Lightlike curves in Lorentz manifolds

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To the memory of Professor András Rapcsák

The purpose of the present paper is to initiate a general study of differential geometry of lightlike curves in Lorentz manifolds. First, we construct a complementary vector subbundle to the tangent bundle of a lightlike curve. Then in section 2 we obtain the Frenet equations with respect to a general Frenet field of frames and prove theorems of reduction of the codimension of a lightlike curve (Theorems 3, 4 and 5). According to the Theorem 5, any lightlike curve of a Minkowski space whose the seventh curvature vanishes, lies in a 5-dimensional plane. This is a surprisingly result and it might have applications in multi-dimensional physical theories. Finally, we prove an existence and uniqueness theorem for lightlike curves in Lorentz manifolds.

§1. A complementary vector subbundle to the tangent bundle of a lightlike curve

Let $M$ be a real $(m + 2)$-dimensional Lorentz manifold, i.e., in $M$ there exists a semi-Riemannian metric $g$ of index $\nu = 1$, (cf. O’Neill [6]). Suppose $C$ is a differentiable curve in $M$ locally given by

$$x^i = x^i(t), \quad t \in [a, b].$$

In case the tangent vector field

$$\frac{d}{dt} = \left( \frac{dx^1}{dt}, \ldots, \frac{dx^{m+2}}{dt} \right),$$

has a non-null length with respect to $g$ we have a complete study of the geometry of $C$ (cf. Spivak [7]). Surprisingly, though theory of curves is one of the intensively studied theory of differential geometry, till now we do not have a method of studying curves whose tangent vector field is
lightlike, i.e., we have
\begin{equation}
g\left(\frac{d}{dt}, \frac{d}{dt}\right) = 0.
\end{equation}

As far as we know, the results obtained on this class of curves refer to the case when the ambient space is one of the Minkowski spaces $R^3_1$ or $R^4_1$ (cf. Cartan [3], Castagnino [2], Bonnor [1], Graves [4], Ikawa [5]).

That is why, we consider as a need a general theory of such curves in a Lorentz manifold. The present paper is concerned with such a study and it might give more insights for a general study of lightlike submanifolds of semi-Riemannian manifolds.

We say that $C$ is a lightlike curve in $M$ if there exists a lightlike vector field $\xi$ tangent to $C$, that is, we have
\begin{equation}
g(\xi, \xi) = 0.
\end{equation}
Certainly, in this case there exists a differentiable function $k_0 \neq 0$ such that
\begin{equation}
\frac{d}{dt} = k_0 \xi,
\end{equation}
and henceforth (1.2) and (1.3) are equivalent with each other. Denote by $TC$ the tangent bundle of $C$ and define as in case of nondegenerate curves
\begin{equation}
TC = \bigcup_{x \in C} TC_x; \quad TC_x^\perp = \{v \in T_x M, \ g(v, \xi_x) = 0\}.
\end{equation}
Then $TC^\perp$ is a vector bundle over $C$ whose fibres are $(m+1)$-dimensional and $\xi$ is a differentiable section of $TC^\perp$. Thus $TC$ is a 1-dimensional vector subbundle of $TC^\perp$. Suppose $sC$ is a complementary vector subbundle to $TC$ in $TC^\perp$, i.e., we have
\begin{equation}
TC^\perp = TC \perp sC,
\end{equation}
where $\perp$ means orthogonal direct sum. It follows that $sC$ is a non-degenerate $m$-dimensional vector subbundle of $TM$. Then denote by $sC^\perp$ the 2-dimensional complementary orthogonal vector subbundle to $sC$ in $TM$, i.e., we have
\begin{equation}
TM = sC \perp sC^\perp.
\end{equation}

Throughout the paper we denote by $F(C)$ the algebra of differentiable functions on $C$ and by $\Gamma(E)$ the $F(C)$-module of differentiable sections of a vector bundle $E$ over $C$. We use the same notation for any other vector bundle.

As in any theory of submanifolds appears as a necessity the construction of a complementary vector bundle of the tangent bundle of the submanifolds in the tangent bundle of the ambient space, we state first the following result.
Theorem 1. Let $C$ be a lightlike curve in a Lorentz manifold $M$. Then for a given vector subbundle $sC$ as in (1.6) there exists a unique 1-dimensional vector subbundle $nC$ of $sC^\perp$ such that on each neighbourhood of coordinates $\mathcal{U} \subset C$, for any $\xi \in \Gamma(TC|_{\mathcal{U}})$ there exists $N \in \Gamma(nC|_{\mathcal{U}})$ satisfying

\begin{equation}
 g(\xi, N) = 1,
\end{equation}

and

\begin{equation}
 g(N, N) = 0.
\end{equation}

Proof. Since $TC$ is a vector subbundle of $sC^\perp$ we may consider a complementary vector subbundle $\delta$ of $TC$ in $sC^\perp$. For any $\xi \in \Gamma(TC|_{\mathcal{U}})$ there exists $V \in \Gamma(\delta|_{\mathcal{U}})$ such that $g(\xi, V) \neq 0$, otherwise $TM$ would be degenerate with respect to $g$. Then it follows that any $N \in \Gamma(nC|_{\mathcal{U}})$ satisfying (1.8) and (1.9) is given by

\begin{equation}
 N = \frac{1}{g(\xi, V)} \left\{ V - \frac{g(V, V)}{2g(\xi, V)} \xi \right\}.
\end{equation}

Moreover, it is easy to check that $N$ depends neither on vector bundle $\delta$ nor on local section $V$. Hence for any $\xi \in \Gamma(TC|_{\mathcal{U}})$ there exists a unique vector field $N \in \Gamma(nC|_{\mathcal{U}})$ satisfying (1.8) and (1.9). Next, consider another neighborhood of coordinates $\mathcal{U}^* \subset C$ such that $\mathcal{U} \cap \mathcal{U}^* \neq \emptyset$. Then $\xi^* = f\xi$, where $f$ is nowhere zero differentiable function on $\mathcal{U} \cap \mathcal{U}^*$, and by using (1.10) obtain $N^* = \frac{1}{f} N$. Therefore we obtain a unique 1-dimensional vector subbundle $nC$ of $sC^\perp$ whose local sections $N$ satisfy (1.8) and (1.9). The proof is complete.

Next we consider the vector bundle

\begin{equation}
 NC = nC \perp sC,
\end{equation}

which according to the proof of Theorem 1 is complementary to $TC$ in $TM$, i.e., we have

\begin{equation}
 TM = (TC \oplus nC) \perp sC,
\end{equation}

where $\oplus$ means direct sum but not orthogonal. It is important to note that the induced metrics on $TC \oplus nC$ and $sC$ are of index $\nu = 1$ (Lorentz metric) and $\nu = 0$ (Riemannian metric) respectively.

§2. The Frenet equations for a lightlike curve in a Lorentz manifold

Let $C$ be a lightlike curve of an $(m+2)$-dimensional Lorentz manifold $M$ and $\{\xi, N\}$ be the lightlike vector fields from Theorem 1. Suppose $\nabla$
is the Levi–Civita connection on $M$. Then by using (1.3), (1.8) and (1.9) we obtain the following general Frenet equations for the lightlike curve $C$:

\[
\begin{align*}
\nabla_{\xi} \xi &= A \xi + k_1 W_1 \\
\nabla_{\xi} N &= -A N + k_2 W_1 + k_3 W_2 \\
\nabla_{\xi} W_1 &= -k_2 \xi - k_1 N + k_4 W_2 + k_5 W_3 \\
\nabla_{\xi} W_2 &= -k_3 \xi - k_4 W_1 + k_6 W_3 + k_7 W_4 \\
\nabla_{\xi} W_3 &= -k_5 W_1 - k_6 W_2 + k_8 W_4 + k_9 W_5 \\
&\vdots \\
\nabla_{\xi} W_{m-2} &= -k_{2m-5} W_{m-4} - k_{2m-4} W_{m-3} + \\
&\quad + k_{2m-2} W_{m-1} + k_{2m-1} W_m \\
\nabla_{\xi} W_{m-1} &= -k_{2m-3} W_{m-3} - k_{2m-2} W_{m-2} + k_{2m} W_m \\
\nabla_{\xi} W_m &= -k_{2m-1} W_{m-2} - k_{2m} W_{m-1},
\end{align*}
\]

where $A$ and $\{k_1, \ldots, k_{2m}\}$ are differentiable functions and $\{W_1, \ldots, W_m\}$ is an orthonormal basis of $\Gamma(sC)$. We call

\[
\{\xi, N, W_1, \ldots, W_m\}
\]

the lightlike Frenet field of frames on $M$ along $C$, and $\{k_1, \ldots, k_{2m}\}$ the curvature functions of $C$. Then by using (1.3), (1.8) and (1.9) we obtain

\[
\begin{align*}
\bar{\xi} &= f \xi; \quad \bar{N} = -\frac{f}{2} \sum_{\alpha=1}^{m} (c_\alpha)^2 + \frac{1}{f} N + \sum_{\alpha=1}^{m} c_\alpha W_\alpha \\
\bar{W}_\alpha &= \sum_{\beta=1}^{m} a_{\alpha\beta} (W_\beta - ac_\beta \xi),
\end{align*}
\]

where $f \neq 0$, $c_\alpha$ and $a_{\alpha\beta}$ are differentiable functions, and for any point $x$ on $C$, $[a_{\alpha\beta}(x)]$ is an element of the orthogonal group $O(m)$, and $\{\bar{\xi}, \bar{N}, \bar{W}_1, \ldots, \bar{W}_m\}$ is another lightlike Frenet field of frames on $M$ along $C$.

Next, from the first Frenet equation of (2.1) written for both Frenet fields of frames we obtain

\[
\begin{align*}
\bar{A} &= k_1 \sum_{\alpha=1}^{m} a_{1\alpha} c_\alpha + \xi(f) + f A; \quad \bar{k}_1 a_{11} = f^2 k_1 \\
\bar{k}_1 a_{12} &= \cdots = \bar{k}_1 a_{1m} = 0.
\end{align*}
\]

Suppose $k_1 = 0$ on $C$. Then $\bar{k}_1 = 0$ on $C$, otherwise from (2.4) obtain $a_{11} = a_{12} = \cdots = a_{1m} = 0$, which is a contradiction because $[a_{\alpha\beta}]$ is an
orthogonal matrix. A lightlike curve $C$ on which $k_1 = 0$ is called a lightlike geodesic of the Lorentz manifold $M$. In this case the first Frenet equation becomes

\begin{equation}
\nabla \bar{\xi} \bar{\xi} = \bar{A} \bar{\xi},
\end{equation}

with respect to the lightlike Frenet field of frames $\{\bar{\xi}, \bar{N}, \bar{W}_1, \ldots, \bar{W}_m\}$. Now, choose $f$ from the first relation (2.3) as a solution of partial differential equation

\begin{equation}
\xi(f) + fA = 0.
\end{equation}

Then $\bar{A} = 0$ and (2.5) becomes

\begin{equation}
\nabla \bar{\xi} \bar{\xi} = 0.
\end{equation}

Suppose $\frac{d}{dt} = \bar{\varepsilon} \bar{\xi}$ and consider $u$ given by $\frac{du}{dt} = \bar{\varepsilon}$ as a new parameter on $C$, provided $\bar{\varepsilon} > 0$ on $C$. Then $\frac{d}{du} = \bar{\xi}$ and (2.7) becomes

\begin{equation}
\frac{d^2 x^i}{du^2} + \Gamma^i_{jk} \frac{dx^j}{du} \frac{dx^k}{du} = 0,
\end{equation}

where $\Gamma^i_{jk}$ are the Christoffel symbols induced by $\nabla$. Then by using (2.8) we obtain

**Theorem 2.** A lightlike curve $C$ of $\mathbb{R}^{m+2}_1$ is a straight line if and only if $k_1 = 0$ on $C$.

For the particular case $m = 2$, Theorem 2 is due to Bonnor [1]. We call $u$ the pseudo-arc on $C$ (cf. Vessiot [8]).

The above study enables us to suppose, from now on, $k_1 \neq 0$ at every point of the lightlike curve $C$. Then $\bar{k}_1 \neq 0$ and (2.4) becomes

\begin{equation}
\begin{cases}
 a_{11} = 1, & a_{1\alpha} = a_{\alpha 1} = 0, \quad \alpha \in \{2, \ldots, m\} \\
 A = f^2 k_1 c_1 + \xi(f) + fA; & \bar{k}_1 = f^2 k_1.
\end{cases}
\end{equation}

**Remark 1.** For the particular case $m = 2$, that is, for a lightlike curve $C$ of a 4-dimensional Lorentz manifold the transformation of lightlike Frenet fields of frames (2.3) becomes

\begin{equation}
\begin{cases}
 \bar{\xi} = f \xi; & \bar{N} = -\frac{f}{2} ((c_1)^2 + (c_2)^2) \xi + \frac{1}{f} N + c_1 W_1 + c_2 W_2, \\
 \bar{W}_1 = W_1 - fc_1 \xi; & \bar{W}_2 = W_2 - fc_2 \xi.
\end{cases}
\end{equation}
According to the above remark we are further concerned with the study of relationships between \( \{ A, k_1, \ldots, k_{2m} \} \) and \( \{ \bar{A}, \bar{k}_1, \ldots, \bar{k}_{2m} \} \) with respect to the transformation of lightlike Frenet field of frames:

\[
\begin{cases}
\bar{\xi} = f \xi; \quad \bar{N} = -\frac{f}{2} \sum_{\alpha=1}^{m} (c_{\alpha})^2 \xi + \frac{1}{f} N + \sum_{\alpha=1}^{m} c_{\alpha} W_{\alpha} \\
\bar{W}_{\alpha} = W_{\alpha} - f c_{\alpha} \xi, \quad \alpha \in \{1, \ldots, m\}
\end{cases}
\]

Thus by direct calculations using (2.11) and the last \( m + 1 \) equations in (2.1) for both lightlike Frenet field of frames we obtain

\[
\begin{align*}
\bar{k}_2 &= k_2 + f c_1 A + \xi (f c_1) + \frac{f^2 k_1}{2} ((c_1)^2 - (c_2)^2 - (c_3)^2) - f (c_2 k_4 + c_3 k_5) \\
\bar{k}_3 &= k_3 + f c_2 A + \xi (f c_2) + f^2 c_1 c_2 k_1 + f c_1 k_4 - f c_3 k_6 \\
\bar{k}_4 &= f (k_4 + f c_2 k_1); \quad \bar{k}_5 = f (k_5 + f c_3 k_1), \\
\bar{k}_{\alpha} &= f k_{\alpha}, \quad \alpha \in \{6, \ldots, m\},
\end{align*}
\]

and

\[
\begin{align*}
f A c_3 + \xi (f c_3) + f^2 c_1 c_3 k_3 + f c_1 k_5 + f c_2 k_6 &= 0 \\
c_2 k_7 + c_3 k_8 &= 0; \quad c_3 k_9 = 0; \quad c_{\alpha} = 0, \quad \alpha \in \{4, \ldots, m\}.
\end{align*}
\]

Now we choose \( c_1 \) given by

\[
c_1 = -\frac{1}{f^2 k_1} (\xi (f) + f A).
\]

Then from (2.9) it follows \( \bar{A} = 0 \). Therefore, we always may consider a lightlike Frenet field of frames \( \left\{ \frac{d}{du} = \xi, N, W_1, \ldots, W_m \right\} \) with respect to with the Frenet equations are given by

\[
\begin{align*}
\frac{D \xi}{Du} &= k_1 W_1 \\
\frac{D N}{Du} &= k_2 W_1 + k_3 W_2 \\
\frac{D W_1}{Du} &= -k_2 \xi - k_1 N + k_4 W_2 + k_5 W_3 \\
\frac{D W_2}{Du} &= -k_3 \xi - k_4 W_1 + k_6 W_3 + k_7 W_4
\end{align*}
\]
\[
\frac{DW_3}{Du} = -k_5 W_1 - k_6 W_2 + k_8 W_4 + k_9 W_5 \\
\vdots
\]
\[
\frac{DW_{m-2}}{Du} = -k_{2m-5} W_{m-4} - k_{2m-4} W_{m-3} + k_{2m-2} W_{m-1} + k_{2m-1} W_m
\]
\[
\frac{DW_{m-1}}{Du} = -k_{2m-3} W_{m-3} - k_{2m-2} W_{m-2} + k_{2m} W_m
\]
\[
\frac{DW_m}{Du} = -k_{2m-1} W_{m-2} - k_{2m} W_{m-1},
\]

where \( \frac{D}{Du} = \nabla \frac{d}{du} \).

**Theorem 3.** Let \( C \) be a lightlike curve of a \((m + 2)\)–dimensional \((m > 4)\) Lorentz manifold \( M \) with \( k_1 \neq 0 \) on \( C \). Then with respect to the lightlike Frenet field of frames \( \{ \xi, N, W_1, \ldots, W_m \} \) we have \( k_8 = k_9 = 0 \) on \( C \).

**Proof.** We choose \( c_3 = 1, c_2 = 0 \) and \( f \) as a non-null solution of (2.6). Then from (2.14) we obtain \( c_1 = 0 \) and the assertion of the theorem follows from (2.13).

**Theorem 4.** Let \( C \) be a lightlike curve of a Minkowski space \( R^{m+2}_1 \) \((m > 3)\) with \( k_\alpha, \alpha \in \{1, \ldots, 2p - 4\}, p \in \{3, \ldots, m\} \) nowhere zero and \( k_{2p-3}, k_{2p-2} \) and \( k_{2p-1} \) everywhere zero on \( C \). Then \( C \) lies in some \( p \)–dimensional Minkowski space of \( R^{m+2}_1 \). In case \( k_{2m-1} \) and \( k_{2m} \) are everywhere zero, \( C \) lies in a Minkowski hyperplane of \( R^{m+2}_1 \).

**Proof.** Let \( \{ \xi = \frac{d}{du}, N, W_1, \ldots, W_{p-2} \} \) be a part of a lightlike Frenet field of frames along \( C \), and \( \Delta(u) \subset T_{x(u)} R^{m+2}_1 \) be the \( p \)–dimensional subspace spanned by \( \{ \xi(u), N(u), W_1(u), \ldots, W_{p-2}(u) \} \). All these subspaces are parallel as \( p \)–dimensional planes of \( R^{m+2}_1 \). In order to prove this we first note that \( \frac{DX}{Du} \) is just \( X'(u) \) in \( R^{m+2}_1 \) and by using (2.15) obtain

\[
W_i'(u) = \sum_{j=1}^{p} A_{ij}(u) W_j(u), \quad i \in \{1, \ldots, p\},
\]

where \( W_{p-1} = N \) and \( W_p = \xi \). Suppose now \( C \) is given by the equations \( x^i = x^i(u), \quad u \in [a, b] \),
and $V$ is a constant vector field on $C$ such that
\begin{equation}
(2.17) \quad g(W_i(a), V) = 0, \quad i \in \{1, \ldots p\}.
\end{equation}

Then by using (2.16) we obtain the system
\begin{equation}
(2.18) \quad \frac{d}{du}(g(W_i(u), V) = g(W'_i(u), V) = \sum_{j=1}^{p} A_{ij}(u)g(W_j(u), V),
\end{equation}
with initial conditions (2.17). By the uniqueness of solutions of (2.18) we infer $g(W'_i(u), V) = 0$ for all $u$. Hence all $p$–planes $\Delta(u)$ are parallel with $\Delta(a)$. The proof is complete by the following general result.

**Proposition 1.** (Spivak [7], p.39). Let $C : x^i = x^i(u), u \in [a, b]$ be an immersed curve of $R^{m+2}$ such that $\frac{dx^i}{du} \in \Delta(u)$ for all $u$, where $\Delta(u)$ are parallel $p$–dimensional planes of $R^{m+2}$. Then $C$ is a curve in some $p$–dimensional plane of $R^{m+2}$.

**Remark 2.** The $p$–dimensional plane $H$ wherein $C$ lies is a Minkowski space since both linear independent lightlike vector fields $\xi$ and $N$ belong to $H$. The second assertion of the theorem follows in a similar way as the first one.

From Theorems 3 and 4 we obtain the following surprising result.

**Theorem 5.** Let $C$ be a lightlike curve of a Minkowski space $R^{m+2}_1$ ($m > 4$) with $\{k_1, \ldots, k_6\}$ nowhere zero and $k_7$ everywhere zero on $C$. Then $C$ lies in a 5-dimensional plane of $R^{m+2}_1$.

§3. The fundamental existence and uniqueness theorem for lightlike curves

Let $M$ be a $(m + 2)$–dimensional Lorentz manifold. In the previous sections we have seen that the lightlike Frenet field of frames $\{\xi, N, W_1, \ldots, W_m\}$ constructed along a lightlike curve is quasi-orthonormal (cf. Vrânceanu–Roşca [9]), that is, $\{W_1, \ldots, W_m\}$ is an orthonormal basis and $\xi$ and $N$ are lightlike vector fields satisfying (1.8).

Consider $R^{m+2}_1$ with the Lorentz metric
\begin{equation}
(3.1) \quad g(x, y) = \sum_{\alpha=1}^{m+1} x^\alpha y^\alpha - x^{m+2} y^{m+2}.
\end{equation}
Then we define the quasi-orthonormal basis

\[
\begin{align*}
\overset{\cdot}{W}_1 &= (1, 0, \ldots, 0), \ldots, \overset{\cdot}{W}_m = (0, \ldots, 1, 0, 0), \\
\overset{\cdot}{W}_{m+1} &= (0, \ldots, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), \quad \overset{\cdot}{W}_{m+2} = (0, \ldots, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})
\end{align*}
\]

that is we have

\[
\begin{align*}
\overset{\cdot}{W}_1 & \cdot \overset{\cdot}{W}_1 = \delta_{1,1}; \quad g(\overset{\cdot}{W}_m, \overset{\cdot}{W}_m) = g(\overset{\cdot}{W}_{m+2}, \overset{\cdot}{W}_{m+2}) = 0 \\
g(\overset{\cdot}{W}_m, \overset{\cdot}{W}_{m+1}) = 1, \quad \alpha, \beta \in \{1, \ldots, m\}.
\end{align*}
\]

It is easy to see that

\[
\sum_{\alpha=1}^{m} \overset{\cdot}{W}_\alpha^i \overset{\cdot}{W}_\alpha^j + \overset{\cdot}{W}_{m+1}^i \overset{\cdot}{W}_{m+2}^j + \overset{\cdot}{W}_{m+1}^j \overset{\cdot}{W}_{m+2}^i = g^{ij},
\]

where we put

\[
g^{ij} = \begin{cases} 
1, & i = j \neq m + 2 \\
-1, & i = j = m + 2 \\
0, & i \neq j.
\end{cases}
\]

**Theorem 6.** Let \( M \) be a Lorentz manifold, let \( k_1, \ldots, k_{2m} : [-\varepsilon, \varepsilon] \to \mathbb{R} \) be everywhere continuous functions and let \( \{\overset{\cdot}{W}_1, \ldots, \overset{\cdot}{W}_{m+2}\} \) from (3.2) as a basis of \( T_{x_0}M \). Then there exists a unique pseudo-arc parametrized lightlike curve \( C : x^i = x^i(u), \ u \in [-\varepsilon, \varepsilon], \) such that \( x^i(0) = x_0^i \), whose curvature functions are \( k_1, \ldots, k_{2m} \) and whose lightlike Frenet field of frames \( \{\xi, N, W_1, \ldots, W_m\} \) satisfies

\[
\xi(0) = \overset{\cdot}{W}_{m+1}, \quad N(0) = \overset{\cdot}{W}_{m+2}, \quad W_\alpha(0) = \overset{\cdot}{W}_\alpha, \quad \alpha \in \{1, \ldots, m\}.
\]

**Proof.** First we note that without loss of generality we may suppose \( M \) is the Minkowski space \( \mathbb{R}^{m+2} \). Then consider the system of differential equations

\[
\begin{align*}
W'_{m+1}(u) &= k_1 W_1 \\
W'_{m+2}(u) &= k_2 W_1 + k_3 W_2 \\
W'_i(u) &= -k_2 W_{m+1} - k_1 W_{m+2} + k_4 W_2 + k_5 W_3 \\
& \quad \vdots \\
W'_{m-2}(u) &= -k_{2m-5} W_{m-4} - k_{2m-4} W_{m-3} + k_{2m-2} W_{m-1} + k_{2m-1} W_m
\end{align*}
\]
\[
W'_{m-1}(u) = -k_{2m-3}W_{m-3} - k_{2m-2}W_{m-2} + k_{2m}W_m
\]
\[
W'_m(u) = -k_{2m-1}W_{m-2} - k_{2m}W_{m-1},
\]
and based on a well known result on the existence and uniqueness of its solutions, there exists a unique solution \((W_1, \ldots, W_{m+2})\) satisfying initial conditions \(W_\alpha(0) = \hat{W}_\alpha, \alpha \in \{1, \ldots, m+2\}\). Now we claim that \(\{W_1(u), \ldots, W_{m+2}(u)\}\) is a quasi-orthonormal basis for any \(u \in [-\varepsilon, \varepsilon]\).

To this end, by direct calculations, using (3.4) we obtain

\[
(3.6) \quad \frac{d}{du} \left( \sum_{\alpha=1}^{m} W^i_\alpha W^j_\alpha + W^i_{m+1}W^j_{m+2} + W^j_{m+1}W^i_{m+2} \right) = 0.
\]

As for \(u = 0\) we have (3.4), from (3.6) it follows

\[
(3.7) \quad \sum_{\alpha=1}^{m} W^i_\alpha(u)W^j_\alpha(u) + W^i_{m+1}(u)W^j_{m+2}(u) + W^j_{m+1}(u)W^i_{m+2}(u) = g^{ij}
\]

Further we construct the field of frames

\[
(3.8) \quad \begin{cases} V_{m+1} = \frac{1}{\sqrt{2}} (W_{m+1} + W_{m+2}); & V_{m+2} = \frac{1}{\sqrt{2}} (W_{m+1} - W_{m+2}) \\ V_\alpha = W_\alpha, & \alpha \in \{1, \ldots, m\}. \end{cases}
\]

Then (3.7) becomes

\[
(3.9) \quad \sum_{\alpha=1}^{m+1} V^i_\alpha(u)V^j_\alpha(u) - V^i_{m+2}(u)V^j_{m+2}(u) = g^{ij}.
\]

Following Bonnor [1], we define the matrix \([b^{ij}]\) as follows

\[
(3.10) \quad \begin{cases} b^{\alpha\beta} = V^\alpha_\beta, & \alpha, \beta \in \{1, \ldots, m+1\}; \quad b^{m+1,m+2} = -\sqrt{-1}V^m_{m+2} \\ b^{(m+2)\alpha} = \sqrt{-1}V^{m+2}_\alpha; \quad b^{(m+2)(m+2)} = V^{m+2}_{m+2}. \end{cases}
\]

It is easy to check that \([b^{ij}]\) is an orthogonal matrix. This implies \(\{V_1, \ldots, V_{m+2}\}\) is an orthonormal basis with respect to the metric (3.1) of \(R^{m+2}_1\).

Hence \(\{W_1, \ldots, W_{m+2}\}\) is a quasi-orthonormal basis for any \(u \in [-\varepsilon, \varepsilon]\). The lightlike curve \(C\) is obtained by integrating the system

\[
(3.11) \quad \frac{dx^i}{du} = W^i_{m+1}(u).
\]

It follows that \(C\) is pseudo-arc parametrized with curvature functions \(\{k_1, \ldots, k_{2m}\}\) with respect to the quasi-orthonormal field of frames \(\{\xi = W_{m+1}, N = W_{m+2}, W_1, \ldots, W_m\}\). The proof is complete.


References


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