K \) satisfy (1) and condition (C). Put \( F(t) := F_t, G(t) := G_t, H(t) := H_t, l_t := l(t), r_t := r(t) \).

From equation (1) we directly obtain
\[
(23) \quad F(st) = F(l_{st}(st)) = H(l_{st}) \circ G(st), \quad (s, t) \in D(K).
\]
\[
(24) \quad F(st) = F((st)r_{st}) = H(st) \circ G(r_{st}),
\]
In view of associativity we have
\[
(25) \quad F(st) = F((l_s s) t) = F(l_s(st)) = H(l_s) \circ G(st), \quad (s, t) \in D(K).
\]
\[
(26) \quad F(st) = F(s(tr_t)) = F((st)t_r) = H(st) \circ G(r_t),
\]
Setting in (1) $t = r_s$ and then $s = l_t$ we get

\[(27) \quad F(s) = H(s) \circ G(r_s), \quad s \in K\]

\[(28) \quad F(t) = H(l_t) \circ G(t), \quad t \in K.\]

Comparing the right hand sides of (27) and (28) for $s = t$ we obtain

\[(29) \quad H(t) \circ G(r_t) = H(l_t) \circ G(t), \quad t \in K.\]

Hence, using (29) and the relation $G(r_t) \in \text{Sur}(X,Y)$, we infer that

\[(30) \quad \text{Ran} \ H(t) \subset \text{Ran} \ H(l_t) \quad \text{for all } t \in K.\]

Moreover, from (25) it follows that

\[(31) \quad \text{Ran} \ F(st) \subset \text{Ran} \ H(l_s) \quad \text{for } (s, t) \in D(K).\]

Note that, in view of (27) and the fact that $G(r_t) \in \text{Sur}(X,Y)$,

\[(32) \quad \text{Ran} \ H(t) = \text{Ran} \ F(t), \quad t \in K.\]

Thus from (30), (31), (32) we have the following relations

\[
\begin{align*}
\text{Dom} \ H(l_t)^{-1} & \supset \text{Ran} \ H(t) = \text{Ran} \ F(t), \quad t \in K, \\
\text{Dom} \ H(l_s)^{-1} & \supset \text{Ran} \ F(st), \quad (s, t) \in D(K).
\end{align*}
\]

Now, on account of (23) and (25) we get

\[(33) \quad H(l_{st})^{-1} \circ F(st) = H(l_s)^{-1} \circ F(st) \quad \text{for } (s, t) \in D(K).\]

Fix a $t \in K$ and introduce an equivalence relation $\sim_t$ on $X$ putting $x \sim_t y$ iff $G(r_t)(x) = G(r_t)(y)$. Denote

\[\tilde{X} := X/\sim_t.\]

Fix an invertible mapping $g_t : \tilde{X} \to X$ such that

\[g_t([x]) \in [x].\]

Then for every $t \in K$ the mapping $G(r_t) \circ g_t : \tilde{X} \to Y$ is a bijection.

From (27) we obtain

\[F(t) \circ g_t = H(t) \circ G(r_t) \circ g_t, \quad \text{whence}\]

\[(34) \quad H(t) = F(t) \circ g_t \circ (G(r_t) \circ g_t)^{-1}.\]

Using (34), (1) may be written as follows:

\[(35) \quad F(st) = F(s) \circ g_s \circ (G(r_s) \circ g_s)^{-1} \circ G(t) \quad \text{for } (s, t) \in D(K).\]
Note that the relations (24) and (26) imply the equalities
\[ F(st) \circ g_{st} = H(st) \circ G(r_{st}) \circ g_{st}, \]
\[ F(st) \circ g_t = H(st) \circ G(r_t) \circ g_t, \quad (s, t) \in D(K). \]
Hence we get
\[ F(st) \circ g_{st} \circ (G(r_{st}) \circ g_{st})^{-1} = F(st) \circ g_t \circ (G(r_t) \circ g_t)^{-1} \]
for \((s, t) \in D(K)\). By (28) we have
\[ G(t) = H(l_t)^{-1} \circ F(t), \quad t \in K. \]
Next, (29) implies
\[ G(t) = H(l_t)^{-1} \circ H(t) \circ G(r_t), \quad t \in K. \]
Putting (37) into (35) we obtain
\[ F(st) = F(s) \circ g_s \circ (G(r_s) \circ g_s)^{-1} \circ H(l_t)^{-1} \circ F(t) \]
for \((s, t) \in D(K)\). Define
\[ T(t) := H(l_t)^{-1} \circ F(t) \circ g_t \circ (G(r_t) \circ g_t)^{-1}, \quad t \in K. \]
Hence by (33), (36), and (39) we can write
\[ T(st) = H(l_{st})^{-1} \circ F(st) \circ g_{st} \circ (G(r_{st}) \circ g_{st})^{-1} = \]
\[ = H(l_s)^{-1} \circ F(st) \circ g_{st} \circ (G(r_s) \circ g_{st})^{-1} = \]
\[ = H(l_s)^{-1} \circ F(st) \circ g_t \circ (G(r_t) \circ g_t)^{-1} = \]
\[ = H(l_s)^{-1} \circ F(s) \circ g_s \circ (G(r_s) \circ g_s)^{-1} \circ H(l_t)^{-1} \circ \]
\[ \circ F(t) \circ g_t \circ (G(r_t) \circ g_t)^{-1} = T(s) \circ T(t). \]
Thus \( T(st) = T(s) \circ T(t) \) for \((s, t) \in D(K)\). From (34) we have
\[ H(t) = H(l_t) \circ T(t), \quad t \in K, \]
and from (40), (38)
\[ G(t) = T(t) \circ G(r_t), \quad t \in K. \]
Setting in (39) and then in (35) \( s = l_t \) we get
\[ F(t) = F(l_t) \circ g_{l_t} \circ (G(r_{l_t}) \circ g_{l_t})^{-1} \circ H(l_t)^{-1} \circ F(t), \quad t \in K \]
and
\[ F(t) = F(l_t) \circ g_{l_t} \circ (G(r_{l_t}) \circ g_{l_t})^{-1} \circ G(t), \quad t \in K, \]
respectively. Hence, using (41), the definition of the function $T$ and (42), we can write

$$F(t) = F(l_t) \circ g_t \circ (G(r_t) \circ g_t)^{-1} \circ T(t) \circ G(r_t) =$$

$$= F(l_t) \circ g_t \circ (G(r_t) \circ g_t)^{-1} \circ H(l_t)^{-1} \circ F(t) \circ g_t \circ$$

$$\circ (G(r_t) \circ g_t)^{-1} \circ G(r_t) =$$

$$= F(t) \circ g_t \circ (G(r_t) \circ g_t)^{-1} \circ G(r_t) = H(l_t) \circ T(t) \circ G(r_t).$$

Thus, the following equality holds:

(43) \[ F(t) = H(l_t) \circ T(t) \circ G(r_t), \quad t \in K. \]

Define

(44) \[ M(t) := H(l_t), \quad N(t) := G(r_t) \quad \text{for} \quad t \in K. \]

It is clear that $M(t) \in \text{In}(Y, Z)$ and $N(t) \in \text{Sur}(X, Y)$ for $t \in K$.

Now we show that the functions $M, N$ satisfy condition (H). Using (33) it is easy to check that

$$M(st) \circ T(st) = M(s) \circ T(st) \quad \text{for} \quad (s, t) \in D(K).$$

Note that (24) and (26) yield

$$H(st) \circ G(r_st) = H(st) \circ G(r_t) \quad \text{for} \quad (s, t) \in D(K),$$

hence by (40)

$$H(l_{st}) \circ T(st) \circ G(r_{st}) = H(l_{st}) \circ T(st) \circ G(r_t), \quad (s, t) \in D(K),$$

and consequently

$$T(st) \circ N(st) = T(st) \circ N(t), \quad (s, t) \in D(K).$$

Finally, formulas (22) result directly from (43), (41) and (40).

The proof of (ii) is easy.

**Theorem 3.** (i) Let $K$ be a groupoid such that the binary operation is associative and $K_l(t) \neq \emptyset$, $K_r(t) \neq \emptyset$ for $t \in K$. Assume that there exist functions $l : K \ni t \to l(t) \in K_l(t)$, $r : K \ni t \to r(t) \in K_r(t)$ such that

(45) \[ l(st) = l(s), \quad r(st) = r(t) \quad \text{for} \quad (s, t) \in D(K) \]

and $\{F_t : t \in K\} \subset Z^X$, $\{G_t : t \in K\} \subset Y^X$, $\{H_t : t \in K\} \subset Z^Y$ satisfy (1). If $H_{l(t)} \in \text{In}(Y, Z)$, $G_{r(t)} \in \text{Sur}(X, Y)$, $t \in K$ then there exist $\{M_t : t \in K\} \subset \text{In}(Y, Z)$, $\{N_t : t \in K\} \subset \text{Sur}(X, Y)$, $\{T_t : t \in K\} \subset Y^Y$ satisfying the condition

(G) \[ M_{st} = M_s, \quad N_{st} = N_t, \quad T_{st} = T_s \circ T_t \quad \text{for} \quad (s, t) \in D(K) \]
and

\[
\begin{align*}
F_t &= M_t \circ T_t \circ N_t, \\
G_t &= T_t \circ N_t, \\
H_t &= M_t \circ T_t, & t \in K.
\end{align*}
\]

(ii) Conversely, if \( \{M_t : t \in K\} \subset Z^Y, \{N_t : t \in K\} \subset Y^X, \{T_t : t \in K\} \subset Y^Y \) satisfy \((G)\) then the functions \(F_t, G_t, H_t\) given by (46) fulfil equation (1).

**Proof.** According to the Lemma there exist families of functions \(\{M_t : t \in K\} \subset \text{In}(Y, Z), \{N_t : t \in K\} \subset \text{Sur}(X, Y), \{T_t : t \in K\} \subset Y^Y\) satisfying condition \((H)\) and such that formulas (46) hold. It is easy to see, using (45), that the families \(\{M_t : t \in K\}, \{N_t : t \in K\}\) defined by (44) satisfy condition \((G)\). So, the proof of (i) is finished.

The proof of (ii) is trivial.

The following example gives an application of Theorem 3.

**Example 1.** Let us consider the following functional equation

\[(47)\]

\[F_{\min\{s, t\}} = H_s \circ G_t \text{ for } (s, t) \in D\]

where \(D := \{(s, t) \in \mathbb{R}^2 : t \leq s \leq c\}, c\) is a fixed real number and \(\{F_t : t \leq c\} \subset Z^X, \{G_t : t \leq c\} \subset Y^X, \{H_t : t \leq c\} \subset Z^Y\) are unknown families of functions. Putting \(l(s) := c\) and \(r(s) := s\) for \(s \in \mathbb{R}, s \leq c\), it is easy to check that (45) holds. Analysing the proof of the Lemma, it is easy to see that the associativity assumption in Theorem 3 can be omitted.

Thus, assuming that \(H_c \in \text{In}(Y, Z), G_s \in \text{Sur}(X, Y)\) for \(s \leq c\), we may use Theorem 3 to get a solution of equation (47). Namely, according to \((G)\), we have \(M_t = M_{\min\{c, t\}} = M_c =: A\) for \(t \leq c\). So, every solution has the form

\[
\begin{align*}
F_t &= A \circ T_t \circ N_t, \\
G_t &= T_t \circ N_t, \\
H_t &= A \circ T_t & t \leq c,
\end{align*}
\]

for some \(A \in Z^Y, \{N_t : t \leq c\} \subset Y^X\) and \(\{T_t : t \leq c\} \subset Y^Y\) such that \(T_{\min\{s, t\}} = T_s \circ T_t\) for \((s, t) \in D\).

The next Proposition gives a condition under which a groupoid \(K\) has the choice functions \(l, r\) satisfying condition (45). To precise the formulation of the Proposition let us denote by \(K^o\) the set of all elements \(e\) from a groupoid \(K\) such that the following condition holds for all \(t \in K\):

\[(48)\]

\[
\begin{align*}
\text{if } et \text{ is defined then } et = t, \\
\text{if } te \text{ is defined then } te = t.
\end{align*}
\]
Define
\[ K_I^o(t) := \{ e \in K^o : \text{et is defined} \}; \]
\[ K_r^o(t) := \{ e \in K^o : te \text{ is defined} \}, \quad t \in K. \]

**Proposition 2.** Let \( K \) be a groupoid such that the binary operation is associative. Suppose that the set \( K_I^o(t) (K_r^o(t)) \) is nonempty for every \( t \in K \). Then there exists a function \( l : K \ni t \rightarrow l(t) \in K_I(t) (r : K \rightarrow K_r(t)) \) such that \( l(st) = l(s) \ (r(st) = r(t)) \) for \((s, t) \in D(K)\).

**Proof.** Let \( l^o : K \ni t \rightarrow l^o(t) := l_t^o \in K_I^o(t) \). From associativity we obtain
\[ l^o_{st}(st) = (l^o_{st}s)t, \quad l^o_{st}(st) = (l^o_{st}s)t \quad \text{for} \ (s, t) \in D(K). \]

Consequently, the products \( l^o_{st}s, l^o_{st}(st) \) are defined for \((s, t) \in D(K)\). Moreover, in virtue of associativity, we may write
\[ l^o_{st}s = l^o_{st}(l^o_{st}s) = (l^o_{st}s)l^o_{st}, \quad l^o_{st}(st) = l^o_{st}(l^o_{st}s) = (l^o_{st}s)l^o_{st} \]
for \((s, t) \in D(K)\). Hence the products \( l^o_{st}l^o_{st}, l^o_{st}s \) are defined and, on account of (48), we get \( l^o_{st} = l^o_s \) for \((s, t) \in D(K)\). Putting \( l(s) := l^o(s) \) for \( s \in K \) we get the Proposition. In the case when \( K_r^o(t) \) is a nonempty set one can proceed in an analogous way.

**Example 2.** Let \( \{ X_i : i \in W \} \) be a family of disjoint sets and let \( S_{ij} \) be the family of all mappings \( f : X_i \rightarrow X_j \) for \( i, j \in W \). It is easy to check that, for the groupoid \( S := \bigcup_{i,j} S_{ij} \) (with composition of functions as a binary operation), the sets \( K_I^o(f), K_r^o(f) \) are nonempty for every \( f \in S \). So, under suitable assumptions, we may use Proposition 2 to reduce the equation
\[ F_{fg} = H_f \circ G_g, \quad f, g \in S, \]
where \( \{ F_f : f \in S \} \subset Z^Y \), \( \{ G_f : f \in S \} \subset Y^X \), \( \{ H_f : f \in S \} \subset Z^Y \) are unknown families of functions, to the Cauchy equation.

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