Regularity theorem for a functional equation involving means

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Dedicated to Professor Zoltán Daróczy on the occasion of his 70th birthday

Abstract. We prove a result improving regularity of solutions of equation

$$\kappa x + (1 - \kappa)y = \lambda \varphi^{-1}(\mu \varphi(x) + (1 - \mu)\varphi(y)) + (1 - \lambda)\psi^{-1}(\nu \psi(x) + (1 - \nu)\psi(y)),$$

and leading to generalizations of some theorems established by D. Głazowska, W. Jarczyk, and J. Matkowski and by Z. Daróczy and Zs. Páles.

Given an interval $I \subset \mathbb{R}$, a continuous strictly monotonic function $\varphi : I \rightarrow \mathbb{R}$ and a real $\mu \in (0, 1)$ we denote by $A^\varphi_\mu(x, y)$ the quasi-arithmetic mean generated by $\varphi$ and weighted by $\mu$:

$$A^\varphi_\mu(x, y) = \varphi^{-1}(\mu \varphi(x) + (1 - \mu)\varphi(y)).$$

In paper [5] D. Głazowska, W. Jarczyk, and J. Matkowski found all the quasi-arithmetic means $A^\varphi_{1/2}$ and $A^\psi_{1/2}$ such that the classical arithmetic mean $A$ is an affine combination of them:

$$A = \lambda A^\varphi_{1/2} + (1 - \lambda)A^\psi_{1/2},$$

assuming that the generators $\varphi, \psi$ are twice continuously differentiable. In other words, they determined all functions $\varphi, \psi : I \rightarrow \mathbb{R}$ of class $C^2$ satisfying the

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The functional equation

\[
\frac{x + y}{2} = \lambda \varphi^{-1} \left( \frac{\varphi(x) + \varphi(y)}{2} \right) + (1 - \lambda) \psi^{-1} \left( \frac{\psi(x) + \psi(y)}{2} \right). \tag{1}
\]

The result of [5] was generalized by Z. Daróczy and Zs. Páles [3, Theorem 6], where the equation

\[
\mu x + (1 - \mu) y = \lambda \varphi^{-1} (\mu \varphi(x) + (1 - \mu) \varphi(y)) + (1 - \lambda) \psi^{-1} (\mu \psi(x) + (1 - \mu) \psi(y)) \tag{2}
\]

with given \( \lambda \in \mathbb{R} \setminus \{0, 1\} \) and \( \mu \in (0, 1) \) was solved in the class \( \mathcal{C}^1 \).

In the present paper we prove the theorem below which allows to generalize the results of both papers [5] and [3]. It shows that continuous functions satisfying the equation

\[
\kappa x + (1 - \kappa) y = \lambda \varphi^{-1} (\mu \varphi(x) + (1 - \mu) \varphi(y)) + (1 - \lambda) \psi^{-1} (\nu \psi(x) + (1 - \nu) \psi(y)), \tag{3}
\]

extending both of (1) and (2), are locally of much higher regularity. The Theorem provides a positive answer to a question posed recently by Z. Daróczy [1].

Results improving regularity of solutions of functional equations have a vast literature (cf. book [6] by A. Járai and the bibliography therein). Some of them will be used below.

The main result of this paper is the following regularity theorem concerning functional equation (3).

**Theorem.** Let \( I \subset \mathbb{R} \) be a non-trivial interval, \( \kappa, \lambda \in \mathbb{R} \setminus \{0, 1\} \) and let \( \mu, \nu \in (0, 1) \). If \( \varphi, \psi : I \to \mathbb{R} \) are continuous strictly monotonic functions and the pair \( (\varphi, \psi) \) satisfies equation (3), then there exists a non-trivial interval \( I_0 \subset I \) such that \( \varphi|_{I_0}, \psi|_{I_0} \) are infinitely many times differentiable and \( \varphi'(x) \neq 0, \psi'(x) \neq 0 \) for every \( x \in I_0 \).

In the proof we shall apply a modification of the method presented in [7]. In particular, we need the following result obtained by Zs. Páles (see [9, Corollary 6 and Example 2]), as well as Lemma 2 which was proved in [7]. The latter is also a consequence of L. Székelyhidi’s results [10] (see also [2], [8]).

**Lemma 1.** Let \( J \subset \mathbb{R} \) be an open interval, \( c \in (0, \infty) \), \( \mu \in (0, 1) \), and let \( f : J \to \mathbb{R} \) be a strictly increasing function such that

\[
J \ni s \mapsto f(s) - cf (\mu s + (1 - \mu) t)
\]
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is strictly monotonic for every \( t \in J \). Then for every \( s_0 \in J \) there exist numbers \( \delta \in (0, \infty) \) and \( K, L \in (0, \infty) \) such that \( (s_0 - \delta, s_0 + \delta) \subset J \) and

\[
K \leq \frac{f(s) - f(t)}{s - t} \leq L
\]

for every \( s, t \in (s_0 - \delta, s_0 + \delta) \), \( s \neq t \).

**Lemma 2.** Let \( J \subset \mathbb{R} \) be an interval and let \( \mu \in (0,1), \vartheta \in \mathbb{R} \). If \( f : J \rightarrow \mathbb{R} \) satisfies

\[
f(\mu s + (1 - \mu)t) = \vartheta f(s) + (1 - \vartheta)f(t)
\]

for all \( s, t \in J \), then there exist an additive function \( a : \mathbb{R} \rightarrow \mathbb{R} \) and a real \( b \) such that

\[
f(s) = a(s) + b, \quad s \in J.
\]

At first we prove the following fact.

**Lemma 3.** Let \( J \subset \mathbb{R} \) be an open interval, \( \kappa, \lambda \in \mathbb{R} \setminus \{0,1\}, \mu, \nu \in (0,1) \), and let \( f, g : J \rightarrow (0, \infty) \) satisfy the equation

\[
f(\mu s + (1 - \mu)t)[\kappa(1 - \nu)g(t) - (1 - \kappa)\nu g(s)] = \lambda(1 - \nu)f(s)g(t) - \lambda(1 - \mu)\nu f(t)g(s).
\]  

(5)

If \( f \) is Lebesgue measurable and \( g \) is of the first Baire class, then \( f \) and \( g \) are infinitely many times differentiable on a non-trivial subinterval of \( J \).

**Proof.** Putting \( s = t \) in (5) it is easy to observe that

\[
\kappa = \lambda \mu + (1 - \lambda)\nu.
\]

(6)

At first assume that \( f \) is constant on a non-trivial subinterval of \( J \). Then, by equation (5), we have

\[
[(1 - \kappa) - \lambda(1 - \mu)]\nu g(s) = [\kappa - \lambda \mu](1 - \nu)g(t)
\]

for \( s, t \) from the same subinterval. Hence, by (6), also \( g \) is constant there.

Now assume that \( g \) is constant on a non-trivial interval \( J_0 \subset J \). Then, by (5), we have

\[
\lambda \mu(1 - \nu)f(s) - \lambda(1 - \mu)\nu f(t) = [\kappa(1 - \nu) - (1 - \kappa)\nu]f(\mu s + (1 - \mu)t)
\]

for all \( s, t \in J_0 \). Using (6) we can rewrite the above condition as

\[
\mu(1 - \nu)f(s) - (1 - \mu)\nu f(t) = (\mu - \nu)f(\mu s + (1 - \mu)t), \quad s, t \in J_0.
\]

(7)
If \( \mu = \nu \) then, by (7), \( f \) is constant on \( J_0 \). Now we assume that \( \mu \neq \nu \). Then (7) is equivalent to the condition

\[
f(\mu s + (1 - \mu)t) = \frac{\mu(1 - \nu)}{\mu - \nu} f(s) - \frac{(1 - \mu)\nu}{\mu - \nu} f(t), \quad s, t \in J_0.
\]

Let \( \vartheta := \frac{\mu(1 - \nu)}{\mu - \nu} \). Then

\[
f(\mu s + (1 - \mu)t) = \vartheta f(s) + (1 - \vartheta) f(t), \quad s, t \in J_0.
\]

Applying Lemma 2 we obtain that there exist additive function \( a : \mathbb{R} \to \mathbb{R} \) and number \( b \in \mathbb{R} \) such that

\[
f(s) = a(s) + b, \quad s \in J_0.
\]

Thus, as \( f \) is Lebesgue measurable, it is continuous.

From that place we assume that neither \( f \), nor \( g \) is constant on a non-trivial subinterval of \( J \). Let

\[
C(g) := \{ v \in J : g \text{ is continuous at } v \}.
\]

As \( g \) is of the first Baire class, \( C(g) \) is a dense \( G_\delta \) subset of \( J \). We show that there exist \( s_0, t_0 \in C(g) \), \( s_0 \neq t_0 \), such that

\[
(1 - \kappa)\nu g(s_0) \neq \kappa(1 - \nu)g(t_0). \quad (8)
\]

Suppose on the contrary that

\[
(1 - \kappa)\nu g(s) = \kappa(1 - \nu)g(t)
\]

for all different \( s, t \in C(g) \). Then \( g \) is constant on \( C(g) \), i.e. there exists a positive \( k \) such that

\[
g(t) = k, \quad t \in C(g). \quad (9)
\]

Therefore \( (1 - \kappa)\nu = \kappa(1 - \nu) \), whence \( \kappa = \nu \) and, by (6), \( \mu = \nu \). Now equation (5) can be rewritten in the form

\[
f(\mu s + (1 - \mu)t)[g(t) - g(s)] = \lambda[f(s)g(t) - f(t)g(s)]. \quad (10)
\]

Thus, by (9),

\[
\lambda k (f(s) - f(t)) = 0, \quad s, t \in C(g),
\]
whence \( f \) is constant on \( C(g) \), i.e. there exists a positive \( l \) such that \( f(t) = l \) for every \( t \in C(g) \).

If there existed an \( s_0 \in J \) such that \( \mu s_0 + (1 - \mu)t \in J \setminus C(g) \) for every \( t \in C(g) \), then \( C(g) \) would be homeomorphic with a subset of \( J \setminus C(g) \). This, however, is impossible, as \( C(g) \) is a dense \( G_\delta \) subset of \( J \) and, consequently, \( J \setminus C(g) \) is of the first Baire category. Therefore, for every \( s \in J \) there exists a \( t \in C(g) \) such that \( \mu s + (1 - \mu)t \in C(g) \). Now, if \( s \in J \) and \( t \in C(g) \) are such that \( \mu s + (1 - \mu)t \in C(g) \), then, by (10), we have

\[
l[k - g(s)] = \lambda[kf(s) - lg(s)].
\]

Hence

\[
f(s) = \frac{kl - l(1 - \lambda)g(s)}{k\lambda}, \quad s \in J.
\]

Using again (10) we obtain

\[
\frac{kl - l(1 - \lambda)g[\mu s + (1 - \mu)t]}{k\lambda}[g(t) - g(s)] = \lambda \left( \frac{kl - l(1 - \lambda)g(s)}{k\lambda}g(t) - \frac{kl - l(1 - \lambda)g(t)}{k\lambda}g(s) \right), \quad s, t \in J,
\]

which, after some calculations, yields

\[
[g(t) - g(s)][k - g[\mu s + (1 - \mu)t]] = 0, \quad s, t \in J. \tag{11}
\]

Since \( g \) is not constant on \( J \), there exists a \( v_0 \in J \) such that \( m := g(v_0) \neq k \).

Take arbitrary \( v \in J \) and \( \varepsilon > 0 \) with \((v - \varepsilon, v + \varepsilon) \subset J \). As \( g \) is not constant on intervals, there exists an \( s \in (v - \varepsilon, v + \varepsilon) \) such that

\[
g[\mu s + (1 - \mu)v_0] \neq k.
\]

By (11) we have \( g(s) = g(v_0) = m \). Therefore, in every neighbourhood of \( v \) there exists an \( s \) with \( g(s) = m \) and, since \( C(g) \) is dense in \( J \), a point \( u \) such that \( g(u) = k \neq m \). Thus \( g \) is not continuous at \( v \) and, consequently, \( C(g) = \emptyset \), which is impossible. This proves the existence of different \( s_0, t_0 \in C(g) \) satisfying (8).

According to (8) there exist open intervals \( U, V \) containing \( s_0, t_0 \), respectively, and such that for every \( s \in U \) and \( t \in V \) we have \( (1 - \kappa)\nu g(s) \neq \kappa(1 - \nu)g(t) \).

Making use of (5) we obtain

\[
f[\mu s + (1 - \mu)t] = \frac{\lambda \mu(1 - \nu)f(s)g(t) - \lambda(1 - \mu)\nu f(t)g(s)}{\kappa(1 - \nu)g(t) - (1 - \kappa)\nu g(s)}, \quad s \in U, \quad t \in V.
\]
Now we are going to apply [6, Th. 8.6] by A. Járai. To this aim put $n = 4,$ $T := J,$ $Z = Z_1 = \cdots = Z_4 = Y := \mathbb{R},$ $X_1 = X_3 = A_1 = A_3 := U$ and $X_2 = X_4 = A_2 = A_4 := V.$ Fix an $\eta > 0$ with $(t_0 - \eta, t_0 + \eta) \subset V$ and define

$$D := \left\{(v, y) \in J \times U : |v - (\mu s_0 + (1 - \mu)t_0)| < \frac{\eta}{2}(1 - \mu)\right\}$$

and $|y - s_0| < \frac{\eta}{2}\left(\frac{1}{\mu} - 1\right).$

Put also $W := \{(v, y, z_1, z_2, z_3, z_4) \in D \times \mathbb{R}^4 : \kappa(1 - \nu)z_4 \neq (1 - \kappa)\nu z_3\}.$

Put also $f := f, f_1 := f|U, f_2 := f|V, f_3 := g|U, f_4 := g|V$ and define $g_1, g_3 : D \to U, g_2, g_4 : D \to V$ by

$$g_1(v, y) = g_3(v, y) = y, \quad g_2(v, y) = g_4(v, y) = \frac{v - \mu y}{1 - \mu},$$

and $h : W \to \mathbb{R}$ by

$$h(v, y, z_1, z_2, z_3, z_4) = \frac{\lambda \mu(1 - \nu)z_1 z_4 - \lambda \nu(1 - \mu)z_2 z_3}{\kappa(1 - \nu)z_4 - \nu(1 - \kappa)z_3}.$$ 

Put $K := [s_0 - \delta, s_0 + \delta],$ where $0 < \delta < \eta\left(\frac{1}{\mu} - 1\right)$ and $[s_0 - \delta, s_0 + \delta] \subset U.$

Making use of [6, Theorem 8.6], applied to the Lebesgue measure, we infer that $f$ is continuous on the interval $J_f := \left\{v \in J : |v - (\mu s_0 + (1 - \mu)t_0)| < \frac{\eta}{2}(1 - \mu)\right\}.$

Fix an $s^* \in J_f.$ Since $f$ is not constant on intervals, there is a $t^* \in J_f$ such that $f(\mu s^* + (1 - \mu)t^*) \neq \frac{\lambda \mu}{\kappa} f(s^*).$ By the continuity of $f$ at $t^*$ we have $f(\mu s^* + (1 - \mu)t) \neq \frac{\lambda \mu}{\kappa} f(s^*)$ for $t$'s from a non-trivial interval $J_g \subset J_f.$ Then, by (5),

$$g(t) = \frac{\nu}{1 - \nu} \frac{(1 - \kappa)f(\mu s^* + (1 - \mu)t) - \lambda(1 - \mu)f(t)}{\kappa f(\mu s^* + (1 - \mu)t) - \lambda \mu f(s^*)} g(s^*), \quad t \in J_g,$$

and, consequently, $g$ is continuous on $J_g.$

Now we show that $f$ is almost everywhere (with respect to the Lebesgue measure) differentiable on some non-trivial subinterval of $J_g$ provided $\mu \neq \nu.$ In that case equation (5) can be rewritten in the form

$$\nu g(s)[(1 - \kappa)f(\mu s + (1 - \mu)t) - \lambda(1 - \mu)f(t)]$$

$$= (1 - \nu)g(t)[\kappa f(\mu s + (1 - \mu)t) - \lambda \mu f(s)].$$
Interchanging $s$ by $t$ here we obtain

$$
\nu g(t)[(1 - \kappa)f(\mu t + (1 - \mu)s) - \lambda(1 - \mu)f(s)]
= (1 - \nu)g(s)[\kappa f(\mu t + (1 - \mu)s) - \lambda \mu f(t)]
$$

for every $s, t \in J$. Multiplying these equalities by sides we have

$$(1 - \nu)^2 g(s)g(t)[\kappa f(\mu s + (1 - \mu)t) - \lambda \mu f(s)]\cdot [(1 - \kappa)f(\mu t + (1 - \mu)s) - \lambda(1 - \mu)f(s)]$$

whence, dividing it by positive $g(s)g(t)$, we get

$$(1 - \nu)^2 [\kappa f(\mu s + (1 - \mu)t) - \lambda \mu f(s)]\cdot [(1 - \kappa)f(\mu t + (1 - \mu)s) - \lambda(1 - \mu)f(s)]$$

for every $s, t \in J$. Put

$$k(s, t) := \lambda(1 - \mu)\nu^2[(1 - \kappa)f(\mu s + (1 - \mu)t) - \lambda(1 - \mu)f(t)]$$

$$- \lambda \mu(1 - \nu)^2[\kappa f(\mu t + (1 - \mu)s) - \lambda \mu f(t)]$$

for every $s, t \in J$. Fix an $s_0 \in J_0$. Then

$$k(s_0, s_0) = \lambda(1 - \mu)\nu^2[(1 - \kappa)f(s_0) - \lambda(1 - \mu)f(s_0)]$$

$$- \lambda \mu(1 - \nu)^2[\kappa f(s_0) - \lambda \mu f(s_0)],$$

which, after using (6) and making some calculations, gives

$$k(s_0, s_0) = \lambda(1 - \lambda)\nu(1 - \nu)(\nu - \mu)f(s_0).$$

Since $f(s_0) > 0$, $\mu \neq 1$, $\nu \neq 1$, and $\mu \neq \nu$, we have $k(s_0, s_0) \neq 0$. Thus there exists an $\varepsilon > 0$ such that $(s_0 - \varepsilon, s_0 + \varepsilon) \subset J_0$ and $k(s, t) \neq 0$ for all $s, t \in (s_0 - \varepsilon, s_0 + \varepsilon)$. Let $J_0 := (s_0 - \varepsilon, s_0 + \varepsilon)$. By (12) we get

$$f(s) = \frac{(1 - \kappa)\nu^2 f(\mu t + (1 - \mu)s)[(1 - \kappa)f(\mu s + (1 - \mu)t) - \lambda(1 - \mu)f(t)]}{k(s, t)}$$

$$- \frac{\kappa(1 - \nu)^2 f(\mu s + (1 - \mu)t)[\kappa f(\mu t + (1 - \mu)s) - \lambda \mu f(t)]}{k(s, t)}.$$
for every $s, t \in J_0$.

Put $s = k = 1, n = 3, Z := \mathbb{R}, T := J_0, Y := \mathbb{R}, D := J_0^2, C := [s_0 - \delta \varepsilon, s_0 + \delta \varepsilon]$ with $\delta := \max \{\mu, 1 - \mu\}, W := D \times G$, where

$$G := \{(w_1, w_2, w_3) \in \mathbb{R}^3 : (1 - \mu)\nu^2[(1 - \kappa)w_2 - \lambda(1 - \mu)w_1]$$

$$\neq \mu(1 - \nu)^2[\kappa w_3 - \lambda \mu w_1]\}.$$  

Define $f := f|_{J_0}, g_1, g_2, g_3 : D \to \mathbb{R}$, by

$$g_1(s, t) = t, \quad g_2(s, t) = \mu s + (1 - \mu)t, \quad g_3(s, t) = \mu t + (1 - \mu)s,$$

and $h : W \to \mathbb{R}$ by

$$h(s, t, w_1, w_2, w_3) := \frac{(1 - \kappa)\nu^2 w_3[(1 - \kappa)w_2 - \lambda(1 - \mu)w_1] - \kappa(1 - \nu)^2 w_2 [\kappa w_3 - \lambda \mu w_1]}{\lambda(1 - \mu)\nu^2[(1 - \kappa)w_2 - \lambda(1 - \mu)w_1] - \lambda \mu(1 - \nu)^2[\kappa w_3 - \lambda \mu w_1]}.$$  

(14)

Then, according to [6, Th. 11.6] by A. Járai, $f$ is locally Lipschitzian on $J_0$, and thus, on account of [4, Th. 3.1.9] it is almost everywhere differentiable on $J_0$.

Now take any positive integer $p$. We prove that $f$ and $g$ are $p$ times continuously differentiable on a non-trivial subinterval of $J_0$. At first assume that $\mu \neq \nu$.

Then, as $k(s_0, s_0) \neq 0$, we have $(f(s_0), f(s_0), f(s_0)) \in G$. Since $G$ is open, there is an open interval $P$ such that $f(s_0) \in P$ and $P^3 \subset G$. Using the continuity of $f$ we find such an open interval $J_1$ that $s_0 \in J_1 \subset J_0$ and $f(J_1) \subset P$. Now let $s = k = 1, n = 3, Z := \mathbb{R}, Z_1 = Z_2 = Z_3 := P, T = T = X_1 = X_2 = X_3 := J_1$, $D := J_1^2$, and take $r_1 = r_2 = r_3 = 1$. Define $f = f_1 = f_2 = f_3 := f|_{J_1}, g_1, g_2, g_3 : D \to \mathbb{R}$ by (12) and $h : D \times Z_1 \times Z_2 \times Z_3 \to \mathbb{R}$ by (14). According to [6, Th. 14.2] $f$ is continuously differentiable on $J_1$. Now, using [6, Th. 15.2] $p$-1 times, we get by induction that $f$ is $p$ times continuously differentiable on $J_1$. As $J_1$ does not depend on $p$, this means that $f$ is infinitely many times differentiable on $J_1$. It follows from (5) that

$$[\kappa(1 - \nu)f(\mu s_0 + (1 - \mu)t) - \lambda \mu(1 - \nu)f(s_0)]g(t)$$

$$= [(1 - \kappa)\nu f(\mu s_0 + (1 - \mu)t) - \lambda(1 - \mu)\nu f(t)]g(s_0), \quad t \in J_1.$$  

(15)

As $f$ is not constant on non-trivial intervals we can find a $t \in J_1$ such that

$$\kappa(1 - \nu)f(\mu s_0 + (1 - \mu)t) - \lambda \mu(1 - \nu)f(s_0) \neq 0.$$
By the continuity of $f$ this is true for $t$’s running through a subinterval of $J_1$. Consequently, we can calculate $g(t)$ by (15) on that subinterval. Clearly, $g$ is infinitely many times differentiable there.

If $\mu = v$ then, by (6), we have $\kappa = \mu$, and thus equation (5) takes the form

$$f(\mu s + (1 - \mu)t)[g(t) - g(s)] = \lambda[f(s)g(t) - f(t)g(s)].$$

Now it is enough to use [3, Th. 5 and 2]. □

The following fact seems to be of interest on its own.

**Lemma 4.** Let $I \subset \mathbb{R}$ be an open interval, $\mu \in (0, 1)$, and let $\varphi : I \to \mathbb{R}$ be a continuous strictly monotonic function. Assume that the mean $A_\mu^\varphi$ is differentiable with respect to one of the variables. Then $\varphi$ is differentiable on a non-trivial interval and $\varphi'$ does not vanish wherever it exists. If, in addition, the partial derivative of $A_\mu^\varphi$ is continuous in the other variable on a non-trivial interval, then $\varphi$ is continuously differentiable on a non-trivial interval.

**Proof.** Assume, for instance, that $A_\mu^\varphi$ is differentiable with respect to the first variable.

Since $\varphi^{-1}$ is strictly monotonic, it is differentiable almost everywhere with respect to the Lebesgue measure. Fix any point $u_0 \in \varphi(I)$ of the differentiability of $\varphi^{-1}$. We prove that $\varphi^{-1}$ is differentiable in the open interval $\mu u_0 + (1 - \mu)\varphi(I)$ and the derivative of $\varphi^{-1}$ does not vanish wherever it exists.

Take any point $v \in \varphi(I)$ and then any $u \in \varphi(I) \setminus \{u_0\}$ such that $\mu u + (1 - \mu)v \in \mu u_0 + (1 - \mu)\varphi(I)$. Then we have

$$\varphi^{-1}(\mu u + (1 - \mu)v) - \varphi^{-1}(\mu u_0 + (1 - \mu)v)$$

$$= \frac{A_\mu^\varphi(\varphi^{-1}(u), \varphi^{-1}(v)) - A_\mu^\varphi(\varphi^{-1}(u_0), \varphi^{-1}(v))}{\mu(u - u_0) \cdot \frac{\varphi^{-1}(u) - \varphi^{-1}(u_0)}{u - u_0}}.$$ 

Now letting $u$ tend to $u_0$ we see that $\varphi^{-1}$ is differentiable at $\mu u_0 + (1 - \mu)v$ and

$$(\varphi^{-1})'((\mu u_0 + (1 - \mu)v) = \frac{1}{\mu} \partial_1 A_\mu^\varphi(\varphi^{-1}(u_0), \varphi^{-1}(v)) \cdot (\varphi^{-1})'(u_0)$$

(16)

for all $v \in \varphi(I)$. If $(\varphi^{-1})'$ vanished anywhere, then, by (16), it would be zero on a non-trivial interval, which is impossible as $\varphi^{-1}$ is one-to-one. The desired properties of the function $\varphi$ follows directly from what we have just proved about $\varphi^{-1}$.

The additional assertion is a direct consequence of formula (16). □
Proof of the Theorem. Replacing $I$ with its interior we may assume that $I$ is open. Without loss of generality we may also confine ourselves to the case of strictly increasing $\varphi$ and $\psi$. Moreover, replacing, if necessary, $\kappa$ with $1 - \kappa$ (consequently, $\mu$ with $1 - \mu$ and $\nu$ with $1 - \nu$) and by interchanging $x$ and $y$, we may assume that $\kappa$ is positive. Of course, at least one of the numbers $\lambda$ and $1 - \lambda$ is positive. Assume, for instance, the first case. Let $J := \varphi(I)$. Clearly, $J$ is an open interval.

At first we show that $\varphi$ and $\varphi^{-1}$ are locally Lipschitzian and their derivatives do not vanish wherever they exist. Putting $s = \varphi(x)$ and $t = \varphi(y)$ in (3) we get

$$(1 - \lambda)\psi^{-1}(\mu \psi(\varphi^{-1}(s)) + (1 - \nu)\psi(\varphi^{-1}(t))) = \kappa \varphi^{-1}(s) + (1 - \kappa)\varphi^{-1}(t) - \lambda \varphi^{-1}(\mu s + (1 - \mu)t)$$

for every $s, t \in J$. Since the left-hand side is strictly monotonic as a function of $s$, so does the right-hand side. Hence

$$J \ni s \mapsto \varphi^{-1}(s) - \frac{\lambda}{\kappa} \varphi^{-1}(\mu s + (1 - \mu)t)$$

is strictly monotonic for every $t \in J$. For every $v_0 \in J$, by Lemma 1, we can find $\delta \in (0, \infty)$ and $K, L \in (0, \infty)$ such that $(v_0 - \delta, v_0 + \delta) \subset J$ and

$$K \leq \frac{\varphi^{-1}(u) - \varphi^{-1}(v)}{u - v} \leq L, \quad u, v \in (v_0 - \delta, v_0 + \delta), \; u \neq v.$$

Then also for every $x_0 \in I$ there exist $\delta > 0$ and $K, L > 0$ such that

$$\frac{1}{L} \leq \frac{\varphi(x) - \varphi(y)}{x - y} \leq \frac{1}{K}, \quad x, y \in (x_0 - \delta, x_0 + \delta), \; x \neq y.$$

In particular, it follows that if the function $\varphi$ is differentiable at a point $x_0 \in I$, then $\varphi'(x_0) \neq 0$ and if the function $\varphi^{-1}$ is differentiable at $v_0 \in \varphi(I)$, then $(\varphi^{-1})'(v_0) \neq 0$.

Now we show that $\varphi$ is differentiable on $I$. For every $v \in J$ put

$$U(v) = \frac{1}{1 - \mu}(J - v) \cap \frac{1}{\mu}(v - J);$$

observe that $U(v)$ is an open interval containing 0. Given any $v \in J$ and $u \in U(v)$ define also

$$V(u) = (J - (1 - \mu)u) \cap (J + \mu u);$$
Observe that \( \varphi^{-1} \) is of full Lebesgue measure in \( A \); consequently, so is their union \( \bigcup_{i=0}^{N} I_{i} \). By the monotonicity of \( \varphi \), for \( v \in J \) and \( u \in U(v) \), we get
\[
\lambda \varphi^{-1}(v) = \kappa \varphi^{-1}(v + (1 - \mu)u) + (1 - \kappa) \varphi^{-1}(v - \mu u) \\
- (1 - \lambda) \psi^{-1}(v) + (1 - \nu) \psi(\varphi^{-1}(v - \mu u))
\]
for every \( v \in J \) and \( u \in U(v) \).

Take any \( v_{0} \in J \) and define functions \( f_{1}, f_{2} : U(v_{0}) \to I \) by
\[
f_{1}(u) = \varphi^{-1}(v_{0} + (1 - \mu)u), \quad f_{2}(u) = \varphi^{-1}(v_{0} - \mu u).
\]
For \( i = 1, 2 \) put
\[
N_{i} = \{ u \in U(v_{0}) : f_{i} \text{ is not differentiable at } u \}.
\]
By the monotonicity of \( f_{1}, f_{2} \), the sets \( N_{1}, N_{2} \) are of Lebesgue measure 0 and, consequently, so is their union \( N \). Since \( \varphi \) and \( \varphi^{-1} \) are locally Lipschitzian, also the function \( A_{x}^{\varphi} \) has that property, and thus, by Rademacher’s theorem [4, Theorem 3.1.9], \( A_{x}^{\varphi} \) is almost everywhere differentiable on \( I^{2} \). In particular, the set
\[
C = \{(x, y) \in I^{2} : A_{x}^{\varphi}(\cdot, y) \text{ is differentiable at } x \text{ and } A_{x}^{\varphi}(x, \cdot) \text{ is differentiable at } y \}
\]
is of full Lebesgue measure in \( I^{2} \). As \( (f_{1}, f_{2})(U(v_{0})) \) is the product of two open intervals and the functions \( f_{1}, f_{2} \) are locally Lipschitzian, the set \( (f_{1}, f_{2})^{-1}(C) \) has a positive measure; otherwise \( C \cap (f_{1}, f_{2})(U(v_{0})) = (f_{1}, f_{2})[(f_{1}, f_{2})^{-1}(C)] \) would be of measure zero. Hence it follows that the set \( (f_{1}, f_{2})^{-1}(C) \setminus N \) is non-empty. Take any \( u_{0} \in (f_{1}, f_{2})^{-1}(C) \setminus N \). Then \( f_{1}, f_{2} \) are differentiable at \( u_{0} \) and the functions \( A_{x}^{\varphi}(\cdot, f_{2}(u_{0})) \) and \( A_{x}^{\varphi}(f_{1}(u_{0}), \cdot) \) are differentiable at \( f_{1}(u_{0}) \) and \( f_{2}(u_{0}) \), respectively.

Now define functions \( g_{1}, g_{2} : V(u_{0}) \to I \) by
\[
g_{1}(v) = \varphi^{-1}(v + (1 - \mu)u_{0}), \quad g_{2}(v) = \varphi^{-1}(v - \mu u_{0}).
\]
Observe that \( g_{1}(v_{0}) = f_{1}(u_{0}) \) and \( g_{2}(v_{0}) = f_{2}(u_{0}) \). Therefore the functions \( A_{x}^{\varphi}(\cdot, g_{2}(v_{0})) \) and \( A_{x}^{\varphi}(g_{1}(v_{0}), \cdot) \) are differentiable at the points \( g_{1}(v_{0}) \) and \( g_{2}(v_{0}) \), respectively, whence, according to (3), \( A_{x}^{\varphi}(\cdot, g_{2}(v_{0})) \) and \( A_{x}^{\varphi}(g_{1}(v_{0}), \cdot) \) are differentiable at \( g_{1}(v_{0}) \) and \( g_{2}(v_{0}) \), respectively. Moreover, as \( f_{1} \) is differentiable at \( u_{0} \), the function \( \varphi^{-1} \) is differentiable at \( v_{0} + (1 - \mu)u_{0} \), and thus \( g_{1} \) is differentiable at \( v_{0} \). Similarly, we infer that the function \( g_{2} \) has the same property. Consequently, the function \( V(u_{0}) \ni v \mapsto A_{x}^{\varphi}(g_{1}(v), g_{2}(v)) \) is differentiable at \( v_{0} \). Now (17) gives
\[
\lambda \varphi^{-1}(v) = \kappa g_{1}(v) + (1 - \kappa) g_{2}(v) - (1 - \lambda) A_{x}^{\varphi}(g_{1}(v), g_{2}(v)), \quad v \in V(u_{0}),
\]
and we get the differentiability of \( \varphi^{-1} \) at \( v_0 \). As \( v_0 \) is an arbitrary point of \( J \) and the derivative of \( \varphi^{-1} \) does not vanish, \( \varphi \) is differentiable on \( I \).

According to (3) and applying Lemma 4 to \( \psi \) and \( \nu \) instead of \( \varphi \) and \( \mu \), respectively, we find a non-empty open interval \( I_0 \subset I \) such that \( \psi \) is differentiable in \( I_0 \); clearly also \( \varphi \) is differentiable in \( I_0 \).

Define functions \( f, g : I_0 \to (0, \infty) \) by
\[
f(s) = \varphi'(\varphi^{-1}(s)), \quad g(s) = \psi'(\varphi^{-1}(s)).
\]
We show that the pair \( (f, g) \) satisfies equation (5). Indeed, differentiating both sides of equality (3) with respect to \( x \) we get
\[
\frac{\lambda \mu \varphi'(x)}{\varphi'(\varphi^{-1}(\mu \varphi(x) + (1-\mu)\varphi(y)))} + \frac{(1-\lambda)\nu \psi'(x)}{\psi'(\psi^{-1}(\nu \psi(x) + (1-\nu)\psi(y)))} = \kappa
\]
for all \( x, y \in I_0 \). On the other hand, differentiating equality (3) with respect to \( y \) we have
\[
\frac{\lambda(1-\mu)\varphi'(y)}{\varphi'(\varphi^{-1}(\mu \varphi(x) + (1-\mu)\varphi(y)))} + \frac{(1-\lambda)(1-\nu)\psi'(y)}{\psi'(\psi^{-1}(\nu \psi(x) + (1-\nu)\psi(y)))} = 1 - \kappa
\]
for all \( x, y \in I_0 \). Multiplying equality (18) by \((1-\nu)\psi'(y)\) and (19) by \(-\nu \psi'(x)\) and adding the obtained equalities by sides we have
\[
\frac{\lambda \mu (1-\nu) \varphi'(x) \psi'(y) - \lambda(1-\mu)\nu \varphi'(y) \psi'(x)}{\varphi'(\varphi^{-1}(\mu \varphi(x) + (1-\mu)\varphi(y)))} = \kappa(1-\nu)\psi'(y) - (1-\kappa)\nu \psi'(x)
\]
for all \( x, y \in I_0 \), whence, setting here \( x = \varphi^{-1}(s) \) and \( y = \varphi^{-1}(t) \), we see that equality (5) holds for every \( s, t \in \varphi(I_0) \). Since \( \varphi^{-1} \) is locally Lipschitzian and \( \varphi' \) is measurable \( \varphi' \circ \varphi^{-1} \) is Lebesgue measurable. Moreover, \( \psi' \) is of the first Baire class and \( \varphi^{-1} \) is continuous whence \( \psi' \circ \varphi^{-1} \) is of the first Baire class. Therefore, due to Lemma 3, we infer that \( f, g \) are infinitely many times differentiable on a non-empty subinterval of \( \varphi(I_0) \). This competes the proof. \( \Box \)

References

Regularity theorem for a functional equation involving means


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