On the Weyl curvature of Deszcz

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Abstract. Geometrical characterizations are given for the \((0,6)\)-tensor \(R \cdot C\) and the \((0,6)\) Tachibana–Weyl tensor \(Q(g, C) := -\wedge g \cdot C\), whereby \(C\) denotes the \((0,4)\) Weyl conformal curvature tensor of a Riemannian manifold \((M, g)\), \(R\) denotes the curvature operator acting on \(C\) as a derivation, and where the natural metrical endomorphism \(\wedge g\) also acts as a derivation on \(C\). By comparison of these \((0,6)\)-tensors \(R \cdot C\) and \(Q(g, C)\), a new scalar valued Riemannian curvature invariant \(L_C(p, \pi, \pi')\) is determined on \((M, g)\), called the Weyl curvature of Deszcz, which in general depends on two tangent 2-planes \(\pi\) and \(\pi'\) at the same point \(p\), and of which the isotropy determines that \(M\) is Weyl pseudo-symmetric in the sense of Deszcz.

1. Introduction

Recently, the parallel transport of Riemann curvatures and Ricci curvatures on a (semi-)Riemannian manifold \((M, g)\) around infinitesimal co-ordinate parallelograms was studied in [13] and [14]. There, amongst others, new geometrical interpretations of the \((0,6)\) curvature tensor \(R \cdot R\), whereby the first \(R\) stands for the curvature operator acting as a derivation on the second \(R\) which stands for the \((0,4)\) Riemann–Christoffel curvature tensor, of the \((0,6)\) Tachibana tensor \(Q(g, R) := -\wedge g \cdot R\), whereby the metrical endomorphism \(\wedge g\) also acts as a derivation on the \((0,4)\) Riemann–Christoffel curvature tensor, as well as of the \((0,4)\) curvature tensor \(R \cdot S\), whereby \(S\) denotes the \((0,2)\) Ricci tensor and of the

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Tachibana-Ricci tensor $Q(g, S) := - \wedge g \cdot S$ are given. By comparison of the $(0, 6)$ tensors $R \cdot R$ and $Q(g, R)$, a new scalar valued Riemannian invariant curvature function was determined on $(M, g)$, the so-called double sectional curvature or the sectional curvature of Deszcz $L_R(p, \pi, \pi)$, which depends on two tangent $2$-planes $\pi$ and $\pi$ at any point $p$ of $M$. The manifolds $(M, g)$ for which the sectional curvature of Deszcz is isotropic, i.e., does not depend on the planes at $p$, but remains a scalar valued function which at most depends only on the points of $M$, are the manifolds which are pseudo-symmetric in the sense of Deszcz (see e.g. [7], [17]).

And similarly, by comparison of the $(0, 4)$-tensors $R \cdot S$ and $Q(g, S)$, another new scalar valued Riemannian invariant curvature function was determined on $(M, g)$, the so-called Ricci curvature of Deszcz, $L_S(p, d, \pi)$, which depends on a tangent direction $d$ and a tangent plane $\pi$ at any point $p$ of $M$. The manifolds $(M, g)$ for which the Ricci curvature of Deszcz is isotropic, i.e., depends at most only on the points of $M$, are the manifolds which are Ricci pseudo-symmetric in the sense of Deszcz (see e.g. [7], [8], [14]).

In the present article, we basically make a similar study concerning the $(0, 4)$ Weyl conformal curvature tensor $C$ on a manifold $(M, g)$ of dimension $n \geq 4$.

New geometrical interpretations of the $(0, 6)$-tensors $R \cdot C$ and $Q(g, C) := - \wedge g \cdot C$ are given, in particular thus characterizing the Weyl semi-symmetric spaces ($R \cdot C = 0$) and the conformally flat spaces ($C = 0$). Then, the conformal sectional curvature of Deszcz or the Weyl curvature of Deszcz, $L_C(p, \pi, \pi)$, is defined. This scalar curvature invariant $L_C(p, \pi, \pi)$ is isotropic with respect to both planes $\pi$ and $\pi$ at all points $p$ of $M$ if and only if the manifold is Weyl pseudo-symmetric in the sense of Deszcz (see e.g. [5], [6], [7]). For dimension $n = 3$, $C$ vanishes identically and therefore hereafter we always assume $n \geq 4$. Further, we recall that when $n \geq 5$, a Riemannian manifold $M$ is pseudo-symmetric if and only if it is Weyl pseudo-symmetric, but that for $n = 4$ the class of Weyl pseudo-symmetric spaces is essentially larger than the class of pseudo-symmetric spaces as shown in [5].

2. A geometrical interpretation of $R \cdot C$

In an $n$-dimensional ($n \geq 4$) Riemannian manifold $M$ with metric tensor $g$, let $\nabla$ denote the Levi–Civita connection. Then, the $(1, 1)$-curvature operator $\mathcal{R}(X,Y)$, the $(0, 4)$ curvature tensor $R$, the $(0, 2)$ Ricci tensor $S$ and the scalar curvature $\tau$ of $(M, g)$ are respectively given by:

$$\mathcal{R}(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$
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\[ R(X_1, X_2, X_3, X_4) = g(\mathcal{R}(X_1, X_2)X_3, X_4), \]

\[ S(X, Y) = \sum_{i=1}^{n} R(E_i, X, Y, E_i), \quad \tau = \sum_{j=1}^{n} S(E_j, E_j), \quad (1) \]

whereby \( \{E_1, E_2, \ldots, E_n\} \) denotes any local orthonormal frame field on \( M \), \( [.,.] \) denotes the Lie bracket of vector fields and \( X_1, X_2, X_3, X_4, X, Y \) denote any tangent vector fields on \( M \). And, for \( n \geq 4 \), the (0,4) Weyl conformal curvature tensor \( C \) is then given by

\[ C(X_1, X_2, X_3, X_4) := R(X_1, X_2, X_3, X_4) \]

\[ + \frac{1}{n-2} \{ g(X_1, X_3)S(X_2, X_4) + g(X_2, X_4)S(X_1, X_3) - g(X_1, X_4)S(X_2, X_3) - g(X_2, X_3)S(X_1, X_4) \} \]

\[ + \frac{\tau}{(n-1)(n-2)} \{ g(X_1, X_4)g(X_2, X_3) - g(X_1, X_3)g(X_2, X_4) \}. \quad (2) \]

The (0,6)-tensor \( R \cdot C \) is obtained by the action of the curvature operator \( \mathcal{R} \) as a derivation on the (0,4) Weyl conformal curvature tensor \( C \):

\[ (R \cdot C)(X_1, X_2, X_3, X_4; X, Y) = (\mathcal{R}(X, Y) \cdot C)(X_1, X_2, X_3, X_4) \]

\[ = -C(\mathcal{R}(X, Y)X_1, X_2, X_3, X_4) - C(X_1, \mathcal{R}(X, Y)X_2, X_3, X_4) \]

\[ - C(X_1, X_2, \mathcal{R}(X, Y)X_3, X_4) - C(X_1, X_2, X_3, \mathcal{R}(X, Y)X_4). \quad (3) \]

Now let \( \mathcal{P} \) be any co-ordinate parallelogram on the manifold \( M \) cornered at the point \( p \) for which the co-ordinate values \( x \) and \( y \) at \( p \) change along the sides by amounts \( \Delta x \) and \( \Delta y \) (Figure 1). Let \( \vec{a} = \frac{\partial}{\partial x} \big|_p \) and \( \vec{b} = \frac{\partial}{\partial y} \big|_p \) be the natural tangent vectors at \( p \) of the \( x \) and \( y \) co-ordinate lines, respectively.

Then, as is well known and which goes back to SCHOUTEN in 1918 [16], after parallel transport of any vector \( \vec{z} \) at \( p \) all around an infinitesimal co-ordinate parallelogram \( \mathcal{P} \) (Figure 2), the resulting vector \( \vec{z}^* \) at \( p \) is given by

\[ \vec{z}^* = \vec{z} + [\mathcal{R}(\vec{a}, \vec{b})\vec{z}] \Delta x \Delta y + O^{\geq 2}(\Delta x, \Delta y). \quad (4) \]

For any plane \( \pi \) tangent to \( M \) at \( p \) the Weyl sectional curvature or, in short, the Weyl curvature, \( K_C(p, \pi) \), is given by

\[ K_C(p, \pi) = C(\vec{v}, \vec{w}, \vec{v}, \vec{w}), \quad (5) \]

whereby \( \vec{v} \) and \( \vec{w} \) is any pair of orthonormal tangent vectors at \( p \) spanning \( \pi = v \wedge w \). Since \( C \) is a curvature tensor, similarly as shown by Cartan for the Riemann–Christoffel tensor \( R \) and the Riemann sectional curvatures \( K \), the knowledge of the
Figure 1. A co-ordinate parallelogram

Figure 2. Parallel transport of a vector around a co-ordinate parallelogram

“full” tensor $C$ is equivalent to the knowledge of the Weyl sectional curvatures $K_C$. Using (2), the Weyl sectional curvature of a plane $\pi = \vec{v} \wedge \vec{w}$ at $p \in M$ can be expressed in terms of the Riemann sectional curvature $K(p, \pi) = R(\vec{v}, \vec{w}, \vec{w}, \vec{v})$ and of the Ricci curvatures of the directions $d$ and $\overline{d}$ corresponding with the vectors $\vec{v}$ and $\vec{w}$, i.e., $\text{Ric}(p, d) = S(\vec{v}, \vec{v})$, $\text{Ric}(p, \overline{d}) = S(\vec{w}, \vec{w})$, as follows,

$$K_C(p, \pi) = K(p, \pi) - \frac{1}{n-2} \left\{ \text{Ric}(p, d) + \text{Ric}(p, \overline{d}) \right\} + \frac{\tau}{(n-1)(n-2)}.$$
By the metrical character of the Levi–Civita connection \( \nabla \), in particular, any pair of orthonormal vectors \( \vec{v} \) and \( \vec{w} \) at \( p \) after parallel transport around any co-ordinate parallelogram \( P \) yields again a pair of orthonormal vectors \( \vec{v}^\ast \) and \( \vec{w}^\ast \) at \( p \). These vectors span the plane \( \pi^\ast = \vec{v}^\ast \wedge \vec{w}^\ast \) which is the parallel transported plane around \( P \) of the plane \( \pi = \vec{v} \wedge \vec{w} \) (Figure 3).

\[
K_C(p, \pi^\ast) = K_C(p, \pi) - \left( [R \cdot C](\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y}) \right) \Delta x \Delta y + O(\Delta x, \Delta y).
\]

We recall that a Riemannian manifold is said to be Weyl semi-symmetric if \( R \cdot C = 0 \). Then, denoting by \( \Delta^2\pi K_C(p, \pi) = K_C(p, \pi) - K_C(p, \pi^\ast) \) the change in Weyl sectional curvature \( K_C(p, \pi) \) under the parallel transport of the plane \( \pi \) around an infinitesimal parallelogram \( P \), we can formulate the following.

**Theorem 1.** In second order approximation, the tensor \( R \cdot C \) of a Riemannian manifold (of dimension \( \geq 4 \)) measures the change of the Weyl sectional curvature \( K_C(p, \pi) \) of a plane \( \pi = \vec{v} \wedge \vec{w} \) at any point \( p \) under parallel transport around any infinitesimal co-ordinate parallelogram \( P \) cornered at \( p \) and tangent to \( \vec{x} \) and \( \vec{y} \), i.e.,

\[
\Delta^2\pi K_C(p, \pi) \approx [R \cdot C](\vec{v}, \vec{w}, \vec{v}, \vec{w}; \vec{x}, \vec{y})] \Delta x \Delta y, \quad (6)
\]
where \( \pi \) is the tangent plane at \( p \) spanned by \( \vec{x} \) and \( \vec{y} \).

**Corollary 2.** A Riemannian manifold (of dimension \( \geq 4 \)) is Weyl semi-symmetric if and only if, up to second order, the Weyl sectional curvature for any 2-plane \( \pi \) at any point \( p \) is invariant under the parallel transport of \( \pi \) around any infinitesimal co-ordinate parallelogram \( P \) cornered at \( p \).

The next properties readily follow from the algebraic symmetries of the Weyl tensor \( C \).

**Lemma 3.** The tensor \( R \cdot C \) has the following algebraic symmetry properties:

(i) \( (R \cdot C)(X_1, X_2, X_3, X_4; X, Y) = -(R \cdot C)(X_2, X_1, X_3, X_4; X, Y) = -(R \cdot C)(X_1, X_2, X_3, X_4; X, Y) = (R \cdot C)(X_3, X_4, X_1, X_2; X, Y) = (R \cdot C)(X_1, X_2, X_3, X_4, Y, X) \)

(ii) \( (R \cdot C)(X_1, X_2, X_3, X_4; X, Y) + (R \cdot C)(X_1, X_3, X_4, X_2; X, Y) + (R \cdot C)(X_1, X_4, X_2, X_3; X, Y) = 0. \)

3. **On the Tachibana–Weyl tensor**

The simplest \((0,6)\)-tensor on a Riemannian manifold which has the same algebraic symmetry properties as the \((0,6)\)-tensor \( R \cdot C \) may well be the \((0,6)\)-tensor \( Q(g, C) := - \wedge g \cdot C \), defined by the action as a derivation on \( C \) of the metrical endomorphism \( X \wedge g Y \). This endomorphism is defined by sending a tangent vector field \( Z \) to the tangent vector field given by

\[
(X \wedge g Y)Z = g(Y, Z)X - g(X, Z)Y.
\]

Then,

\[
Q(g, C)(X_1, X_2, X_3, X_4; X, Y) := - [(X \wedge g Y) \cdot C](X_1, X_2, X_3, X_4)
= C((X \wedge g Y)X_1, X_2, X_3, X_4) + C(X_1, (X \wedge g Y)X_2, X_3, X_4)
+ C(X_1, X_2, (X \wedge g Y)X_3, X_4) + C(X_1, X_2, X_3, (X \wedge g Y)X_4).
\]

By analogy with the action of the natural metrical endomorphism as a derivation on the \((0,4)\) curvature tensor \( R \), i.e., \( Q(g, R) := - \wedge g \cdot R \), which is called the Tachibana tensor of the Riemannian manifold \((M, g)\), we will call \( Q(g, C) := - \wedge g \cdot C \) the Tachibana–Weyl tensor of \((M, g)\).

Concerning the geometrical meaning of this tensor we first state the following.
Theorem 4. A Riemannian manifold $(M, g)$ of dimension $n \geq 4$ is conformally flat if and only if its Tachibana–Weyl tensor vanishes identically.

Proof. By a classical result of Weyl, a Riemannian manifold of dimension $\geq 4$ is conformally flat if and only if its conformal curvature tensor $C$ vanishes identically [18]. And, of course, $C \equiv 0$ automatically implies that $Q(g, C) \equiv 0$.

Conversely, if $Q(g, C) \equiv 0$ we need to show that $C \equiv 0$. Algebraically, just like the fact that $Q(g, R) \equiv 0$ implies that the sectional curvature $K$ of the Riemannian manifold $(M, g)$ is constant (see e.g. [9]), $Q(g, C) \equiv 0$ straightforwardly implies that $K_C$ is constant. And, since the trace of $C$ is zero, the result follows.

A different kind of geometrical meaning of the tensor $Q(g, C)$ corresponds somewhat to the one given in Theorem 1 for the tensor $R \cdot C$. It is related to the geometrical meaning of the endomorphism $\wedge g$ according to which

$$\tilde{\mathbf{z}} = \mathbf{z} - [(\tilde{\mathbf{x}} \wedge g \tilde{\mathbf{y}}) \mathbf{z}] \Delta \varphi + O^1(\Delta \varphi),$$

whereby $\tilde{\mathbf{z}}$ is the vector obtained from a tangent vector $\mathbf{z}$ at $p$ after the rotation over an angle $\Delta \varphi$ of the projection of $\mathbf{z}$ on $\pi = \tilde{\mathbf{x}} \wedge \tilde{\mathbf{y}}$, while keeping the component of $\mathbf{z}$ perpendicular to $\pi$ fixed (Figure 4) [13].

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4}
\caption{A geometrical interpretation of the vector $(\tilde{\mathbf{x}} \wedge g \tilde{\mathbf{y}}) \mathbf{z}$}
\end{figure}

Now, consider at $p$ any two orthonormal vectors $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{w}}$ and let $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{w}}$ be the vectors obtained from $\mathbf{v}$ and $\mathbf{w}$ after such a rotation over an infinitesimal angle $\Delta \varphi$ of the projections of $\mathbf{v}$ and $\mathbf{w}$ on $\pi = \tilde{\mathbf{x}} \wedge \tilde{\mathbf{y}}$. Comparing the Weyl sectional...
Then, denoting by ΔCpK(p, π) = K(p, π) − K(p, ˜π) the change in Weyl sectional curvature K(p, π) under the above kind of rotations over an infinitesimal angle Δϕ, we can formulate the following.

**Theorem 5.** In first order approximation, the Tachibana–Weyl tensor Q(g, C) of a Riemannian manifold (of dimension ≥ 4) measures the change of the Weyl sectional curvature K(p, π) of a plane π = ˜v ∧ ˜w under an infinitesimal rotation over an angle Δϕ of the projections of ˜v and ˜w on π = ˜x ∧ ˜y, i.e.,

$$ΔCpK(p, π) ≈ \left[ Q(g, C)(v, w, w, v; x, y) \right] Δϕ. \quad (7)$$

**4. Definition and properties of the Weyl curvature of Deszcz**

Let (M, g) be an n(≥ 4)-dimensional Riemannian manifold which is not conformally flat and denote by UC the set of points where the Tachibana–Weyl tensor Q(g, C) is not identically zero, i.e., UC = \{p ∈ M | Q(g, C)p ≠ 0\}. Then, at a point p ∈ UC, a plane π = ˜v ∧ ˜w is said to be Weyl curvature-dependent with respect to a plane ˜π = ˜x ∧ ˜y when Q(g, C)(v, w, w, v; x, y) ≠ 0. This definition is independent of the choice of bases for π and ˜π.

At a point p ∈ UC, let a plane π = ˜v ∧ ˜w be Weyl curvature-dependent with respect to a plane ˜π = ˜x ∧ ˜y. Then, we define the Weyl curvature of Deszcz L(p, π, ˜π) at the point p as the scalar

$$L(p, π, ˜π) = \frac{(R · C)(v, w, w, v; x, y)}{Q(g, C)(v, w, w, v; x, y)}.$$

This definition is again independent of the choice of bases for the planes π and ˜π.

**Theorem 6.** At any point p ∈ UC, the tensor R · C of a Riemannian manifold M is completely determined by the Weyl curvatures of Deszcz L(p, π, ˜π).

**Proof.** Assume there exists a (0, 6)-tensor W with the same algebraic symmetries as R · C and so that for any two Weyl curvature-dependent planes π = ˜v ∧ ˜w and ˜π = ˜x ∧ ˜y at p,

$$\frac{(R · C)(v, w, w, v; x, y)}{Q(g, C)(v, w, w, v; x, y)} = \frac{W(v, w, w, v; x, y)}{Q(g, C)(v, w, w, v; x, y)}.$$
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We have to prove that \( \forall \vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, \vec{x}_5, \vec{x}_6 \in T_p M, \)

\[
(R \cdot C)(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4; \vec{x}_5, \vec{x}_6) = W(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4; \vec{x}_5, \vec{x}_6).
\]

Let \( T \) be the \((0,6)\)-tensor \( T = R \cdot C - W \). Obviously, \( T \) has the same algebraic symmetries as \( R \cdot C \) and \( W \). Further, for every pair of Weyl curvature-dependent planes \( \pi = \vec{v} \wedge \vec{w} \) and \( \overline{\pi} = \vec{x} \wedge \vec{y} \),

\[
T(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y}) = 0. \quad (8)
\]

When two planes \( \pi = \vec{v} \wedge \vec{w} \) and \( \overline{\pi} = \vec{x} \wedge \vec{y} \) are not Weyl curvature-dependent there holds that \( Q(g, C)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y}) = 0 \). Using the following argument from [10] we show that also in this case \( T(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y}) = 0 \). Namely, since \( Q(g, C) \) is a polynomial in the components of \( \vec{v}, \vec{w}, \vec{x} \) and \( \vec{y} \), the zero set does not contain any open subset (for otherwise \( Q(g, C) \equiv 0 \), which would be a contradiction with \( p \in U_C \)). We can then choose series of tangent vectors \( \vec{v}_l \rightarrow \vec{v}, \vec{w}_l \rightarrow \vec{w}, \vec{x}_l \rightarrow \vec{x} \) and \( \vec{y}_l \rightarrow \vec{y} \) such that for any \( l \), \( \vec{v}_l \wedge \vec{w}_l \) is Weyl curvature-dependent with respect to \( \vec{x}_l \wedge \vec{y}_l \). We have for every \( l \) that \( T(\vec{v}_l, \vec{w}_l, \vec{v}_l, \vec{x}_l; \vec{x}_l, \vec{y}_l) = 0 \) and thus in the limit we find that also \( T(\vec{v}, \vec{w}, \vec{v}, \vec{x}, \vec{y}) = 0 \). Hence, (8) holds for all \( \vec{v}, \vec{w}, \vec{x}, \vec{y} \in T_p M \). Using polarization and the symmetric properties of \( R \cdot C \) completes the proof.

**Corollary 7.** The Weyl semi-symmetric spaces are characterized by the vanishing of the Weyl curvatures of Deszcz.

A Riemannian manifold \( M \) \((n \geq 4)\) is said to be Weyl pseudo-symmetric at a point \( p \in U_C \) if there exists a scalar \( L_C(p) \) such that,

\[
R \cdot C|_p = L_C(p) \cdot Q(g, C)|_p.
\]

The manifold \((M, g)\) is called **Weyl pseudo-symmetric in the sense of Deszcz** if it is Weyl pseudo-symmetric at every point of \( U_C \subset M \).

**Theorem 8.** A Riemannian manifold \((M, g)\) \((n \geq 4)\) is Weyl pseudo-symmetric in the sense of Deszcz if and only if at all of its points \( p \in U_C \) all the Weyl curvatures of Deszcz are the same, i.e., for all Weyl curvature-dependent planes \( \pi \) and \( \overline{\pi} \) at \( p \), \( L_C(p, \pi, \overline{\pi}) = L_C(p) \).

**Proof.** If \( R \cdot C|_p = L_C(p) \cdot Q(g, C)|_p \) at \( p \), the statement is obvious. So assume that \( L_C(p, \pi, \overline{\pi}) = L_C(p) \) for every two Weyl curvature-dependent planes \( \pi \) and \( \overline{\pi} \). Then,

\[
(R \cdot C)(\vec{v}, \vec{w}, \vec{v}, \vec{w}, \vec{x}, \vec{y}) = L_C(p) \cdot Q(g, C)(\vec{v}, \vec{w}, \vec{v}, \vec{w}, \vec{x}, \vec{y}).
\]
The tensor \( T = R \cdot C - L \cdot C \) has the same algebraic symmetries as \( R \cdot C \). For two Weyl curvature-dependent planes \( \pi = \vec{v} \wedge \vec{w} \) and \( \overline{\pi} = \vec{x} \wedge \vec{y} \), there holds that \( T(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y}) = 0 \). If both planes are not Weyl curvature-dependent, an argument as in the proof of Theorem 6 shows that \( T(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4; \vec{x}_5, \vec{x}_6) = 0, \forall \vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4; \vec{x}_5, \vec{x}_6 \in T_pM \), and by polarization it then follows that \( T(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4; \vec{x}_5, \vec{x}_6) = 0, \forall \vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4; \vec{x}_5, \vec{x}_6 \in T_pM \). □

5. Pseudo-symmetry and the squaroids of Levi–Civita

Next, we give a geometrical interpretation of the Weyl curvature of Deszcz \( L_C \) in terms of the squaroids of Levi–Civita (see e.g. [2], [9], [13, 15]). Starting from any two tangent vectors \( \vec{v} \) and \( \vec{w} \) at any point \( p \) of \( M \), Levi–Civita constructed in 1917 his so-called parallelogramoids as follows. Consider through \( p \) the geodesic \( \alpha \) with tangent \( \vec{v} \) and let \( q \) be the point on this geodesic at an infinitesimal distance \( A \) from \( p \). Denote by \( \vec{w}^* \) the vector obtained after parallel transport of \( \vec{w} \) from \( p \) to \( q \) along \( \alpha \). Then, through \( p \) and \( q \) consider the geodesics \( \beta_p \) and \( \beta_q \) which are tangent to \( \vec{w} \) and \( \vec{w}^* \), respectively. Fix on them the points \( \overline{p} \) and \( \overline{q} \) at a same infinitesimal distance \( B \) from \( p \) and \( q \), respectively. The parallelogramoid cornered at \( p \) with sides tangent to \( \vec{v} \) and \( \vec{w} \) is then completed by the geodesic \( \overline{\pi} \) through \( \overline{p} \) and \( \overline{q} \). Let \( A' \) be the geodesic distance between \( \overline{p} \) and \( \overline{q} \). Levi–Civita showed that, in first order approximation, the sectional curvature \( K(p, \pi) \) of the plane \( \pi = \vec{v} \wedge \vec{w} \) can be expressed as

\[
K(p, \pi) \approx \frac{A^2 - A'^2}{A^2B^2 \sin^2(\theta)}
\]

whereby \( \theta \) is the angle between the vectors \( \vec{v} \) and \( \vec{w} \).

Let \( \vec{v} \) and \( \vec{w} \) be orthonormal vectors at any point \( p \in M \). Consider the Levi–Civita squaroid based on \( \vec{v} \) and \( \vec{w} \) with side \( \varepsilon \), i.e., the parallelogramoid for which \( A = B = \varepsilon \) (Figure 5). Then, when \( \varepsilon' \) is the length of the closing geodesic, the sectional curvature \( K(p, \pi) \) is given by

\[
K(p, \pi) \approx \frac{\varepsilon^2 - \varepsilon'^2}{\varepsilon^4}.
\]

Consider at \( p \in M \) an orthonormal basis \( \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\} \) of \( T_pM \) and construct for every plane \( \vec{v} \wedge \vec{e}_j \) \((j \neq 1)\) the squaroid of Levi–Civita, all with the same sides \( \varepsilon \). Let us denote the lengths of the completing geodesics by \( \varepsilon'_{j'} \). The
Ricci curvatures $\text{Ric}(p, d)$, with $d$ the direction of the vector $\vec{v}$, can then, up to first order approximation, be expressed as

$$\text{Ric}(p, d) \approx \sum_{j \neq 1} \frac{\varepsilon^2 - \varepsilon_j'^2}{\varepsilon^4}.$$

Now, consider two planes $\pi = \vec{v} \wedge \vec{w}$ and $\vec{w} \wedge \vec{y}$ at a point $p$ of $M$ and parallelly transport the frame $\{\vec{v} = \vec{e}_1, \vec{w} = \vec{e}_2, \vec{e}_3, \ldots, \vec{e}_n\}$ to $\{\vec{v}^*, \vec{w}^* = \vec{e}_2^*, \vec{e}_3^*, \ldots, \vec{e}_n^*\}$ around an infinitesimal co-ordinate parallelogram $\mathcal{P}$. We construct for every plane $\vec{v}^* \wedge \vec{e}_j^* (j \neq 1)$ and for every plane $\vec{w}^* \wedge \vec{e}_k^* (k \neq 2)$, the squaroids of Levi–Civita, all with the same sides $\varepsilon$ and denote the lengths of the completing geodesics by $\varepsilon_j'^* \varepsilon_k'^*$, respectively.

Then, according to the formulas for the tensors $R \cdot R$ and $R \cdot S$ which are analogous to formula (6) for the tensor $R \cdot C$ [13], [14], we find, up to second order approximation with respect to $\Delta x$ and $\Delta y$, that

$$(R \cdot R)(\vec{v}, \vec{w}, \vec{v}, \vec{w}; \vec{x}, vecy) \approx \frac{(\varepsilon_j'^*)^2 - (\varepsilon_j'^*')^2}{\varepsilon^4} \frac{1}{\Delta x \Delta y},$$

and

$$(R \cdot S)(\vec{v}, \vec{v}; \vec{x}, \vec{y}) \approx \sum_{j \neq 1} \frac{(\varepsilon_j'^*)^2 - (\varepsilon_j'^*')^2}{\varepsilon^4} \frac{1}{\Delta x \Delta y}.$$

Let $\{\vec{v} = \vec{e}_1, \vec{w} = \vec{e}_2, \vec{e}_3, \ldots, \vec{e}_n\}$ be the frame which is obtained after an infinitesimal rotation as before of the frame $\{\vec{v} = \vec{e}_1, \vec{w} = \vec{e}_2, \vec{e}_3, \ldots, \vec{e}_n\}$ with respect to the plane $\vec{w} = \vec{x} \wedge \vec{y}$, and construct for every plane $\vec{v} \wedge \vec{e}_j^* (j \neq 1)$ and for every
plane \( \widetilde{\omega} \wedge \widetilde{e}_k \) \((k \neq 2)\) the squaroids of Levi–Civita, all with the same side \(\varepsilon\), and denote the lengths of the completing geodesics by \(\varepsilon_j^*\) and \(\varepsilon_k^*\), respectively.

In this case, according to the formulas for the Tachibana tensors \(Q(g, R)\) and \(Q(g, S)\) which are analogous to formula (7) for the Tachibana–Weyl tensor \(Q(g, C)\), we find, up to first order with respect to the angle \(\Delta \varphi\) of infinitesimal rotation, that

\[
Q(g, R)(\vec{v}, \vec{u}, \vec{w}, \vec{v}, \vec{x}, \vec{y}) \approx \frac{(\varepsilon_j^*)^2 - (\varepsilon_j')^2}{\varepsilon^4} \frac{1}{\Delta \varphi},
\]

and

\[
Q(g, S)(\vec{v}, \vec{v}, \vec{x}, \vec{y}) \approx \frac{\sum_{j \neq 1} (\varepsilon_j')^2 - (\varepsilon_j')^2}{\varepsilon^4} \frac{1}{\Delta \varphi}.
\]

We recall from [13], [14] that a plane \(\pi = \vec{v} \wedge \vec{w}\) is said to be curvature-dependent with respect to a plane \(\bar{\pi} = \vec{x} \wedge \vec{y}\) if \(Q(g, R)(\vec{v}, \vec{w}, \vec{v}, \vec{x}, \vec{y}) \neq 0\), and that a direction \(d\) spanned by a vector \(\vec{v}\) is Ricci curvature-dependent with respect to a plane \(\bar{\pi} = \vec{x} \wedge \vec{y}\) if \(Q(g, S)(\vec{v}, \vec{v}, \vec{x}, \vec{y}) \neq 0\). These definitions are independent of the choices of bases for the planes \(\pi\) and \(\bar{\pi}\) and the direction \(d\), respectively.

Then, the sectional curvature of Deszcz \(L_R(p, \pi, \bar{\pi})\) of the plane \(\pi\) which is curvature-dependent with respect to \(\bar{\pi}\) at \(p \in \mathcal{U}_R = \{x \in M \mid Q(g, R)x \neq 0\}\), and the Ricci curvature of Deszcz \(L_S(p, d, \bar{\pi})\) of the direction \(d\) which is Ricci curvature-dependent with respect to the plane \(\bar{\pi}\) at a point \(p \in \mathcal{U}_S = \{x \in M \mid Q(g, S)x \neq 0\}\), can respectively be expressed as

\[
L_R(p, \pi, \bar{\pi}) \approx \frac{(\varepsilon_j')^2 - (\varepsilon_j')^2}{(\varepsilon_j^*)^2 - (\varepsilon_j')^2} \frac{\Delta \varphi}{\Delta x \Delta y},
\]

and

\[
L_S(p, d, \bar{\pi}) \approx \frac{\sum_{j \neq 1} (\varepsilon_j')^2 - (\varepsilon_j')^2}{\sum_{k \neq 1} (\varepsilon_k^*)^2 - (\varepsilon_k')^2} \frac{\Delta \varphi}{\Delta x \Delta y}.
\]

Because the tensor \(R \cdot C\) can be written in terms of the tensors \(R \cdot R\) and \(R \cdot S\) as,

\[
(R \cdot C)(X_1, X_2, X_3, X_4; X, Y) = (R \cdot R)(X_1, X_2, X_3, X_4; X, Y) - \frac{1}{n - 2} \{g(X_1, X_4)(R \cdot S)(X_2, X_3; X, Y) + g(X_2, X_3)(R \cdot S)(X_1, X_4; X, Y) \}
\]

\[
- g(X_1, X_3)(R \cdot S)(X_2, X_4; X, Y) - g(X_2, X_4)(R \cdot S)(X_1, X_3; X, Y),
\]

the Weyl curvature of Deszcz \(L_C(p, \pi, \bar{\pi})\) of the plane \(\pi\) which is Weyl curvature-dependent with respect to the plane \(\bar{\pi}\) at a point \(p \in \mathcal{U}_C\) can be expressed as

\[
L_C(p, \pi, \bar{\pi}) \approx \frac{(\varepsilon_j^*)^2 - (\varepsilon_j')^2}{(\varepsilon_j^*)^2 - (\varepsilon_j')^2} \frac{\sum_{j \neq 1} (\varepsilon_j')^2 - (\varepsilon_j')^2}{\sum_{k \neq 1} (\varepsilon_k^*)^2 - (\varepsilon_k')^2} \frac{\Delta \varphi}{\Delta x \Delta y}.
\]


Thus, calibrating the changes of the Riemann, Ricci and Weyl curvatures under parallel translation (⋆) around a parallelogram \( P \) with infinitesimal parameter growths \( \Delta x \) and \( \Delta y \) by the changes of the same curvatures under rotation (∼) over an infinitesimal angle \( \Delta \varphi = \Delta x \Delta y \) with respect to \( \overline{\pi} \), we find the following approximate geometrical expressions in terms of the squared Levi–Civita of sides \( \varepsilon \), for, respectively, the Riemann sectional curvature of Deszcz \( L_{R} \), the Ricci curvature of Deszcz \( L_{S} \) and the Weyl curvature of Deszcz \( L_{C} \).

**Theorem 9.** Let \((M, g)\) be a Riemannian manifold, \( p \in U_{R} \) and \( \pi \subset T_{p}M \) curvature-dependent with respect to a tangent plane \( \overline{\pi} \subset T_{p}M \). Under the above calibration of infinitesimal parallel translation by associated infinitesimal rotation, the sectional curvature of Deszcz \( L_{R}(p, \pi, \overline{\pi}) \) can be expressed in terms of the lengths of closing sides in squaroids of Levi–Civita as follows:

\[
L_{R}(p, \pi, \overline{\pi}) \approx \frac{(\varepsilon^{'*}_j)^2 - (\varepsilon'_j)^2}{(\varepsilon'_2)^2 - (\varepsilon'_2)^2}.
\]

**Theorem 10.** Let \((M, g)\) be a Riemannian manifold, \( p \in U_{S} \) and \( d \) a tangent direction which is Ricci curvature-dependent with respect to a tangent plane \( \overline{\pi} \subset T_{p}M \). Under the above calibration of infinitesimal parallel translation by associated infinitesimal rotation, the Ricci curvature of Deszcz \( L_{S}(p, d, \overline{\pi}) \) can be expressed in terms of the lengths of closing sides in squaroids of Levi–Civita as follows:

\[
L_{S}(p, d, \overline{\pi}) \approx \sum_{j \neq 1} \left[ \frac{(\varepsilon^{'*}_j)^2 - (\varepsilon'_j)^2}{(\varepsilon'_k)^2 - (\varepsilon'_k)^2} \right] \cdot \sum_{k \neq 1} \left[ \frac{(\varepsilon^{'*}_k)^2 - (\varepsilon'_k)^2}{(\varepsilon'_k)^2 - (\varepsilon'_k)^2} \right] - \frac{1}{n-2} \left[ \sum_{j \neq 1} \frac{(\varepsilon^{'*}_j)^2 - (\varepsilon'_j)^2}{(\varepsilon'_j)^2 - (\varepsilon'_j)^2} \right] - \frac{1}{n-2} \left[ \sum_{k \neq 2} \frac{(\varepsilon^{'*}_k)^2 - (\varepsilon'_k)^2}{(\varepsilon'_k)^2 - (\varepsilon'_k)^2} \right].
\]

**Theorem 11.** Let \((M, g)\) be a Riemannian manifold, \( p \in U_{R} \) and \( \pi \subset T_{p}M \) Weyl curvature-dependent with respect to a tangent plane \( \overline{\pi} \subset T_{p}M \). Under the above calibration of infinitesimal parallel translation by associated infinitesimal rotation, the Weyl curvature of Deszcz \( L_{C}(p, \pi, \overline{\pi}) \) can be expressed in terms of the lengths of closing sides in squaroids of Levi–Civita as follows:

\[
L_{C}(p, \pi, \overline{\pi}) \approx \frac{(\varepsilon^{'*}_j)^2 - (\varepsilon'_j)^2}{(\varepsilon'_2)^2 - (\varepsilon'_2)^2} - \frac{1}{n-2} \left[ \sum_{j \neq 1} \frac{(\varepsilon^{'*}_j)^2 - (\varepsilon'_j)^2}{(\varepsilon'_j)^2 - (\varepsilon'_j)^2} \right] - \frac{1}{n-2} \left[ \sum_{k \neq 2} \frac{(\varepsilon^{'*}_k)^2 - (\varepsilon'_k)^2}{(\varepsilon'_k)^2 - (\varepsilon'_k)^2} \right].
\]

**Remarks 12.** If a manifold \((M, g)\) is pseudo-symmetric, then it is automatically Ricci pseudo-symmetric as well as Weyl pseudo-symmetric, but the converse statements are not true in general. The warped products of a 1-dimensional manifold and a non pseudo-symmetric Einstein manifold of dimension \( \geq 3 \), are non pseudo-symmetric, Ricci pseudo-symmetric manifolds. All Cartan hypersurfaces in the spheres \( \mathbb{S}^{n+1} \) with \( n = 6, 12 \) or 24 are non pseudo-symmetric, Ricci pseudo-symmetric manifolds. The warped products of Riemannian spheres of dimension
≥ 2 with Einstein spaces of dimension ≥ 4 are non conformally flat, non pseudo-symmetric, non Einstein, but Ricci pseudo-symmetric manifolds. Examples of non pseudo-symmetric, non conformally flat, but Weyl pseudo-symmetric Riemannian manifolds were obtained in [5] by applying suitable conformal deformations on a non semi-symmetric, non conformally flat, but Weyl semi-symmetric, Riemannian 4-dimensional manifold given by DErdziński in [4]. Also, of course, every conformally flat manifold of dimension ≥ 4 is Weyl pseudo-symmetric, but there do exist conformally flat manifolds of dimension ≥ 4 which are not pseudo-symmetric. For more information on various pseudo-symmetries, see e.g. [1], [3], [5], [7], [11], [12].

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