Extension theory and the $\Psi^\infty$ operator

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Abstract. We are going to define for each simplicial complex $K$, an operator $\Psi^\infty$ on the subcomplexes of $K$. If one is given a collection of spaces, closed subspaces of them, and maps of the closed subspaces to a subpolyhedron of $|K|$ that extend to maps into $|K|$, then we are going to use the $\Psi^\infty$ operator to help determine a subcomplex of minimal cardinality into which the maps can be extended simultaneously.

The question (raised by A. Dranishnikov and J. Dydak) of whether the extension dimension, $\text{extdim}_{(C,T)}X$, has a countable representative when $X$ is compact and metrizable, $C$ is the class of compact metrizable spaces, and $T$ is the class of CW-complexes is an unsolved problem. We shall define an “anti-basis” for a CW-complex and use this along with the $\Psi^\infty$ operator to allow one to view this problem from another perspective.

1. Introduction

Extension theory, which was first introduced by A. DRANISHNIKOVA in 1994, is based on the following notion. If $K$ is a CW-complex and $X$ is a space, then one says that $K$ is an absolute extensor for $X$, $K \in \text{AE}(X)$, or $X$ is an absolute co-extensor for $K$, $X \tau K$, if for each closed subset $A$ of $X$ and map (i.e. continuous function) $f : A \rightarrow K$, there exists a map $F : X \rightarrow K$ such that $F$ is an extension of $f$. For example, if $X$ is a normal space and $K = I = [0, 1]$, then Tietze’s extension theorem yields that $I \in \text{AE}(X)$, or $X \tau I$.

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Key words and phrases: absolute co-extensor, absolute extensor, cardinality of complex, cohomological dimension, CW-complex, compactification, covering dimension, extension theory, finitely-bounded, Hausdorff $\sigma$-compactum, $\lambda$-bounded, $\sigma$-compactum, Stone-Čech compactification, universal compactum, weight.
Suppose that $X$ is either a metrizable space or a compact Hausdorff space. A classical result from the theory of covering dimension is that $\text{dim} \, X \leq n$ if and only if $X \tau S^n$. For cohomological dimension $\text{dim}_G$ over an abelian group $G$, a similar fact is true: $\dim_G X \leq n$ if and only if $X \tau K$ where $K$ is an Eilenberg–Mac Lane CW-complex in the class $K(G, n)$. For these and other reasons, A. DRANISHNIKOV [2] defined the notions of extension theory and extension dimension. Given a class $\mathcal{C}$ of spaces and a class $\mathcal{T}$ of CW-complexes, one defines (see Section 5) an equivalence relation $\sim_{(\mathcal{C}, \mathcal{T})}$ on the CW-complexes. For a given space $X$, not necessarily in $\mathcal{C}$, its extension dimension, $\text{extdim}_{(\mathcal{C}, \mathcal{T})} X$, may exist. The latter, when it exists, is a uniquely determined equivalence class under $\sim_{(\mathcal{C}, \mathcal{T})}$. He and J. DYDAK asked in [4] (Problem 5.4 below) whether with respect to the classes $\mathcal{C}$ of compact Hausdorff spaces and $\mathcal{T}$ of CW-complexes, the extension dimension of every metrizable compactum has a countable representative. We shall show, Proposition 5.5, that for certain universal compacta the answer is yes.

Whenever $X$ is a Tychonoff space, then by $\beta X$ we shall mean the Stone-\v{C}ech compactification of $X$. Let $X = \sum \{X_s \mid s \in S\}$ be a topological sum of compact Hausdorff spaces, $K$ be a CW-complex, and assume that $X_s \tau K$ for each $s \in S$. Suppose that one is given a collection $\{A_s \mid s \in S\}$ of closed subsets $A_s$ of $X_s$ along with maps $f_s : A_s \to K$. Under what conditions can these maps $f_s$ be extended to maps $F_s : X_s \to K$ so that for some finite subcomplex $K_0$ of $K$, $F_s(X_s) \subseteq K_0$ for all $s \in S$? Definitions needed to describe such a problem and others more general than it along with some rudimentary results, e.g., Corollary 2.9, about such extensions can be found in Section 2 below. A situation like this was encountered in [16] where the author (see Proposition 2.4 of that citation) determined a relationship between that kind of extension problem and whether $\beta X \tau K$.

To deal simultaneously with the problems outlined above, we shall introduce in Section 3, for each simplicial complex $K$, the operator $\Psi^\infty$ on the subcomplexes of $K$. It will be true that $\Psi^\infty$ is idempotent and that if $X$ is a space, $X \tau |K|$, and $L$ is a subcomplex of $K$, then $X \tau |\Psi^\infty(L)|$. Moreover, if $L$ is of infinite cardinality, then the cardinality of $\Psi^\infty(L)$ equals the cardinality of $L$. We apply the $\Psi^\infty$ operator in Section 4. Our main result in that section is Theorem 4.7 which covers as a special case metrizable $\sigma$-compacta. One might also view Corollary 4.5 to see one of the fundamental properties of the $\Psi^\infty$ operator.

In Section 5 we introduce the concept of an anti-basis for a polyhedron $|K|$. Roughly speaking, it consists of a set of subcomplexes of $K$ that detect when a space $Y$ is not an absolute co-extensor for $K$. Theorem 5.10 states that for certain classes of spaces the existence of a countable anti-basis consisting of finite
subcomplexes implies the existence of a countable representative of extension dimension as in the question of Dranishnikov and Dydak.

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2. \( \lambda \)-bounded collections of maps

If \( K \) is a simplicial complex, then \(|K|\) will be endowed with the weak topology and we shall treat \(|K|\) as a CW-complex whose CW-structure is determined in the usual way by the triangulation \( K \). By \( \text{card } K \) we of course mean the cardinal number of the set of simplexes of \( K \). If \( K \) is a CW-complex, then by \( \text{card } K \) we mean the cardinal number of the set of cells of \( K \). If \( X \) is a space, then \( \text{wt } X \) will designate the weight of \( X \). The next lemma will be used implicitly below.

**Lemma 2.1.** Let \( f : K \to L \) be a map of CW-complexes and \( K_0 \) a subcomplex of \( K \):

1. if \( K_0 \) is finite, then \( f(K_0) \) is contained in a finite subcomplex of \( L \);
2. if \( K_0 \) is infinite, then \( f(K_0) \) is contained in a subcomplex \( M \) of \( L \) with \( \text{card } M \leq \text{card } K_0 \). \( \square \)

Let us recall Definition 2.1 of [16]; we use a slightly different terminology in order to conform to the needs of this paper.

**Definition 2.2.** Let \( K \) be a CW-complex and \( \mathcal{X} = \{(X_s, A_s) \mid s \in S\} \) a collection of pairs of spaces. Suppose that for each \( s \in S \), \( A_s \) is closed in \( X_s \) and a map \( f_s : A_s \to K \) has been given. We shall say that \( \{f_s \mid s \in S\} \) is \( \mathcal{X} \) finitely-bounded in \( K \) if there exists a finite subcomplex \( K_0 \) of \( K \) such that each map \( f_s \) can be extended to a map of \( X_s \) into \( K_0 \). If \( A_s = A, X_s = X, \) and \( f_s = f \) for all \( s \in S \), then we shall refer to the map \( f \) as being \((X, A)\) finitely-bounded in \( K \) with the obvious meaning.

Let us first note:

**Lemma 2.3.** Let \( f : K \to L \) be a map between CW-complexes \( K \) and \( L \). Suppose that \( \{X_s \mid s \in S\} \) is a set of spaces. Let \( \{A_s \mid s \in S\} \) and \( \{f_s \mid s \in S\} \) be collections such that for each \( s \in S \), \( A_s \) is a closed subspace of \( X_s \) and \( f_s : A_s \to K \) is a map. Put \( \mathcal{X} = \{(X_s, A_s) \mid s \in S\} \).

1. If \( \{f_s \mid s \in S\} \) is \( \mathcal{X} \) finitely-bounded in \( K \), then \( \{f \circ f_s \mid s \in S\} \) is \( \mathcal{X} \) finitely-bounded in \( L \).
Suppose that \( g : L \to K \) is a map, \( g \circ f \simeq 1_K \), and for each \( s \in S \), \( X_s \) has the homotopy extension property with respect to CW-complexes. Assume also that there is a finite subcomplex \( M \) of \( K \) such that \( f_s(A_s) \subset M \) for all \( s \in S \) and that \( \{ f \circ f_s \mid s \in S \} \) is \( X \) finitely-bounded in \( L \). Then \( \{ f_s \mid s \in S \} \) is \( X \) finitely-bounded in \( K \).

**Proof.** (1) Let \( K_0 \) be a finite subcomplex of \( K \), and \( \{ F_s \mid s \in S \} \) a collection such that for each \( s \in S \), \( F_s : X_s \to K_0 \) is a map having the property that \( F_s \mid A_s \neq f_s \). There exists a finite subcomplex \( L_0 \) of \( L \) (see Lemma 2.1(1)) such that \( f(K_0) \subset L_0 \). Then \( \{ f \circ F_s \mid s \in S \} \) witnesses the fact that \( \{ f \circ f_s \mid s \in S \} \) is \( X \) finitely-bounded in \( L \).

(2) Let \( L_0 \) be a finite subcomplex of \( L \) and \( \{ G_s \mid s \in S \} \) a collection of maps \( G_s : X_s \to L_0 \) such that \( G_s \mid A_s = f \circ f_s \) for all \( s \in S \). Again applying Lemma 2.1(1), choose a finite subcomplex \( K^* \) of \( K \) such that \( g(L_0) \subset K^* \). We may assume that \( M \subset K^* \). Hence, \( \{ G_s \mid s \in S \} \) is a collection such that \( g \circ G_s : X_s \to K^* \) is a map for each \( s \in S \).

Let \( F : K \times [0,1] \to K \) be a homotopy such that \( F(x,0) = x \) and \( F(x,1) = g \circ f(x) \) for all \( x \in K \). Put \( K' = F(K^* \times [0,1]) \). Then \( K' \) is contained in a finite subcomplex \( K_0 \) of \( K \). Moreover, \( F(K^* \times \{0\}) = K^* \), so \( K^* \subset K_0 \). Putting \( F^* = F \mid (K^* \times [0,1]) : K^* \times [0,1] \to K_0 \) one gets a deformation \( F^* \) of \( K^* \) in \( K_0 \) having the property that \( F^*(x,1) = g \circ f(x) \) for all \( x \in K^* \).

Notice that if \( s \in S \) and \( a \in A_s \), then \( f_s(a) \in M \subset K^* \). So there is a homotopy \( Q_s : A_s \times I \to K_0 \) given by \( Q_s(a,t) = F^*(f_s(a),t) \). We see that \( Q_s(a,0) = F^*(f_s(a),0) = f_s(a) \) and \( Q_s(a,1) = F^*(f_s(a),1) = g \circ f \circ f_s(a) \). But \( g \circ G_s \mid A_s = g \circ f \circ f_s \) and \( g \circ G_s : X_s \to K^* \subset K_0 \). The homotopy extension property shows that \( f_s \) extends to a map of \( X_s \) into \( K_0 \). Since \( K_0 \) is finite and is independent of the choice of \( s \in S \), our proof of (2) is complete. \( \square \)

We now extend Definition 2.2 to consider maps to CW-complexes whose images must land in subcomplexes of infinite cardinalities.

**Definition 2.4.** Let \( K \) be a CW-complex, \( \lambda \) be an infinite cardinal, and \( \mathcal{X} = \{(X_s, A_s) \mid s \in S \} \) a collection of pairs of spaces. Suppose that for each \( s \in S \), \( A_s \) is closed in \( X_s \) and a map \( f_s : A_s \to K \) has been given. We shall say that \( \{ f_s \mid s \in S \} \) is \( \mathcal{X} \) \( \lambda \)-bounded in \( K \) if there exists a subcomplex \( K_0 \) of \( K \) such that \( \text{card} \; K_0 \leq \lambda \) and each map \( f_s \) can be extended to a map of \( X_s \) into \( K_0 \). If \( A_s = A \), \( X_s = X \), and \( f_s = f \) for all \( s \in S \), then we shall refer to the map \( f \) as being \( (X,A) \) \( \lambda \)-bounded in \( K \) with the obvious meaning.
Lemma 2.5. Let $Y$ be a space, $A$ a closed subspace of $Y$, and $K$ a CW-complex. Let $X$ be a space, $g : X \to Y$ a map, $B$ a closed subspace of $g^{-1}(A)$, and $f : A \to K$ a map.

1. If $f$ is $(Y, A)$ finitely-bounded, then $f \circ (g | B)$ is $(X, B)$ finitely bounded.
2. If $\lambda$ is an infinite cardinal and $f$ is $(Y, A)$ $\lambda$-bounded in $K$, then $f \circ (g | B)$ is $(X, B)$ $\lambda$-bounded in $K$. \hfill $\square$

As pointed out in [16], nontrivial examples of CW-complexes $K$ along with a non-finitely-bounded collection of maps in $K$ can be extrapolated from the proof of Theorem 1.5 of [12]. In that proof, the author produces a countably infinite set $T$ and a collection, $\{X_T \mid T \in T\}$ of metrizable compacta. Each $X_T$ has a specified closed subspace $S_T$ homeomorphic to $S^2$. It is true that $\dim_G X_T \leq 2$ for every abelian group $G$. In the last paragraph of the proof, select $K$ (designated $P$ there) to be $K(G, 2)$ for any nontrivial abelian group $G$. Then for each $T \in T$, let $f_T : S_T \to K$ be a map such that $f_T(S_T) = f_T(S_{T'})$ and $(f_T)_*(H_2(S_T)) = (f_T)_*(H_2(S_{T'})) \neq 0$ for each $T, T' \in T$. With this and an examination of the finale of the proof of Theorem 1.5 in [12], we have,

Proposition 2.6. For every nontrivial abelian group $G$ and $K = K(G, 2)$, there exist a countably infinite set $S$, collections $\mathcal{X} = \{(X_s, A_s) \mid s \in S\}$, and $\{f_s \mid s \in S\}$ where for each $s \in S$, $X_s$ is a compact metrizable space with $X_s \cap K$, $A_s$ is a closed subspace of $X_s$, and $f_s : A_s \to K$ is a map whose image lies in a fixed finite subcomplex of $K$, chosen in such a manner that, $\{f_s \mid s \in S\}$ is $\mathcal{X}$ finitely-bounded in $K$ but not $\mathcal{X}$ finitely-bounded in $K$. \hfill $\square$

The next lemma can be proved using the same techniques found in our proof of Lemma 2.3.

Lemma 2.7. Let $f : K \to L$ be a map between CW-complexes $K$ and $L$. Suppose that $\{X_s \mid s \in S\}$ is a set of spaces. Let $\{A_s \mid s \in S\}$ and $\{f_s \mid s \in S\}$ be collections such that for each $s \in S$, $A_s$ is a closed subspace of $X_s$ and $f_s : A_s \to K$ is a map. Put $\mathcal{X} = \{(X_s, A_s) \mid s \in S\}$ and suppose that $\lambda$ is an infinite cardinal.

1. If $\{f_s \mid s \in S\}$ is $\mathcal{X}$ $\lambda$-bounded in $K$, then $\{f \circ f_s \mid s \in S\}$ is $\mathcal{X}$ $\lambda$-bounded in $L$.
2. Suppose that $g : L \to K$ is a map, $g \circ f \simeq 1_K$, and for each $s \in S$, $X_s$ has the homotopy extension property with respect to CW-complexes. Assume also that there is a subcomplex $M$ of $K$ with card $M \leq \lambda$ such that $f_s(A_s) \subseteq M$ for all $s \in S$ and that $\{f \circ f_s \mid s \in S\}$ is $\mathcal{X}$ $\lambda$-bounded in $L$. Then $\{f_s \mid s \in S\}$ is $\mathcal{X}$ $\lambda$-bounded in $K$. \hfill $\square$
A proof similar to that of Lemma 3.1 of [6], which applies to polyhedra, can be used to obtain the following stronger result applying to CW-complexes:

**Lemma 2.8.** Let $X$ be a space with $\text{wt} X \leq \lambda$ for some infinite cardinal $\lambda$. Suppose $f : X \to K$ is a map where $K$ is a CW-complex. Then $f(X) \subset L$ for some subcomplex $L$ of $K$ where $\text{card} L \leq \lambda$. □

**Corollary 2.9.** Let $K$ be a CW-complex and $\lambda$ an infinite cardinal. Suppose that $S$ is a set with $\text{card} S \leq \lambda$, $\{X_s \mid s \in S\}$, $\{A_s \mid s \in S\}$ are collections of spaces with $X_s \tau K$, $\text{wt} X_s \leq \lambda$, and $A_s$ is a closed subset of $X_s$ for each $s \in S$. Put $X = \{(X_s, A_s) \mid s \in S\}$. Then every collection $\{f_s \mid s \in S\}$ of maps $f_s : A_s \to K$ is $X$ $\lambda$-bounded in $K$. □

This leads to a result showing that “gluing” together such a collection of spaces does not change the “$\lambda$-bounded in $K$” condition.

**Corollary 2.10.** Let $\lambda$ be an infinite cardinal, $S$ a set, $\{X_s \mid s \in S\}$, $\{A_s \mid s \in S\}$ collections of spaces, and $A_s$ a closed subset of $X_s$ for each $s \in S$. Put $X = \{(X_s, A_s) \mid s \in S\}$. Assume that there is a space $A$ and for each $s \in S$, a homeomorphism $h_s : A_s \to A$. Let $X$ be the quotient set formed from $\sum \{X_s \mid s \in S\}$ by gluing the sets $X_s$ to $A$ via the homeomorphisms $h_s$ and let $q$ be the quotient function. Let $X$ be given a topology such that $A$ is closed in $X$ and $q \mid X_s : X_s \to X$ is a map for each $s \in S$. Then for each CW-complex $K$ the following are true:

1. If $f : A \to K$ is a map that is $(X, A)$ $\lambda$-bounded, then $\{ f \circ (q|A_s) \mid s \in S\}$ is $X$ $\lambda$-bounded in $K$.
2. If $\text{card} S \leq \lambda$ and for each $s \in S$, $\text{wt} X_s \leq \lambda$ and $X_s \tau K$, then every map $f : A \to K$ is $(X, A)$ $\lambda$-bounded in $K$. □

An example of a space $X$ as in Corollary 2.10 could be obtained as follows. Suppose that $\{X_s \mid s \in S\}$ is a collection of Hausdorff spaces each containing a closed subspace $A_s$ homeomorphic to say $S^n$. Then form $X$ by gluing these spaces together along $S^n$ and applying the weak topology to $X$.

### 3. $\Psi$ Operators

For each simplicial complex $K$, denote by $\mathcal{F}_K$ the set of nonempty finite subcomplexes of $K$. Fix a simplicial complex $K$. Suppose that $M \in \mathcal{F}_K$; let $\mathcal{D}_{(M,K)}$ be the set of $D \in \mathcal{F}_K$ such that $M \subset D$. Define a relation $\sim_{(M,K)}$ on $\mathcal{D}_{(M,K)}$ by declaring that if $D, C \in \mathcal{D}_{(M,K)}$, then $D \sim_{(M,K)} C$ if there exists a
simplicial isomorphism of \(D\) to \(C\) which is the identity on \(M\). Plainly \(\sim_{(M,K)}\) is an equivalence relation on \(D_{(M,K)}\), and we shall write the equivalence class of an element \(D\) of \(D_{(M,K)}\) as \([D]_{(M,K)}\). The equivalence class \([M]_{(M,K)}\) is just \(\{M\}\).

Let \(E_{(M,K)}\) be the set of equivalence classes of \(D_{(M,K)}\) under the relation \(\sim_{(M,K)}\) and \(q_{(M,K)} : D_{(M,K)} \to E_{(M,K)}\) the quotient function. The set \(E_{(M,K)}\) is countable. Using the axiom of choice, fix once and for all a function \(\theta_{(M,K)} : E_{(M,K)} \to D_{(M,K)}\) such that \(\theta_{(M,K)}([D]_{(M,K)}) \in [D]_{(M,K)}\) for each \(D \in D_{(M,K)}\), i.e., \(\theta_{(M,K)}(E) \in q_{(M,K)}^{-1}(E)\) for each \(E \in E_{(M,K)}\). We point out that \(M \subset \theta_{(M,K)}(E)\). Assume that the preceding construction has been applied to each \(M \in \mathcal{F}_K\).

For \(M \in \mathcal{F}_K\) and \(E \in E_{(M,K)}\), \(\theta_{(M,K)}(E)\) is a subcomplex of \(K\). Thus, (i) for all \(M \in \mathcal{F}_K\), \(\bigcup \theta_{(M,K)}(E_{(M,K)})\) is a subcomplex of \(K\) containing the subcomplex \(M\).

We now define the function \(\Psi\) from the set of subcomplexes \(L\) of \(K\) to the set of subcomplexes of \(K\) by,

\[
\Psi(L) = \bigcup \left\{ \bigcup \theta_{(M,K)}(E_{(M,K)}) \mid M \in \mathcal{F}_L \right\}.
\]

An application of (i) shows that for each pair \(L \subset L'\) of subcomplexes of \(K\), (ii) \(\Psi(L)\) is a subcomplex of \(K\), and \(L \subset \Psi(L)\), and (iii) \(\Psi(L) \subset \Psi(L')\).

Let us denote \(\Psi^0(L) = L\); inductively for each \(k \in \mathbb{N}\) if \(\Psi^{k-1}(L)\) has been defined, then by \(\Psi^k(L)\) we mean \(\Psi(\Psi^{k-1}(L))\). Put,

\[
\Psi^\infty(L) = \bigcup \{ \Psi^k(L) \mid k \in \mathbb{N} \}.
\]

Of course \(\Psi^\infty(L)\) is a subcomplex of \(K\). We shall show that \(\Psi^\infty\) is an idempotent operator on the set of subcomplexes of a given simplicial complex \(K\).

**Lemma 3.1.** Let \(K\) be a simplicial complex and \(L\) a subcomplex of \(K\). Then \(\Psi^1(\Psi^\infty(L)) = \Psi^\infty(L)\), and hence \(\Psi^\infty(\Psi^\infty(L)) = \Psi^\infty(L)\).

**Proof.** Suppose that \(M\) is a finite subcomplex of \(\Psi^\infty(L)\). Then for some \(k \in \mathbb{N}\), \(M \subset \Psi^k(L)\), so by (iii), \(\Psi(M) \subset \Psi(\Psi^k(L)) = \Psi^{k+1}(L) \subset \Psi^\infty(L)\). \(\square\)

From the construction of \(\Psi\), it is not difficult to see that, (iv) in case \(L\) is infinite, then \(\text{card}(\Psi^1(L)) = \text{card}(L)\).

We therefore may state the following lemma.

**Lemma 3.2.** Let \(K\) be a simplicial complex and \(L\) a subcomplex of \(K\). If
Let a nonintersecting countable set of vertices. Indeed, for any choice of 
the Ψ homotopy having the property that if 

$$L = \emptyset, \text{ then } \Psi^1(L) = \emptyset,$$

(2) $L$ is finite, then $\text{card}(\Psi^1(L)) \leq \aleph_0$, 

(1) $L$ is infinite and $k \in \mathbb{N}$, then we may conclude that 
$\text{card}(\Psi^k(L)) = \text{card}(\Psi^\infty(L)) = \text{card}(L)$. 

We now provide an example to illustrate the operator $\Psi^1$. Let $K$ be an 
infinite wedge of one-simplexes with vertex $v$ and $L$ the subcomplex consisting 
of the vertex $v$. The finite subcomplexes of $K$ that contain $L$ consist of finite 

wedges of 1-simplexes along with a nonintersecting discrete, 

nonempty, finite or countably infinite set of vertices. Simply fix in advance a subcomplex 
of the vertex $v$ that contain $L$, and for each $n \in \mathbb{N}$, $M_n$ is a wedge of $n$ 1-simplexes and $M_n \subset M_{n+1}$. Make all 

choices of values of $\theta_{(L,K)}$ to be subcomplexes of $M$ and so that for each $n \in \mathbb{N}$, 
$\theta_{(L,K)}([M_n]) = M_n$. Then $\Psi^1(L) = M$. 

On the other hand, we may also choose $\theta_{(L,K)}$ in a way that $\Psi^1(L)$ 
consists of a countable wedge of 1-simplexes along with a nonintersecting discrete, 
nonempty, finite or countably infinite set of vertices. Indeed, for any choice of 
$\theta_{(L,K)}$, $\Psi^1(L)$ will always consist of a countable wedge of 1-simplexes along with 
a nonintersecting countable set of vertices.

The situation with $\Psi^2(L)$ will again depend on $\theta_{(L,K)}$. If $\Psi^1(L) = M$ as 
above, then $\Psi^2(L)$ consists of $M$ along with a countable (possibly empty) set of 
vertices outside $M$. If $\Psi^1(L)$ contains some discrete nonempty set of vertices, 
then $\Psi^2(L)$ could consist of $\Psi^1(L)$ along with some additional 1-simplexes and 
perhaps an additional countable discrete set of vertices.

When a homotopy $F : |M| \times I \rightarrow |M|$ is treated then we in addition define 
the $\Psi_F^k$ operator, derive its properties (see Lemma 3.3), and use it to give a short 
proof of Proposition 3.4.

Let $M$ be a simplicial complex and $F : |M| \times I \rightarrow |M|$ a homotopy having 
the property that if $x \in |M|$, $\sigma \in M$, and $x \in \text{int} \sigma$, then $F(x,0) \in \sigma$. For each 
finite subcomplex $Q$ of $M$, note that $F([Q] \times \{0\}) \subset |Q|$. Let $S_Q$ be the smallest 
subcomplex of $M$ such that $F([Q] \times I) \subset |S_Q|$. Then $Q$ is a subcomplex of $S_Q$ and $S_Q$ is finite. For any subcomplex $L$ of $M$, put $\Psi_F(L) = \bigcup \{S_Q \mid Q \in \mathcal{F}_L\}$. 

Then $\Psi_F(L)$ is a subcomplex of $M$. Let $\Psi_F^0(L) = L$, and for each $k \in \mathbb{N}$, 
if $\Psi_F^{k-1}(L)$ has been defined, then we let $\Psi_F^k(L) = \Psi_F(\Psi_F^{k-1}(L))$. Finally, let 
$\Psi_F^\infty(L) = \bigcup \{\Psi_F^k(L) \mid k \in \mathbb{N}\}$. Then it is easy to check the next result.

**Lemma 3.3.** Let $M$ be a simplicial complex and $F : |M| \times I \rightarrow |M|$ a 
homotopy having the property that if $x \in |M|$, $\sigma \in M$, and $x \in \text{int} \sigma$, then
Then for each subcomplex $L$ of $M$, $\Psi_f^1(\Psi_f^\infty(L)) = \Psi_f^\infty(L)$, and hence $\Psi_f^\infty$ is an idempotent operator on the set of subcomplexes of $M$. Moreover, (1) if $\text{card } L$ is finite, then $\text{card}(\Psi_f^\infty(L)) \leq \aleph_0$, (2) if $\text{card } L$ is infinite, then $\text{card}(\Psi_f^\infty(L)) = \text{card}(L)$, and (3) $F([\Psi_f^\infty(L)] \times I) \subset [\Psi_f^\infty(L)]$. □

In Proposition 3.4, for completeness we state (1) without proof since this is a standard fact in the theory of CW-complexes, and our current techniques are useful only for proving (2).

**Proposition 3.4.** Let $K$ be a CW-complex of cardinality $\alpha$.

1. If $\alpha$ is finite, then there exists a finite simplicial complex $T$ and a homotopy equivalence between $K$ and $|T|$.

2. If $\alpha$ is infinite, then there exists a simplicial complex $T$ of cardinality $\leq \alpha$ and a homotopy equivalence between $K$ and $|T|$.

**Proof.** As mentioned above, we only prove (2). There exists a simplicial complex $M$ and a homotopy equivalence $h : K \to |M|$. Let $f : |M| \to K$ be a homotopy inverse of $h$ and $F : |M| \times I \to |M|$ a homotopy from the identity of $|M|$ to the map $h \circ f$. Choose a subcomplex $L$ of $M$ with $\text{card } L \leq \alpha$ such that $h(K) \subset |L|$. Let $T = \Psi_f^\infty(L) \subset M$ and $f^* = f \mid |T| : |T| \to K$. Apply Lemma 3.3(3) to see that $h \circ f^*$ is homotopic to the identity on $|T|$. It is routine to check that $f^* \circ h$ is homotopic to the identity on $K$. Apply Lemma 3.3(1,2) to see that $\text{card } T \leq \alpha$. □

### 4. $X$-connectedness and $\lambda$-boundedness

By a pair $(U, V)$ of spaces we mean a space $U$ along with a subspace $V$ of $U$. Next is Definition 6.1 of [16]. As mentioned there, this should be compared with similar ones given in [9], [10], and [11].

**Definition 4.1.** Let $X$ be a space and $(U, V)$ a pair of spaces. We shall say that $(U, V)$ is $X$-connected if for each closed subset $A$ of $X$ and map $f : A \to V$, there exists a map $F : X \to U$ that extends $f$.

The term $\sigma$-compactum usually refers to a metrizable space that can be written as a countable union of compact subspaces of itself. Such a space is obviously normal and Hausdorff; moreover, every CW-complex is an absolute neighborhood extensor for it. Let us generalize that definition.
Definition 4.2. Let $X$ be a space. Then we shall say that $X$ is a Hausdorff $\sigma$-compactum if $X$ is a normal Hausdorff space, every CW-complex is an absolute neighborhood extensor for $X$, and $X$ can be written as a countable union of compact Hausdorff subspaces.

Proposition 4.3. Let $K$ be a simplicial complex and $L$ a subcomplex of $K$. Suppose that $\{X_s \mid s \in S\}$ is a collection of Hausdorff $\sigma$-compacta and that for each $s \in S$, $X_s \tau[K]$. The following are true.

1. The pair $(|\Psi^\infty(L)|, |\Psi^\infty(L)|)$ is $X_s$-connected for each $s \in S$.
2. If $s \in S$ and $X_s$ is compact Hausdorff, then $(|\Psi^{n+1}(L)|, |\Psi^n(L)|)$ is $X_s$-connected.
3. If $\lambda$ is an infinite cardinal, $\text{card} L \leq \lambda$, for each $s \in S$, $A_s$ is a closed subset of $X_s$, $f_s : A_s \rightarrow |L|$ is a map, and $X = \{(X_s, A_s) \mid s \in S\}$, then $\{f_s \mid s \in S\}$ is $X$ $\lambda$-bounded in $|K|$.

Proof. Statement (3) of this proposition will follow from Statement (1) along with an application of Lemma 3.2(2,3). We proceed with a proof of (1).

Consider $s \in S$ and a map $f_s : A_s \rightarrow |\Psi^\infty(L)|$. Write $X_s = \bigcup\{Z_i \mid i \in \mathbb{N}\}$ with $Z_1 \subset Z_2 \subset \ldots$, and for each $i \in \mathbb{N}$, $Z_i$ is a compact Hausdorff space. We shall proceed with an induction argument.

Since $X_s \tau[K]$, then $Z_i \tau[K]$. Let $g_1 : A_s \cup Z_1 \rightarrow |K|$ be a map such that $g_1 | A_s = f_s$. There exists $M \in \mathcal{F}_{\Psi^\infty(L)}$ such that $g_1(Z_1 \cap A_s) \subset |M|$. Now $g_1(Z_i) \subset |M'|$ for some finite subcomplex $M'$ of $K$, where $M \subset M'$. By the definition of $\Psi^1(\Psi^\infty(L))$, we may as well assume that $M' \subset \Psi^1(\Psi^\infty(L)) = \Psi^\infty(L)$. Hence we may treat $g_1 : A_s \cup Z_1 \rightarrow |\Psi^\infty(L)|$.

Using the ANE property of $|\Psi^\infty(L)|$, there exists a closed neighborhood $D_1$ of $A_s \cup Z_1$ in $X_s$ and a map $h_1 : D_1 \rightarrow |\Psi^\infty(L)|$ that extends $g_1$.

Suppose that $k \in \mathbb{N}$ and we have found $D_1 \subset \cdots \subset D_k$, and $h_1, \ldots, h_k$ such that for $1 \leq i \leq k$:

(a) $D_i$ is a closed neighborhood of $A_s \cup Z_i$ in $X_s$,
(b) $h_i$ is a map of $D_i$ to $|\Psi^\infty(L)|$,
(c) $h_i | A_s = f_s$, and
(d) if $1 \leq i < j \leq k$, then $D_i \subset D_j$ and $h_j | D_i = h_i$.

Choose a map $g_{k+1} : D_k \cup Z_{k+1} \rightarrow |K|$ such that $g_{k+1} | D_k = h_k$. Now $h_k(Z_k \cup (Z_{k+1} \cap D_k)) = g_{k+1}(Z_k \cup (Z_{k+1} \cap D_k)) \subset |\Psi^\infty(L)|$. There exists $N \in \mathcal{F}_{\Psi^\infty(L)}$ such that $g_{k+1}(Z_k \cup (Z_{k+1} \cap D_k)) \subset |N|$.

Now $g_{k+1}(Z_{k+1}) \subset |N'|$ for some finite subcomplex $N'$ of $K$, where $N \subset N'$. By the definition of $\Psi^1(\Psi^\infty(L)) = \Psi^\infty(L)$, we may as well assume that $M' \subset \Psi^\infty(L)$. 


There exists a closed neighborhood $D_{k+1}$ of $D_k \cup Z_{k+1}$ in $X_s$ and a map $h_{k+1} : D_{k+1} \to |\Psi^\infty(L)|$ that extends $g_{k+1}$.

This completes the induction. Observe that $\bigcup \{ \text{int}_{X_s} D_k \mid k \in \mathbb{N} \} = X_s$. Define a function $F_s : X_s \to |\Psi^\infty(L)|$ to be $\bigcup \{ h_k \mid k \in \mathbb{N} \}$. Clearly $F_s$ is a map, and $F_s \mid A_s = f_s$. Hence for all $s \in S$, there exists a map $X_s$ into $|\Psi^\infty(L)| \subset |K|$ that extends $f_s$. This completes our proof of (1).

In the special case that $X_s$ is a compact Hausdorff space, start with a map $f_s : A_s \to |\Psi^n(L)|$. Just apply the first step of the above inductive argument and see that $F_s(X_s) \subset |\Psi^1(\Psi^n(L))| = |\Psi^{n+1}(L)|$, so (2) is true. □

Corollary 4.4. Let $K$ be a simplicial complex, $\lambda$ an infinite cardinal, and $L$ a subcomplex of $K$ with card $L \leq \lambda$. Suppose that $X$ is a Hausdorff $\sigma$-compactum such that $X \tau|K|$; then for each closed subspace $A$ of $X$ every map $f : A \to |L|$ is $(X, A)$ $\lambda$-bounded in $|K|$. □

Corollary 4.5. Let $K$ be a simplicial complex and $X$ a Hausdorff $\sigma$-compactum with $X \tau|K|$. Then for every subcomplex $L$ of $K$, $X \tau|\Psi^\infty(L)|$.

We shall use Proposition 4.6 in our proof of Theorem 4.7. It appears as Proposition 3.1 of [8] where we explain that this result differs from E. Michael’s Proposition 3.6(a) of [14], but is an improved version based on Lemma 1 of [15].

Proposition 4.6. Let $X$ be a paracompact space and $\mathcal{G}$ a collection of subsets of $X$. Suppose that the following are true:

1. $\mathcal{G}$ contains an open cover of $X$,
2. if $U \in \mathcal{G}$ and $W$ is open in $U$, then $W \in \mathcal{G}$,
3. if $U, Q$ are open elements of $\mathcal{G}$, then $U \cup Q \in \mathcal{G}$, and
4. if $K \subset \mathcal{G}$ is a discrete collection of open subsets of $X$, then $\bigcup K \in \mathcal{G}$.

Then the entire space $X$ is in $\mathcal{G}$. □

Theorem 4.7. Let $K$ be a simplicial complex, $\lambda$ an infinite cardinal, and $L$ a subcomplex of $K$ with card $L \leq \lambda$. Suppose that $X$ is a paracompact space, $\{ X_s \mid s \in S \}$ a locally finite cover of $X$ consisting of closed subspaces that are Hausdorff $\sigma$-compacta, and $X_s \tau|K|$ for each $s \in S$. Then $(|\Psi^\infty(L)|, |\Psi^\infty(L)|)$ is $X$-connected, and hence every map $f : A \to |\Psi^\infty(L)|$ of a closed subset $A$ of $X$ is $\lambda$-bounded in $|K|$. If for each $s \in S$, $X_s$ is compact and Hausdorff, then we may state additionally that for all $n \in \mathbb{N}$, $(|\Psi^{n+1}(L)|, |\Psi^n(L)|)$ is $X$-connected.

Proof. Let $\mathcal{G}$ be the collection of open subsets $G$ of $X$ such that if $A$ is a closed subset of $\text{cl}_X G$, and $f : A \to |\Psi^\infty(L)|$ is a map, then $f$ extends to a
map of $\text{cl}_X G$ to $|\Psi^\infty(L)|$. We will show that $G$ satisfies conditions (1)–(4) of Proposition 4.6. Then we will be assured that $X \in G$. The proof of the first part will be concluded by referring to Lemma 3.2(2,3).

Let $U$ be an open cover of $X$ with the property that if $U \in U$, then $\text{cl}_X(U)$ intersects $X_s$ for only finitely many $s \in S$. Fix $U \in U$, let $A$ be a closed subset of $\text{cl}_X U$, and $f : A \to |\Psi^\infty(L)|$ a map.

Let $T \subset S$ be the finite subset having the property that $X_s \cap \text{cl}_X U \neq \emptyset$ if and only if $s \in T$. Define $T_1$ to be the subset of $T$ such that if $s \in T$ and $X_s$ is a compact Hausdorff space, then $s \in T_1$. Let $T_2 = T \setminus T_1$.

Put $Y = \bigcup \{X_s \mid s \in T_1\}$. Then $Y$ is a compact Hausdorff space. By Proposition 4.3, there exists a map $h : Y \to |\Psi^1(\Psi^\infty(L))| = |\Psi^\infty(L)|$ that extends $f|(A \cap Y) : A \cap Y \to |\Psi^\infty(L)|$. Let $h^* : A \cup Y \to |\Psi^\infty(L)|$ be the map such that $h^*|A = f$ and $h^*|Y = h$.

Let $s \in T_2$. By Proposition 4.3, there exists a map $f_s : X_s \to |\Psi^\infty(\Psi^\infty(L))| = |\Psi^\infty(L)|$ that extends $h^* | ((A \cup Y) \cap X_s) : ((A \cup Y) \cap X_s) \to |\Psi^\infty(L)|$. Put $f^*_s : A \cup Y \cup X_s \to |\Psi^\infty(L)|$ such that $f^*_s | (A \cup Y) = h^*$ and $f^*_s | X_s = f_s$. Using the fact from Lemma 3.1 that $\Psi^\infty(\Psi^\infty(L)) = \Psi^\infty(L)$, one may, step by step, add the remaining $\sigma$-compacta indexed by $T_2$ to end up with a map of $\text{cl}_X U$ to $|\Psi^\infty(L)|$ that extends $f$. This shows that $G$ contains an open cover of $X$. Part (2) of Proposition 4.6 is easily seen to be true.

Since for open elements $U$ and $Q$ of $G$, $\text{cl}_X(U \cup Q) = \text{cl}_X U \cup \text{cl}_X Q$, the reader can see how to prove (3) of Proposition 4.6 by using the same techniques we just employed above. That $G$ satisfies Part (4) of Proposition 4.6 is obvious.

In case $X_s$ is compact and Hausdorff for each $s \in S$, then one may simply change the definition of $G$ to require that $f$ extends to a map of $\text{cl}_X G$ to $|\Psi^{n+1}(L)|$.

\[\square\]

5. Extension Dimension and Anti-Bases

We shall now recall the notion of extension dimension. Let $\mathcal{C}$ be a class of spaces, $T$ a class of CW-complexes, and $K, K' \in T$. If it is true that for all $X \in \mathcal{C}$, $X \tau K$ implies that $X \tau K'$, then we write $K \leq_{(\mathcal{C},T)} K'$. This defines a preorder on $T$ (see [4] or [7]). One specifies $K \sim_{(\mathcal{C},T)} K'$ if and only if $K \leq_{(\mathcal{C},T)} K'$ and $K' \leq_{(\mathcal{C},T)} K$; then $\sim_{(\mathcal{C},T)}$ is an equivalence relation on $T$. An equivalence class $[K]_{(\mathcal{C},T)}$ under this relation is called an extension type relative to $(\mathcal{C},T)$. For any space $X$, we write $X \tau[K]_{(\mathcal{C},T)}$ to mean that $X \tau K'$ for all $K' \in [K]_{(\mathcal{C},T)}$. 
Let $T$ be a class of CW-complexes and $C$ be a class of spaces $X$ having the homotopy extension property with respect to $K$ for any element $K$ of $T$. Whenever $K$, $L$ are homotopy equivalent elements of $T$, then $[K]_{(C,T)} = [L]_{(C,T)}$. □

Let $X$ be a space. Consider $S = \{[K]_{(C,T)} \mid X\tau[K]_{(C,T)}\}$. If $S$ has an initial element\(^1\) with respect to the relation $\leq_{(C,T)}$, then that element is called the \textit{extension dimension} of $X$ relative to $(C, T)$, written $\text{extdim}_{(C,T)} X$.

In the sequel we shall use,

$T_{\text{CW}}$ = the class of CW-complexes,

$T_{\text{POL}}$ = the class of polyhedra,

$\mathcal{K}$ = the class of compact Hausdorff spaces, \footnote{By an initial element of $S$, we mean $s_0 \in S$ having the property that $s_0 \leq_{(C,T)} s$ for all $s \in S$. If such $s_0$ exists, it is unique.}

$\mathcal{K}_m$ = the class of compact metrizable spaces.

Theorem 11 of [3] along with Lemma 1.1 of [6] can be used to obtain the next information.

**Theorem 5.2.** For each $L, K \in T_{\text{CW}}$, it is true that $L \leq_{(\mathcal{K},T_{\text{CW}})} K$ if and only if $L \leq_{(\mathcal{K}_m,T_{\text{POL}})} K$. Hence, $[K]_{(\mathcal{K},T_{\text{CW}})} = [K]_{(\mathcal{K}_m,T_{\text{POL}})}$. Similarly, if $K \in T_{\text{POL}}$, then $[K]_{(\mathcal{K},T_{\text{POL}})} = [K]_{(\mathcal{K}_m,T_{\text{POL}})}$. □

It is remarked in Theorem 5.5 of [6] (see also [5]) that for any compact Hausdorff space $X$, $\text{extdim}_{(\mathcal{K},T_{\text{CW}})} X$ exists. This extension dimension has a special type of representative. Let us cite Theorem 13 of [3].

**Theorem 5.3.** For each $X \in \mathcal{K}$, there exists $L = \bigvee \{L_a \mid a \in A\}$ where $\text{card } A \leq 2^{\aleph_0}$, for each $a \in A$, $L_a \in T_{\text{CW}}$, $L_a$ is countable, and,

$$\text{extdim}_{(\mathcal{K},T_{\text{CW}})} X = [L]_{(\mathcal{K},T_{\text{CW}})}.$$ □

Now we state Problem 2.19.2 of [4], noting that it has also been posed as Problem 2 of [3] and Problem 2.1 of [1].
Problem 5.4. Determine whether for each compact metrizable space $X$, there is a countable CW-complex $M$ such that \( \text{extdim}_{(\mathcal{K},\tau_{\text{CW}})} X = [M]_{(\mathcal{K},\tau_{\text{CW}})} \).

The next fact is immediate from Corollary 1.3 of [7]

Proposition 5.5. Let $K$ be a countable CW-complex and $\alpha$ an infinite ordinal. Suppose that $X$ is a compact Hausdorff space with $\text{wt} X \leq \alpha$ having the property that $X \tau K$ and each compact Hausdorff space $Y$ with $Y \tau K$ and $\text{wt} Y \leq \alpha$ embeds in $X$. Then $\text{extdim}_{(\mathcal{K},\tau_{\text{CW}})} X = [K]_{(\mathcal{K},\tau_{\text{CW}})}$.

This provides many examples of compact Hausdorff spaces with “countable” extension dimension, since by Corollary 1.9 of [13], every finite CW-complex admits a universal Hausdorff compactum of a given weight.

Lemma 5.6. Let $K$ be a CW-complex and $X$ a Hausdorff $\sigma$-compactum. Suppose that $K$ is not an absolute extensor for $X$. Then there exists a compact subset $A$ of $X$ and a map $f : A \to K$ that does not extend to a map of $X$ to $K$.

Proof. There exists a closed subspace $B$ of $X$ and a map $g : B \to K$ that does not extend to a map of $X$ to $K$. Write $X = \bigcup \{X_i \mid i \in \mathbb{N}\}$ where for each $i \in \mathbb{N}$, $X_i$ is a compact Hausdorff space.

If $g \mid (B \cap X_1) : B \cap X_1 \to K$ does not extend to a map of $X_1 \to K$, then define $A = B \cap X_1$ and $f = g \mid A : A \to K$. Otherwise, choose a map $h_1 : B \cup X_1 \to K$ that extends $g$. We may as well assume that the domain of $h_1$ is a closed neighborhood $N_1$ of $B \cup X_1$. Suppose that $k \in \mathbb{N}$ and we have found closed subsets $N_1 \subset \cdots \subset N_k$, of $X$, and maps $h_i : N_i \to K$, $1 \leq i \leq k$, such that for $1 \leq i \leq j \leq k$,

(i) $h_j \mid N_i = h_i$,

(ii) $X_i \subset \text{int}_X N_i$, and

(iii) $h_i \mid B = g$.

If $h_k \mid (N_k \cap X_{k+1})$ does not extend to a map of $X_{k+1}$ to $K$, then choose $A = N_k \cap X_{k+1}$ and $f = h_k \mid A : A \to K$. If it does extend, there exists a closed neighborhood $N_{k+1}$ of $N_k \cup X_{k+1}$ and a map $h_{k+1} : N_{k+1} \to K$ such that $h_{k+1} \mid N_k = h_k : N_k \to K$.

If this recursive process ends after finitely many steps, then our proof is complete. If it does not end, then put $G = \bigcup \{h_i \mid i \in \mathbb{N}\} : X \to K$. Then $G$ is a map that extends $g$, and we have reached a contradiction. \(\square\)

We have the following statement in case $K$ is a CW-complex, $Y$ a space, and $K \notin \text{AE}(Y)$. 
Lemma 5.7. Let $K$ be a CW-complex, $Y$ a space, $A$ a closed subspace of $Y$, $L$ a subcomplex of $K$, and $f : A \to L$ a map that does not extend to a map of $Y$ to $K$. Then for any subcomplex $M$ of $K$ with $L \subset M$, the map $f : A \to M$ does not extend to a map of $Y$ to $M$. □

This motivates us to define the notion of an “anti-basis” and show how this is related to Problem 5.4.

Definition 5.8. Let $\mathcal{K}^*$ be a class of spaces, $K$ be a simplicial complex, and $F$ a collection of subcomplexes of $K$ having the property that whenever $Y \in \mathcal{K}^*$ and $|K|$ is not an absolute extensor for $Y$, then there exist a closed subspace $A$ of $Y$, $F \in F$, and map $f : A \to |F|$ that does not extend to a map of $Y$ into $|K|$. Then we shall call $F$ an anti-basis for $K$ relative to $\mathcal{K}^*$.

An application of Lemma 5.6 shows the following.

Example 5.9. Let $\mathcal{K}^*$ be a class of Hausdorff $\sigma$-compacta and $K$ a simplicial complex. Then $\mathcal{F}_K$ is an anti-basis for $K$ relative to $\mathcal{K}^*$.

Now we have the following theorem.

Theorem 5.10. Let $\mathcal{K}^*$ be a class of Hausdorff $\sigma$-compacta, $X \in \mathcal{K}^*$, and $K$ a simplicial complex. Suppose that $\text{extdim}(\mathcal{K}^*, \tau_{\text{CW}}) X$ exists and equals $|K|_{(\mathcal{K}^*, \tau_{\text{CW}})}$. If $K$ has a countable anti-basis $F$ relative to $\mathcal{K}^*$ such that $F$ consists of finite subcomplexes of $K$, then there is a countable representative of $\text{extdim}(\mathcal{K}^*, \tau_{\text{CW}}) X$. Indeed, $M = \Psi^\infty(\bigcup F)$ is a countable subcomplex of $K$ and $|M|$ represents $\text{extdim}(\mathcal{K}^*, \tau_{\text{CW}}) X$.

Proof. Put $L = \bigcup F$. Then $L$ is a countable subcomplex of $K$. Let $M = \Psi^\infty(L)$. By Lemma 3.2(3), $M$ is a countable subcomplex of $K$. Moreover since $X \tau |K|$, by Corollary 4.5, $X \tau |M|$. We know that $|K| \leq_{(\mathcal{K}^*, \tau_{\text{CW}})} |M|$. It remains to prove the opposite inequality.

Suppose that $Y \in \mathcal{K}^*$, $Y \tau |M|$, and $Y \tau |K|$ is false. By Definition 5.8, there is an element $L$ of $F$, a closed subspace $A$ of $Y$, and a map $f : A \to |L|$ that does not extend to a map of $Y$ to $K$. But $|L| \subset |M|$ and $Y \tau |M|$, so $f$ extends to a map of $Y$ to $|M|$. This contradicts Lemma 5.7. □

References


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