On \((\psi, \gamma)\)-stability of Cauchy equation on some noncommutative groups

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Dedicated to Professor Zoltán Daróczy on the occasion of his seventieth birthday

Abstract. In this paper, the \((\psi, \gamma)\)-stability of the Cauchy functional equation is investigated on some noncommutative groups. It is shown that if \(\gamma\) is invariant with respect to inner automorphisms of a step-two solvable group \(G\), then the Cauchy equation \(f(xy) = f(x) + f(y)\) is \((\psi, \gamma)\)-stable on \(G\). If \(\psi\) satisfies the condition \(\lim_{n \to \infty} \frac{\psi(n^2)}{n} = 0\), then the Cauchy equation is \((\psi, \gamma)\)-stable on step-two solvable groups and also on step-three nilpotent groups.

1. Introduction

In 1940, S. M. Ulam [17] posed the following fundamental problem. Given a group \(G_1\), a metric group \((G_2, d)\) and a positive number \(\varepsilon\), does there exist a number \(\delta > 0\) such that if \(f : G_1 \to G_2\) satisfies \(d(f(xy), f(x)f(y)) < \delta\) for all \(x, y \in G_1\), then a homomorphism \(T : G_1 \to G_2\) exists with \(d(f(x), T(x)) < \varepsilon\) for all \(x, y \in G_1\)? See S. M. Ulam [17] for a discussion of such problems, as well as D. H. Hyers [8], [9], D. H. Hyers and S. M. Ulam [11], [12], Aoki [2], Th. M. Rassias [15], [16], G. L. Forti [7], and J. Aczél and J. Dhombres [1]. The first affirmative answer was given by D. H. Hyers [8] in 1941.

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Theorem 1.1 (Hyers [8]). Let $E_1$ and $E_2$ be Banach spaces. If the function $f : E_1 \rightarrow E_2$ satisfies the inequality
\[
\|f(x + y) - f(x) - f(y)\| < \varepsilon
\]  
(1.1)
for some $\varepsilon > 0$ and for all $x, y \in E_1$, then there exists a unique function $T : E_1 \rightarrow E_2$ such that
\[
T(x + y) - T(x) - T(y) = 0 \quad \text{for all } x, y \in E_1
\]  
(1.2)
and
\[
\|f(x) - T(x)\| < \varepsilon \quad \text{for all } x \in E_1.
\]  
(1.3)

Aoki [2] proved a generalized version of Hyers’ result which permitted the Cauchy difference to become unbounded. That is, he assumed that
\[
\|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)
\]  
for all $x, y \in E_1$,
where $\varepsilon$ and $p$ are constants satisfying $\varepsilon > 0$ and $0 \leq p < 1$. By making use of the direct method of Hyers [8], he proved in this case too, that there is an additive function $T$ from $E_1$ into $E_2$ given by the formula
\[
T(x) = \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)
\]
such that
\[
\|T(x) - f(x)\| \leq k\varepsilon\|x\|^p,
\]
where $k$ depends on $p$ as well as $\varepsilon$. Independently, Th. M. Rassias [15] in 1978 rediscovered the above result and proved that the mapping $T$ is not only additive, under certain conditions, it is also linear. Rassias’ paper [15] provided an impetus for a lot of activities in the development of what we now call Hyers–Ulam–Rassias stability theory of functional equations. On an arbitrary group $G$, the Cauchy functional equation $f(xy) = f(x) + f(y)$ takes the form $f(xy) = f(x) + f(y)$ for all $x, y \in G$. The first paper to extend Rassias’s result to a class nonabelian groups and semigroups was [5]. In [5] among other results, it was proven that the Cauchy functional equation $f(xy) = f(x) + f(y)$ is $(\psi, \gamma)$-stable on any abelian group as well as any metabelian (step-two nilpotent) group. It was also shown that any group $A$ can be embedded into a group $G$, where the Cauchy functional equation is $(\psi, \gamma)$-stable. This paper is a continuation of the study of $(\psi, \gamma)$-stability initiated in [5]. In this paper, we study the $(\psi, \gamma)$-stability of the Cauchy functional equation on step-two solvable groups and step-three nilpotent groups.
2. The space of \((\psi, \gamma)\)-pseudoadditive mappings

In this section, we recall some important notions from [5] that we need for this paper. We will denote the set of real numbers by \(\mathbb{R}\) and the set of natural numbers by \(\mathbb{N}\). Let \(\mathbb{R}^+_0 = [0, \infty)\) be the set of non-negative numbers and \(\mathbb{R}^+ = (0, \infty)\) be the set of positive numbers. Let \(S\) be an arbitrary semigroup and \(G\) be a group. Throughout this paper, the function \(\psi : \mathbb{R}^+_0 \to \mathbb{R}^+_0\) is considered to be an increasing function satisfying the following three additional conditions:

1. \(\psi(t_1 + t_2) \leq \psi(t_1) + \psi(t_2)\) for all \(t_1, t_2 \in \mathbb{R}^+_0\),
2. \(\lim_{n \to \infty} \psi(n) = 0, n \in \mathbb{N}\).

Throughout this paper, by \(\gamma\) we will mean a function \(\gamma : S \to \mathbb{R}^+_0\) satisfying the inequality

\[\gamma(xy) \leq \gamma(x) + \gamma(y)\] for all \(x, y \in S\).

It is obvious that for any \(x \in S\) and for any \(m \in \mathbb{N}\) the inequality

\[\gamma(x^m) \leq m \gamma(x)\] holds.

**Definition 2.1.** Let \(S\) be an arbitrary semigroup and \(E\) a Banach space. Further, let \(\psi : \mathbb{R}^+_0 \to \mathbb{R}^+_0\) and \(\gamma : S \to \mathbb{R}^+_0\) be the functions as described above. The mapping \(f : S \to E\) is said to be a \((\psi, \gamma)\)-quasiadditive mapping if there exists a \(\theta \in \mathbb{R}^+_0\) such that

\[\|f(xy) - f(x) - f(y)\| \leq \theta[\psi(\gamma(x)) + \psi(\gamma(y))]\] \(\forall x, y \in S\) (2.2)

holds.

It is clear that the set of all \((\psi, \gamma)\)-quasiadditive mappings from \(S\) to \(E\) is a real linear space relative to the usual operations. Let us denote it by \(KAM_{\psi,\gamma}(S; E)\).

**Definition 2.2.** Let \(\varphi : S \to E\) be a mapping from the semigroup \(S\) to a Banach space \(E\). The mapping \(\varphi\) is said to be a \((\psi, \gamma)\)-pseudoadditive mapping if it is a \((\psi, \gamma)\)-quasiadditive mapping satisfying \(\varphi(x^n) = n\varphi(x)\) for all \(x \in S\) and for each \(n \in \mathbb{N}\).

We denote the space of all \((\psi, \gamma)\)-pseudoadditive mappings from a semigroup \(S\) to a Banach space \(E\) by \(PAM_{\psi,\gamma}(S; E)\). By \(HOM(S; E)\) we mean the set of all homomorphisms from \(S\) to \(E\). By \(B_{\psi,\gamma}(S; E)\) we denote the linear space of functions from \(S\) to \(E\) over reals satisfying the relation:

\[\|f(x)\| \leq c\psi(\gamma(x))\] \(\text{for some } c > 0\) and for all \(x \in S\).
3. Stability

In this section, we prove some general results related to the \((\psi, \gamma)\)-stability of the Cauchy functional equation. In [5] the following theorem was established.

**Theorem 3.1.** The linear space \(KAM_{\psi, \gamma}(S; E)\) is a direct sum of the subspaces \(PAM_{\psi, \gamma}(S; E)\) and \(B_{\psi, \gamma}(S; E)\), that is

\[
KAM_{\psi, \gamma}(S; E) = PAM_{\psi, \gamma}(S; E) \oplus B_{\psi, \gamma}(S; E).
\]

**Definition 3.2.** The Cauchy functional equation

\[
f(xy) = f(x) + f(y), \quad \forall x, y \in S
\]  
(3.1)

is said to be \((\psi, \gamma)\)-stable for the pair \((S; E)\) if for any \(f : S \to E\) satisfying the functional inequality

\[
\|f(xy) - f(x) - f(y)\| \leq \theta[\psi(\gamma(x)) + \psi(\gamma(y))] \quad \forall x, y \in S
\]  
(3.2)

there is a solution \(g : S \to E\) of functional equation (3.1) such that the function \(f(x) - g(x)\) belongs to the space \(B_{\psi, \gamma}(S; E)\).

It was shown in [5] that the equation (3.1) is \((\psi, \gamma)\)-stable for the pair \((S; E)\) if and only if \(PAM_{\psi, \gamma}(S; E) = HOM(S; E)\).

The following theorem and its proof are generalizations of a similar result proved in [6].

**Theorem 3.3.** Let \(E_1\) and \(E_2\) be Banach spaces over reals. Then the equation (3.1) is \((\psi, \gamma)\)-stable for the pair \((S, E_1)\) if and only if it is \((\psi, \gamma)\)-stable for the pair \((S, E_2)\).

**Proof.** Let \(E\) be a Banach space over reals and \(\mathbb{R}\) be the set of reals. Let the equation (3.1) be stable for the pair \((S, E)\). Suppose (3.1) is not stable for the pair \((S, \mathbb{R})\). Then there is a nontrivial \((\psi, \gamma)\)-pseudocharacter \(f\) on \(S\). So, for some \(\theta \geq 0\) we have

\[
\|f(xy) - f(x) - f(y)\| \leq \theta[\psi(\gamma(x)) + \psi(\gamma(y))] \quad \forall x, y \in S.
\]

Now let \(e \in E\) and \(\|e\| = 1\). Consider the function \(\varphi : S \to E\) given by the formula \(\varphi(x) = f(x) \cdot e\). It is clear that \(\varphi\) is a nontrivial \((\psi, \gamma)\)-pseudoadditive \(E\)-valued function, and we obtain a contradiction.
Now suppose that the equation (3.1) is stable for the pair \((S, \mathbb{R})\), that is, \(PAM_{\psi, \gamma}(S; \mathbb{R}) = HOM(S; \mathbb{R})\). Denote by \(E^\ast\) the space of linear bounded functionals on \(E\) endowed by functional norm topology. It is clear that for any \(\varphi \in PAM_{\psi, \gamma}(S; E)\) and any \(\lambda \in E^\ast\) the function \(\lambda \circ \varphi\) belongs to the space \(PAM_{\psi, \gamma}(S; \mathbb{R})\). Indeed, for some nonnegative \(\theta\) and any \(x, y \in S\) we have
\[
\|\varphi(xy) - \varphi(x) - \varphi(y)\| \leq \theta|\psi(\gamma(x)) + \psi(\gamma(y))|.
\]
Hence
\[
|\lambda \circ \varphi(xy) - \lambda \circ \varphi(x) - \lambda \circ \varphi(y)| = |\lambda(\varphi(xy) - \varphi(x) - \varphi(y))| \\
\leq \|\lambda\| (\theta|\psi(\gamma(x)) + \psi(\gamma(y))|) = \|\lambda\|\theta|\psi(\gamma(x)) + \psi(\gamma(y))|.
\]
Obviously, \(\lambda \circ \varphi(x^n) = n\lambda \circ \varphi(x)\) for any \(x \in S\) and for any \(n \in \mathbb{N}\). Hence the function \(\lambda \circ \varphi\) belongs to the space \(PAM_{\psi, \gamma}(S; \mathbb{R})\). Let \(f : S \to E\) be a nontrivial \((\psi, \gamma)\)-pseudoadditive mapping. Then there are \(x, y \in S\) such that \(f(xy) - f(x) - f(y) \neq 0\). Hahn–Banach Theorem implies that there is a \(\ell \in E^\ast\) such that \(\ell(f(xy) - f(x) - f(y)) \neq 0\), and we see that \(\ell \circ f\) is a nontrivial \((\psi, \gamma)\)-pseudoadditive real-valued function on \(S\). This contradiction proves the theorem.

In view of Theorem 3.3, it is not important which Banach space is used on the range. Thus one may consider the \((\psi, \gamma)\)-stability of the functional equation (3.1) on the pair \((S, \mathbb{R})\). Let us simplify the following notations: In the case \(E = \mathbb{R}\) the spaces \(KAM_{\psi, \gamma}(S; \mathbb{R}), PAM_{\psi, \gamma}(S; \mathbb{R})\), and \(HOM(S; \mathbb{R})\) will be denoted by \(KX_{\psi, \gamma}(S), PX_{\psi, \gamma}(S), X(S)\), respectively. Further, we will call a \((\psi, \gamma)\)-additive map a \((\psi, \gamma)\)-quasicharacter, and a \((\psi, \gamma)\)-pseudoadditive map a \((\psi, \gamma)\)-pseudocharacter. We also will use the following properties of the \((\psi, \gamma)\)-pseudocharacter
\begin{enumerate}
  \item \(f(xy) = f(yx)\) for any \(x, y \in S\),
  \item \(f(ab) = f(a) + f(b)\), if \(ab = ba\)
\end{enumerate}
established in [5]. From the first property it follows that if \(S\) is a group, then for any \(x, y \in S\), the relation \(f(y^{-1}xy) = f(x)\) holds. This implies that every \((\psi, \gamma)\)-pseudocharacter \(f\) is invariant under inner automorphisms of group \(S\). As usually by pseudocharacter we mean a real-valued function \(f : S \to \mathbb{R}\) satisfying conditions:
\begin{enumerate}
  \item the set \(\{f(xy) - f(x) - f(y) \mid \forall x, y \in S\}\) is bounded, and
  \item \(f(x^n) = nf(x)\) for any \(x \in S\) and any \(n \in \mathbb{N}\).
\end{enumerate}
The set of pseudocharacters of a semigroup \(S\) will be denoted by \(PX(S)\). It is clear that if \(\gamma\) is a constant function then \(PX_{\psi, \gamma}(S) = PX(S)\).

**Lemma 3.4.** Let the group \(G\) be the union of its subgroups, \(G = \bigcup_{\alpha \in I} G_\alpha\), such that for any \(x, y \in G\) there is \(\alpha \in I\) such that \(x, y \in G_\alpha\). Suppose that the
equation (3.1) is \((\psi, \gamma)\)-stable for any \(G_\alpha\). Then the equation (3.1) is \((\psi, \gamma)\)-stable on \(G\).

**Proof.** Let \(f \in PX_{\psi, \gamma}(G)\). Then for some \(\theta > 0\) and for any \(x, y \in G\) we have the inequality
\[
|f(xy) - f(x) - f(y)| \leq \theta [\psi(\gamma(x)) - \psi(\gamma(x))].
\]
For any \(x, y \in G\) there is an \(\alpha\) such that \(x, y \in G_\alpha\). The equation (3.1) is stable on \(G_\alpha\). Therefore \(f(xy) = f(x) + f(y)\). It means that (3.1) is stable on \(G\), and the proof of the lemma is now complete. \(\square\)

In [5], it was shown that if \(G\) is a group and \(f \in PAM_{\psi, \gamma}(G; E)\), then (i) \(f(e) = 0\), and (ii) \(f(x^{-1}) = -f(x)\) for any \(x \in G\).

Now for any group \(G\) we introduce the following function \(\gamma\). Let \(G'\) be commutator subgroup of \(G\) and \(g \in G'\). Then \(g\) can be represented as a product \(g = c_1c_2\ldots c_k\) of commutators \(c_i\). By commutator length \(|g|\) of \(g\) we mean the minimum number of commutators we need to represent \(g\) as a product of commutators. For unit element \(e\) we set \(|e| = 0\). Suppose \(G = G'\). Then we define
\[
\gamma(g) = |g|.
\]
We define \(\gamma(G) = \sup \{\gamma(g) \mid g \in G\}\). Therefore, \(\gamma(G)\) is a nonnegative integer or \(+\infty\).

**Theorem 3.5.** Let the group \(G\) be the union of its subgroups, \(G = \cup_{\alpha \in I} G_\alpha\), such that for any \(x, y \in G\) there is an \(\alpha \in I\) such that \(x, y \in G_\alpha\). Suppose that \(G = G'\), and that for any \(\alpha\) there is \(\beta\) such that \(G_\alpha \subseteq G'_\beta\). Let the function \(\gamma\) be defined by (3.3). Assume that \(\gamma(G'_\alpha) < \infty\) for any \(\alpha \in I\). Then the equation (3.1) is \((\psi, \gamma)\)-stable on \(G\).

**Proof.** Since \(G = G' = \cup_{\alpha \in I} G_\alpha = \cup_{\alpha \in I} G'_\alpha\), by Lemma 3.4 it is necessary and sufficient to show that (3.1) is \((\psi, \gamma)\)-stable on \(G'_\alpha\) for any \(\alpha \in I\).

Let \(\gamma(G'_\alpha) = k_\alpha \in \mathbb{N}\). Then for any \(x \in G'_\alpha\) we have \(\gamma(x) \leq k_\alpha\) and \(\psi(\gamma(x)) \leq \psi(k_\alpha)\). Therefore if \(f \in PX_{\psi, \gamma}(G'_\alpha)\), then
\[
|f(xy) - f(x) - f(y)| \leq \theta [\psi(\gamma(x)) + \psi(\gamma(y))], \quad x, y \in G'_\alpha,
\]
which yields
\[
|f(xy) - f(x) - f(y)| \leq 2\theta \psi(k_\alpha), \quad x, y \in G'_\alpha.
\]
From the last relation it follows that \( f \in PX(G'_\alpha) \). Consider \( f \) on \( G'_\alpha \). Let \( a, b \in G_\alpha \) and \( w = a^{-1}b^{-1}ab \) their commutator. Let \( G_\alpha \subseteq G'_\beta \) for some \( \beta \in I \).

Then we have \( a, b, w \in G'_\beta \) and

\[
|f(a^{-1}b^{-1}ab) - f((ba)^{-1}) - f(ab)| \leq 2\theta \psi(k_\beta),
\]

which simplifies to

\[
|f(a^{-1}b^{-1}ab) + f((ba)^{-1}) - f(ab)| \leq 2\theta \psi(k_\beta),
\]

Thus \( f \) is uniformly bounded on the set of commutators \( \{[a, b] \mid a, b \in G_\alpha \} \).

Now let \( g = w_1w_2 \ldots w_k \), where \( w_i \) is a commutator for \( i = 1, \ldots, k_\alpha \). Then \( |f(w_1w_2 \ldots w_k)| \leq 2k_\alpha \theta \psi(k_\beta) \). Thus \( f \) is a bounded function on \( G'_\alpha \). Now from the relation \( f(x^n) = nf(x), \forall x \in G'_\alpha, \forall n \in \mathbb{N} \) it follows that \( f \equiv 0 \) on \( G'_\alpha \). But it is known that if a pseudocharacter is zero on commutator subgroup of a group \( B \) then it is an additive character of \( B \) (see [4]). Therefore \( f \) is a character of \( G_\alpha \) and \( f(xy) = f(x) + f(y) \). This completes the proof of the theorem.

\[ \square \]

4. Stability on step-two solvable groups

Let \( [x, y] \) denotes commutator of two group elements \( x \) and \( y \), that is \( [x, y] = x^{-1}y^{-1}xy \). A group \( G \) is said to be step-two solvable group if for any \( x, y, u, v \) in \( G \) we have the equality \( [[x, y], [u, v]] = e \), where \( e \) is the unit element of \( G \) (see [13]). It is obvious that any abelian group is a step-two solvable group. Any extension of an abelian group by another abelian group is a step-two solvable group.

Let \( F = F(X) \) be a free group of an arbitrary rank with the set of free generators \( X \). Then a subgroup of \( F \) generated by all elements of the form \( [[x, y], [u, v]] \), where \( x, y, u, v \in F \) is a normal subgroup of \( F \). Let us denote it by \( F'' \). Then quotient group \( F^{[2]}(X) = F/F'' \) is a free step-two solvable group with the free set of generators \( X \). Then for any step-two solvable group \( H \) any mapping \( \tau : X \to H \) can be extended as an homomorphism of \( F^{[2]} \) onto the subgroup of \( H \) generated by the set \( \tau(X) \).
Let $G$ be a free step-two solvable group with two generators $a$ and $b$. It is well known (see [3]) that $G'$ is a free abelian group with the set of free generators: $w_{i,j} = a^{-i}b^{-j}[a,b]^{j}a^{i}$ for $i, j \in \mathbb{Z}$. When there is no confusion, we will write $w_{i,j}$ simply as $w_{ij}$. Let $w = w_{00}$.

**Lemma 4.1.** For any $i, j \in \mathbb{Z}$, we have the following relations:

(1) $a^{-k}w_{i,j}a^{k} = w_{i+k,j}$,

(2) $b^{-k}w_{0,j}b^{k} = w_{0,j+k}$.

**Proof.** The proof is obvious. □

**Lemma 4.2.** For any $k \in \mathbb{N}$, we have

$$a^{-1}b^{-k}ab^{k} = w_{00}w_{01}w_{02} \ldots w_{0(k-1)}.$$  

**Proof.** We prove this lemma by induction on $k$. If $k = 1$, then we have $a^{-1}b^{-1}ab^{1} = w_{00}$. Suppose that for any $k \leq n$ lemma has been established and let us establish it for $n + 1$. Since

$$a^{-1}b^{-n+1}ab^{n+1} = a^{-1}b^{-1}b^{n}ab^{n}b = a^{-1}b^{-1}aa^{-1}b^{-n}ab^{n}b$$
$$= a^{-1}b^{-1}a[a,b]^{n}b = a^{-1}b^{-1}ab^{n}[a,b]^{n}b$$
$$= [a,b]^{-1}[a,b]^{n}b = w_{00}b^{-1}w_{00}w_{01} \ldots w_{0(n-1)}b$$

(by induction hypothesis)
$$= w_{00}w_{01}w_{02} \ldots w_{0n} \quad (\text{by Lemma 4.1 (2)})$$

the proof of the lemma is now complete. □

In the last two sections, as usual, for $x, y \in G$, the conjugate of $x$ by $y$ will be denoted by $x^{y}$ and hence $x^{y} = y^{-1}xy$.

**Theorem 4.3.** Let $D$ be an arbitrary step-two solvable group. Suppose that function $\gamma$ is invariant with respect to inner automorphism of group $D$. Then the equation (3.1) is $(\psi, \gamma)$-stable on $D$.

**Proof.** First let $D = G$ be a step-two solvable free group with two generators $a$ and $b$. Let $f \in PX_{\psi,\gamma}(G)$. Thus for some $\theta > 0$, the map $f : G \to \mathbb{R}$ satisfies the relation

$$|f(xy) - f(x) - f(y)| \leq \theta[\psi(\gamma(x)) + \psi(\gamma(y))], \quad x, y \in G. \quad (4.1)$$

We should show that $f \in X(G)$. Since $G$ is a free step-two solvable group there is an additive character $\xi$ of $G$ such that $\xi(a) = f(a)$ and $\xi(b) = f(b)$. Then function
\( \phi = f - \xi \) is an element of \( PX_{\psi,\gamma}(G) \) such that \( \phi(a) = \phi(b) = 0 \). It is clear that \( f \in X(G) \) if and only if \( \phi \in X(G) \). So, from the beginning we can assume that \( f(a) = f(b) = 0 \). Then for any \( k \in \mathbb{N} \), letting \( x = a^{-1} \) and \( y = b^{-k}ab^k \) in the last inequality, we obtain

\[
|f(a^{-1}b^{-k}ab^k) - f(a^{-1}) - f(b^{-k}ab^k)| \leq \theta[\psi(\gamma(a^{-1})) + \psi(\gamma(b^{-k}ab^k))]
\]

and using relations \( f(a) = 0 \) and \( \gamma(b^{-k}ab^k) = \gamma(a) \) we get

\[
|f(a^{-1}b^{-k}ab^k)| \leq \theta[\psi(\gamma(a^{-1}) + \psi(\gamma(a))], \quad k \in \mathbb{N}. \tag{4.2}
\]

Taking into account that \( f|_{G'} \) is an additive character (since \( G' \) is commutative) invariant with respect inner automorphisms of \( G \) we get

\[
f(a^{-1}b^{-k}ab^k) = f(w_{00}w_{01} \ldots w_{0k-1}) = kf(w_{00}). \tag{4.3}
\]

Now from (4.2) and (4.3) we obtain

\[
|kf(w_{00})| \leq \theta[\psi(\gamma(a^{-1}) + \psi(\gamma(a))], \quad k \in \mathbb{N}
\]

which implies \( f(w_{00}) = 0 \). Therefore, \( f(w_{ij}) = 0 \) for any \( i, j \in \mathbb{Z} \) and \( f|_{G'} \equiv 0 \).

Let \( A \) and \( B \) be subgroup of \( G \) generated by \( a \) and \( b \) respectively. Let \( \overline{B} \) be the subgroup of \( G \) generated by \( B \) and \( G' \). Then \( \overline{B} \) is the semidirect product of \( B \) and \( G' \), that is \( \overline{B} = B \times G' \). Let us verify that \( f|_{\overline{B}} \equiv 0 \).

For any \( n \in \mathbb{N} \), any \( c \in B \) and any \( v \in G' \) we have

\[
(cv)^n = c^n v^{e_n-1} v^{e_n-2} \ldots v^e v. \tag{4.4}
\]

Letting \( x = c^n \) and \( y = v^{e_n-1} v^{e_n-2} \ldots v^e v \) in (4.1), we have

\[
|f(c^n v^{e_n-1} v^{e_n-2} \ldots v^e v) - f(c^n)| - f(v^{e_n-1} v^{e_n-2} \ldots v^e v)|
\leq \theta[\psi(\gamma(c^n)) + \psi(\gamma(v^{e_n-1} v^{e_n-2} \ldots v^e v))]
\]

for each \( n \in \mathbb{N} \). Hence

\[
|f(c^n v^{e_n-1} v^{e_n-2} \ldots v^e v)| \leq \theta[\psi(\gamma(c^n)) + \psi(\gamma(v^{e_n-1} v^{e_n-2} \ldots v^e v))].
\]

Using the subadditivity of \( \gamma \), we have

\[
|f(c^n v^{e_n-1} v^{e_n-2} \ldots v^e v)| \leq \theta \left[ \psi(n \gamma(c)) + \psi \left( \sum_{k=0}^{n-1} \gamma(v^k) \right) \right].
\]
From the last inequality and the fact that $\gamma$ is invariant with respect to inner automorphisms, we obtain

\[ |f(c^n v^{\alpha_{n-1}} v^{\alpha_{n-2}} \ldots v^\alpha)| \leq \theta \psi(n)[\psi(\gamma(c)) + \psi(\gamma(v))] \]

The last relation and (4.4) imply

\[ |f((cv)^n)| \leq \theta \psi(n)[\psi(\gamma(c)) + \psi(\gamma(v))]. \]

Since $f(x^n) = nf(x)$, we obtain

\[ nf(f(cv)) \leq \theta \psi(n)[\psi(\gamma(c)) + \psi(\gamma(v))], \]

and therefore

\[ |f(cv)| \leq \frac{\theta \psi(n)}{n} [\psi(\gamma(c)) + \psi(\gamma(v))] \]

for each $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality and using the fact that $\lim_{n \to \infty} \frac{\psi(n)}{n} = 0$, we obtain $f(cv) = 0$ and therefore $f|_{\mathbb{F}} \equiv 0$.

Now consider group $G$. This group is a semidirect product of $A$ and $B$, that is $G = A \times B$. Every element $g$ of $G$ can be represented in the form $g = du$, where $d \in A$ and $u \in B$. Arguing as above we can show that $f(g) = 0$. Therefore $f \equiv 0$ on the group $G$. It means that equation (3.1) is $(\psi, \gamma)$-stable on $G$.

Now suppose that $H$ be an arbitrary step-two solvable group with two generators $a$ and $b$. The group $G$ is a free step-two solvable with two generators $a$ and $b$. Then there is an epimorphism $\tau : G \to H$ such that $\tau(a) = \alpha$ and $\tau(b) = \beta$. Define $\gamma^*$ by the rule $\gamma^*(x) = \gamma(\tau(x))$ for any $x \in G$. It is clear that $\gamma^*$ satisfies conditions:

$\gamma^*(xy) \leq \gamma^*(x) + \gamma^*(y)$ and $\gamma^*(x^{-1}yx) = \gamma^*(y)$

for any $x, y \in G$.

Let $f \in PX_{\psi, \gamma^*}(H)$. Then for some $\theta > 0$, the map $f$ satisfies

\[ |f(xy) - f(x) - f(y)| \leq \theta[\psi(\gamma(x)) + \psi(\gamma(y))], \quad \forall x, y \in H. \]

Let us verify that $f \in X(H)$. Suppose that there are $c, d \in H$ such that $f(cd) - f(c) - f(d) \neq 0$. Then function $f^*$ defined by the rule $f^*(x) = f(\tau(x))$ belongs to the space $PX_{\psi, \gamma^*}(G)$. But for elements $u$ and $v$ such that $\tau(u) = c$ and $\tau(v) = d$ we get $f^*(uv) - f^*(u) - f^*(v) = f(cd) - f(c) - f(d) \neq 0$. This contradicts the relation $PX_{\psi, \gamma^*}(G) = X(G)$. Therefore, $f \in X(H)$. So, every step-two solvable group generated by two generators has the $(\psi, \gamma)$-stability property.
Now let $D$ be an arbitrary step-two solvable group. Then $D = \cup_{x,y} D(x,y)$, where $D(x,y)$ is a subgroup generated by elements $x, y \in D$. Equation (3.1) is $(\psi, \gamma)$-stable on any $D(x,y)$. Therefore by Lemma 3.4 equation (3.1) is $(\psi, \gamma)$-stable on $D$. This completes the proof of the theorem. □

**Theorem 4.4.** Let $D$ be an arbitrary step-two solvable group. Suppose the function $\psi$ satisfies an additional condition: $\lim_{n \to \infty} \frac{\psi(n^2)}{n} = 0$. Then the equation (3.1) is $(\psi, \gamma)$-stable on $D$.

**Proof.** As it was done in the previous theorem it is enough to prove this theorem for the case $D = G$, where $G$ is a free step-two solvable group with two generators $a$ and $b$. Let $f \in PX_{\psi,\gamma}(G)$. Then for some $\theta > 0$, the function $f : G \to \mathbb{R}$ satisfies the relation

$$|f(xy) - f(x) - f(y)| \leq \theta[\psi(\gamma(x)) + \psi(\gamma(x))], \quad x, y \in G.$$ 

Let us assume that $f(a) = f(b) = 0$. Then for any $k \in \mathbb{N}$ we have

$$|f(a^{-1}b^{-k}ab^k) - f(a^{-1}) - f(b^{-k}ab^k)| \leq \theta[\psi(\gamma(a^{-1})) + \psi(\gamma(b^{-k}))].$$

From the last inequality, we see that

$$|f(a^{-1}b^{-k}ab^k)| \leq \theta[\psi(\gamma(a^{-1})) + \psi(\gamma(a)) + \psi(\gamma(b^{-k})) + \psi(\gamma(b))].$$

which is

$$|f(a^{-1}b^{-k}ab^k)| \leq \theta[\psi(\gamma(a^{-1})) + \psi(\gamma(a)) + \psi(k\gamma(b^{-1})) + \psi(k\gamma(b))].$$

Since $\psi(t_1 t_2) \leq \psi(t_1) \psi(t_2)$ for all $t_1, t_2 \in \mathbb{R}^+$, we have

$$|f(a^{-1}b^{-k}ab^k)| \leq \theta[\psi(\gamma(a^{-1})) + \psi(\gamma(a))] + \theta \psi(k)[\psi(\gamma(b^{-1})) + \psi(\gamma(b))].$$

Taking into account that $f|_{G'}$ is an additive character invariant with respect inner automorphisms of $G$ we get

$$f(a^{-1}b^{-k}ab^k) = f(w_0 w_0 w_1 \ldots w_{0(k-1)}) = kf(w_0).$$

Therefore, for each $k \in \mathbb{N}$, we have

$$|kf(w_0)| \leq \theta[\psi(\gamma(a^{-1})) + \psi(\gamma(a))] + \theta \psi(k)[\psi(\gamma(b^{-1})) + \psi(\gamma(b))].$$
and hence
\[ |f(w_{00})| \leq \frac{\theta}{k} \left[ \psi(\gamma(a^{-1}) + \psi(\gamma(a))) + \frac{\theta}{k} \left[ \psi(\gamma(b^{-1}) + \psi(\gamma(b))) \right] \right]. \]
The last inequality implies that \( f(w_{00}) = 0 \). Therefore, \( f(w_{ij}) = 0 \) for any \( i, j \in \mathbb{Z} \) and \( f \big|_{G'} \equiv 0 \).

Let \( \overline{B} \) be a subgroup of \( G \) generated by \( B \) and \( G' \). Then \( \overline{B} \) is a semidirect product of \( B \) and \( G' \), that is \( \overline{B} = B \rtimes G' \). Let us verify that \( f \big|_{\overline{B}} \equiv 0 \). For any \( c \in B \) and any \( v \in G' \) we have
\[(cv)^n = c^n v^{c^n-1} v^{c^n-2} \ldots v^c v \quad (4.5)\]
and for each \( n \in \mathbb{N} \)
\[
|f(c^n v^{c^n-1} v^{c^n-2} \ldots v^c v) - f(c^n) - f(v^{c^n-1} v^{c^n-2} \ldots v^c v)|
\leq \theta \left[ \psi(\gamma(c^n)) + \psi(\gamma(v^{c^n-1} v^{c^n-2} \ldots v^c v)) \right].
\]
Hence
\[
|f(c^n v^{c^n-1} v^{c^n-2} \ldots v^c v)| \leq \theta \left[ \psi(\gamma(c^n)) + \psi(\gamma(v^{c^n-1} v^{c^n-2} \ldots v^c v)) \right].
\]
Simplifying the above inequality, we obtain
\[
|f(c^n v^{c^n-1} v^{c^n-2} \ldots v^c v)| \leq \theta \left[ \psi(n \gamma(c)) + \psi \left( \sum_{k=0}^{n-1} \gamma(v^k) \right) \right].
\]
Using the fact that \( v^c = c^{-k}v^c \) and the last inequality, we get
\[
|f(c^n v^{c^n-1} v^{c^n-2} \ldots v^c v)| \leq \theta \left[ \psi(n \gamma(c)) + \psi \left( \sum_{k=0}^{n-1} (\gamma(c^{-k}) + \gamma(v) + \gamma(c^k)) \right) \right]
\]
which implies
\[
|f(c^n v^{c^n-1} v^{c^n-2} \ldots v^c v)| \leq \theta \left[ \psi(n \gamma(c)) + \psi(n \gamma(v)) + \psi \left( \sum_{k=0}^{n-1} (\gamma(c^{-k}) + \gamma(c)) \right) \right].
\]
Further, simplifying the last inequality, we have
\[
|f(c^n v^{c^n-1} v^{c^n-2} \ldots v^c v)|
\leq \theta \left[ \psi(n \gamma(c)) + \psi(n \gamma(v)) + \psi \left( \sum_{k=0}^{n-1} (k [\gamma(c^{-1}) + \gamma(c)]) \right) \right].
\]
The function \( \psi(t) = t^q + 1 \) with \( 0 < q < 1/2 \) satisfies condition \( \lim_{n \to \infty} \frac{\psi(n^2)}{n} = 0 \).
5. Stability on step-three nilpotent groups

A group $G$ is said to be a step-two nilpotent (or metabelian) group if for any
$x, y, u \in G$ we have equality $[[x, y], u] = e$, where $e$ is the unit element of $G$. A
group $G$ with unit element $e$ is said to be a step-three nilpotent group if for any
$x, y, u, v \in G$ the equality $[[[x, y], u], v] = e$ holds (see [13]). It is obvious that any
abelian group is a step-two nilpotent group, and any step-two nilpotent group is
a step-three nilpotent group.

Let $K$ be a commutative field. The set
\[
\begin{cases}
1 & x_1 y_1 z \\
0 & x_1 y_2 z \\
0 & x_3 y_1 z \\
0 & 0 0 1
\end{cases}
\]
\[
x_i, y_i, z \in K, i = 1, 2
\]
of all $4 \times 4$ upper triangular matrices forms a group under matrix multiplication.
This group is denoted by $UT(4, K)$, and any subgroup of this group is a step-three
nilpotent group. The group $UT(4, K)$ is also known as Heisenberg group $H_4(K)$.

Let $F = F(X)$ be a free group an arbitrary rank with the set of free gen-
erators $X$. Denote by $[[[F, F], F], F]$ the normal subgroup of $F$ generated by all
elements of the form $[[[x, y], u], v]$, where $x, y, u, v \in F$. Then the quotient group
$F^{(3)}(X) = F/[[[F, F], F], F]$ is a free step-three nilpotent group with a free set of
generators $X$. It means that for any step-three nilpotent group $H$ any mapping
$\tau : X \to H$ can be extended to a homomorphism of $F^{(3)}$ onto the subgroup of $H$
generated by the set $\tau(X)$.

Let $G$ be a free step-three nilpotent group with two free generators $a$ and $b$.
It is well known that $G$ has the following presentation (see [14]):
\[
G = \langle a, b \mid b^{-1}ab = ac, b^{-1}cb = cd, a^{-1}ca = ch, \]
\[
ad = da, bd = db, ah = ha, hh = hb \rangle.
\] (5.1)

From (5.1) it follows that for any integers $n, m$ the following relations
\[
a^{-n}c^m a^n = c^m a^{nm},
\] (5.2)
\[
b^{-n}c^m b^n = c^m b^{nm},
\] (5.3)
hold. Suppose that $\varphi \in PX_{\psi, \gamma}(G)$.

**Theorem 5.1.** Suppose the function $\psi$ satisfies $\lim_{n \to \infty} \frac{\psi(n^2)}{n} = 0$. Then
for any step-three nilpotent group $G$ the equation (3.1) is $(\psi, \gamma)$-stable.
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**Proof.** As we know we can consider only the case when \(G\) is free step-three nilpotent group with two generators \(a, b\). Let \(\varphi \in PX_{\psi, \gamma}(G)\). We must show that \(\varphi \in X(G)\). We can assume that \(\varphi(a) = \varphi(b) = 0\).

Then from (5.2), we get
\[
\varphi(a^{-n}c^m a^n) = \varphi(c^m h^{nm}). \tag{5.4}
\]

From Theorem 2.11 from [5] it follows that \(\varphi(u^{-1}vu) = \varphi(v)\) for any \(u\) and \(v\). Now taking into account this relation, Theorem 2.10 from [5] and (5.4) we get
\[
\varphi(c^m) = \varphi(c^m) + \varphi(h^{nm}).
\]

So \(\varphi(h) = 0\). Similarly, we get \(\varphi(d) = 0\). From presentation (5.3) it follows that \(b^{-n}ab^n = ac^nd^{\frac{n(n+1)}{2}}\), for any \(n \in \mathbb{N}\). Therefore, \(\varphi(ac^nd^{\frac{n(n+1)}{2}}) = 0\) and \(\varphi(ac^n) = 0\). Thus from
\[
|\varphi(ac^n) - \varphi(a) - \varphi(c^n)| \leq \theta[\psi(\gamma(a)) + \psi(\gamma(c^n))]
\]
we have
\[
|\varphi(c^n)| \leq \theta[\psi(\gamma(a)) + \psi(\gamma(c^n))].
\]

Since \(\varphi \in PX_{\psi, \gamma}(G)\), we have
\[
n|\varphi(c)| \leq \theta[\psi(\gamma(a)) + \psi(n\gamma(c))]
\]
and hence
\[
|\varphi(c)| \leq \theta \left[\frac{\psi(\gamma(a))}{n} + \frac{\psi(n\gamma(c))}{n}\right].
\]

The last inequality implies that \(\varphi(c) = 0\). So, we have \(\varphi(a) = \varphi(b) = \varphi(c) = \varphi(d) = \varphi(h) = 0\).

Now let us show that \(\varphi \equiv 0\) on \(G\). First note that \(\varphi\) is a function on factor group \(G/Z(G)\), where \(Z(G)\) denotes center of \(G\). Indeed, \(Z(G)\) is a free abelian group generated by elements \(d\) and \(h\). From relations \(\varphi(d) = \varphi(h) = 0\) it follows that \(\varphi \equiv 0\) on \(Z(G)\) and for any \(u \in G\) and any \(w \in Z(G)\) we have \(\varphi(uw) = \varphi(u)\).

Taking into account this note we get the following relations:
\[
a^n b^m c^k a^{n_1} b^{m_1} c^{k_1} = a^{n+n_1} b^{m+m_1} c^{n_1+m+k+k_1} \pmod{Z(G)},
\]
and
\[
(a^n b^m c^k)^p = a^{pn p} b^{pm} c^{nm \frac{n(n+1)}{2} + pk} \pmod{Z(G)}. \tag{5.5}
\]
For any \( x, y, z \in G \), we have
\[
|\varphi(xyz) - \varphi(xy) - \varphi(z)| \leq \theta[|\psi(\gamma(xy)) + \psi(\gamma(z))|]
\]
and
\[
|\varphi(xy) - \varphi(x) - \varphi(y)| \leq \theta[|\psi(\gamma(x)) + \psi(\gamma(y))|].
\]
Therefore
\[
|\varphi(xyz) - \varphi(x) - \varphi(y) - \varphi(z)| \leq \theta[|\psi(\gamma(xy)) + \psi(\gamma(z)) + \psi(\gamma(x)) + \psi(\gamma(y))|].
\]
Since \( \psi(\gamma(xy)) \leq \psi(\gamma(x) + \gamma(y)) \leq \psi(\gamma(x)) + \psi(\gamma(y)) \), the last inequality yields
\[
|\varphi(xyz) - \varphi(x) - \varphi(y) - \varphi(z)| \leq 2\theta[|\psi(\gamma(x)) + \psi(\gamma(y)) + \psi(\gamma(z))|]. 
\] (5.6)
Now let \( v = a^n b^m c^k d^h \) be an arbitrary element of \( G \). From (5.5), it follows that for any \( p \in \mathbb{N} \) there is a \( w_p \in Z(G) \) such that \( v^p = a^{pn} b^{pm} c^{nm \frac{p(p-1)}{2} + pk} w_p \). Hence we have
\[
\varphi(v^p) = \varphi(a^{pn} b^{pm} c^{nm \frac{p(p-1)}{2} + pk} w_p) = \varphi(a^{pn} b^{pm} c^{nm \frac{p(p-1)}{2} + pk}).
\]
From (5.6), we get
\[
|\varphi(v^p) - \varphi(a^{pn}) - \varphi(b^{pm}) - \varphi(c^{nm \frac{p(p-1)}{2} + pk})| 
\leq 2\theta[|\psi(\gamma(a^{pn})) + \psi(\gamma(b^{pm})) + \psi(\gamma(c^{nm \frac{p(p-1)}{2} + pk}))|].
\]
Hence from the last inequality, we have
\[
|p\varphi(v)| \leq 2\theta[|\psi(\gamma(a^{pn})) + \psi(\gamma(b^{pm})) + \psi(\gamma(c^{nm \frac{p(p-1)}{2} + pk}))|]
\]
which simplifies to
\[
|p\varphi(v)| \leq 2\theta[|\psi(p\gamma(a^n)) + \psi(p\gamma(b^m)) + \psi(c^{nm \frac{p(p-1)}{2} + pk}))|].
\]
Thus simplifying further, we see that
\[
|\varphi(v)| \leq 2\theta \frac{\psi(p)}{p} [\psi(\gamma(a^n)) + \psi(\gamma(b^m)) + \psi(c^k)] + 2\theta \frac{\psi(p(p-1))}{p} \psi(1/2) \psi(c^k).
\]
Since \( \lim_{p \to \infty} \frac{\psi(p)}{p} = 0 \) and \( \lim_{p \to \infty} \frac{\psi(p(p-1))}{p} = 0 \) the last inequality implies \( \varphi(v) = 0 \). Therefore \( \varphi \equiv 0 \) on \( G \) and equation (3.1) is \( (\psi, \gamma) \)-stable on \( G \). The proof of the theorem is now complete. \( \square \)
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References


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