Stability of a quadratic functional equation on semigroups

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Dedicated to Professor Zoltán Daróczy on his seventieth birthday

Abstract. The stability problem of the functional equation of the form

\[ f(x + 2y) + f(x) = 2f(x + y) + 2f(y), \]

is investigated. We prove that if the norm of the difference between left-hand side and right-hand side of the equation is majorized by a function \( \omega \) of two variables having some standard properties then there exists a unique solution \( F \) of our equation and the norm of differences between \( F \) and the given function \( f \) is controlled by a function depending on \( \omega \).

1. Introduction

In the last years the stability of quadratic functional equations are widely investigated. The most important quadratic functional equation has the following form

\[ f(x + y) + f(x - y) = 2f(x) + 2f(y). \]  

First stability result concerning this equation was obtained by P. W. Cholewa [2]. The theorem states that if \( g \) is a function transforming an abelian group \( G \) into a Banach space \( X \) and satisfying the inequality

\[ \|g(x + y) + g(x - y) - 2g(x) - 2g(y)\| \leq \delta, \quad x, y \in G, \]  

\( \delta \)
where $\delta$ is a given nonnegative constant, then there exists a unique quadratic function $f : G \to X$ such that 
\[
\|f(x) - g(x)\| \leq \frac{\delta}{2}, \quad x \in G.
\] (3)

In 2006 W. Fechner [3] considered a modified inequality (2) where on the right-hand-side of (2) $\delta = \delta(x, y)$ was a function satisfying some additionally conditions and he obtained estimation like (3) depending on this function $\delta$.

Some authors investigated the stability of functional equations which are equivalent to equation (1) ([1], [7]). It make a sense because there are known examples of some equivalent functional equations: one of them stable in the sense of Hyers and Ulam and another one not stable in this sense. A nontrivial pair of functional equations of this type is Jensen functional equation (which is stable on the interval $(0, 1)$ [4]) and Hosszú functional equation (which is nonstable on the interval $(0, 1)$ [8]). The case of Hosszú functional equation is interesting also therefore, that it is stable on the space of all real numbers [6].

The natural domains to the considerations of quadratic functional equation are groups. But if we put $x + y$ instead of $x$ in (1) we obtain 
\[
f(x + 2y) + f(x) = 2f(x + y) + 2f(y),
\] (4)
and this equation can be investigated for functions defining on semigroups. Moreover, it is easily seen that equations (1) and (4) are equivalent in the class of functions acting from a group to an another one. The equation (4) may be treated as a quadratic functional equation on a semigroup. In this paper we will prove the stability of functional equation (4).

2. Results

In our two theorems on the stability of functional equation (4) we will assume that $(S, +)$ is an abelian semigroup, $X$ is a Banach space and $g$ is a function acting from $S$ into $X$. Let us put $S^* = S \setminus \{0\}$ and let $\omega : S^* \times S^* \to \mathbb{R}$ be a function satisfying the following assumptions 
\[
\lim_{k \to \infty} \rho^{-2k}\omega(\rho^kx, \rho^ky) = 0, \quad x, y \in S^*;
\]
\[
\sum_{k=0}^{\infty} \rho^{-2k}\omega(\rho^ku, \rho^kv) \text{ is convergent for all}
\]
\[
(u, v) \in \{(x, x), (x, 2x), (2x, x), (3x, x)\}, \quad x \in S^*,
\] (5)
where $\rho \in \{2, \frac{1}{2}\}$. For the simplicity we define a function $\varphi$ by the following way

$$\varphi(x, y) = \frac{1}{2}[\omega(x, y) + \omega(x, 2y) + \omega(3x, y) + 2\omega(2x, y)], \quad x, y \in S^*$$

and we observe that

$$\lim_{k \to \infty} \rho^{-2k}\varphi(\rho^k x, \rho^k y) = 0, \quad x, y \in S^*;$$

$$\sum_{k=0}^{\infty} \rho^{-2k}\varphi(\rho^k x, \rho^k x) \text{ is convergent for all } x \in S^*. \quad (5')$$

From a lemma which was originally proved in [5] for functions defined on a group, but its proof is literally the same in our case, follows the following lemma.

**Lemma.** Let $g : S \to X$ be a function satisfying the inequality

$$\left\| \sum_{i=1}^{r} \alpha_i g(\gamma_i x + \delta_i y) \right\| \leq \varphi(x, y), \quad x, y \in S^*, \quad (6)$$

where $r, \gamma_i, \delta_i, i \in \{1, \ldots, r\}$ are given positive integers $\alpha_i, i \in \{1, \ldots, r\}$ are given real constants. If there exist constants $K > 0$ and $\rho > 0$ such that

$$\left\| \rho^{-2(n+1)}g(\rho^{n+1} x) - \rho^{-2n}g(\rho^n x) \right\| \leq K \rho^{-2n}\varphi(\rho^n x, \rho^n x), \quad x \in S^*, \quad n \in \mathbb{N} \cup \{0\}. \quad (7)$$

and $\varphi$ satisfies conditions $(5')$ then for every $x \in S^*$ the sequence $(\rho^{-2n}g(\rho^n x))_{n \in \mathbb{N}}$ is convergent to a function $f : S^* \to X$ fulfilling the equation

$$\sum_{i=1}^{r} \alpha_if(\gamma_i x + \delta_i y) = 0, \quad x, y \in S^* \quad (8)$$

and the estimation

$$\left\| g(x) - f(x) \right\| \leq K \sum_{n=0}^{\infty} \rho^{-2n}\varphi(\rho^n x, \rho^n x), \quad x \in S^*. \quad (9)$$

**Theorem 1.** Let $S$ be an abelian semigroup, $X$ be a Banach space, and let $g : S \to X$ be a function satisfying the following inequality

$$\|g(x + 2y) + g(x) - 2g(x + y) - 2g(y)\| \leq \omega(x, y), \quad x, y \in S^*, \quad (10)$$

where $\omega : S^* \times S^* \to [0, \infty)$ fulfilled conditions $(5)$ with $\rho = 2$. Then there exists a unique function $f : S \to X$ satisfying equation $(4)$ and the estimation

$$\|f(x) - g(x)\| \leq \sum_{k=0}^{\infty} \frac{1}{4^{k+1}}\varphi(2^k x, 2^k x), \quad x \in S^*. \quad (11)$$
Proof. According to (10) we obtain
\[ \|g(3x) - 2g(2x) - g(x)\| \leq \omega(x, x), \quad x \in S^*; \]
\[ \|g(5x) + g(3x) - 2g(4x) - 2g(x)\| \leq \omega(3x, x), \quad x \in S^*; \]
\[ \| - g(5x) - g(x) + 2g(3x) + 2g(2x)\| \leq \omega(x, 2x), \quad x \in S^*, \]
whence
\[ \| - 2g(4x) - 4g(3x)\| \leq \omega(x, x) + \omega(x, 2x) + \omega(3x, x), \quad x \in S^*. \quad (12) \]
Moreover, putting 2x instead of x, and x instead of y in (10) we get
\[ \|2g(4x) + 2g(2x) - 4g(3x) - 4g(x)\| \leq 2\omega(2x, x), \quad x \in S^*. \quad (13) \]
It follows from (12) and (13) that
\[ \|g(2x) - 4g(x)\| \leq \varphi(x, x), \quad x \in S^*, \quad (14) \]
and, consequently,
\[ \left\| \frac{1}{4^{n+1}}g(2^{n+1}x) - \frac{1}{4^n}g(2^n x) \right\| \leq \frac{1}{4^{n+1}} \varphi(2^n x, 2^n x), \quad x \in S^*. \]
On account of Lemma there exists a function \( f : S \to X \) satisfying equation (4) and estimation (11).
Let \( f_1 : S \to X \) be a function satisfying equation (4) and the estimation
\[ \|g(x) - f_1(x)\| \leq \sum_{k=0}^{\infty} \frac{1}{4^k} \varphi(2^k x, 2^k x), \quad x \in S^*. \quad (15) \]
Quite similar as in the proof of (14), using (4) instead of (10), one can prove that
\[ f(2x) = 4f(x) \] as well as \( f_1(2x) = 4f_1(x), \quad x \in S^*. \]
Now for arbitrary positive integer \( m \) we have
\[
\|f(x) - f_1(x)\| = \frac{1}{4^m} \|f(2^m x) - f_1(2^m x)\|
\leq \frac{1}{4^m} \left[ \|f(2^m x) - g(2^m x)\| + \|g(2^m x) - f_1(2^m x)\| \right]
\leq \frac{2}{4^m} \sum_{k=0}^{\infty} \frac{1}{4^k} \varphi(2^{m+k} x, 2^{m+k} x) = 2 \sum_{k=m}^{\infty} \frac{1}{4^k} \varphi(2^k x, 2^k x)
\]
and hence \( f(x) = f_1(x), \quad x \in S^*. \) \( \square \)
In a similar way one can prove the following theorem.

**Theorem 2.** Let $S$ be an abelian semigroup uniquely divisible by two and let $X$ be a Banach space. If $g : S \to X$ is a function satisfying the following inequality

$$
\|g(x + 2y) + g(x) - 2g(x + y) - 2g(y)\| \leq \omega(x, y), \quad x, y \in S^*,
$$

where $\omega : S^* \times S^* \to [0, \infty)$ fulfilled conditions (5) with $\rho = \frac{1}{2}$ then there exists a unique function $g : S \to X$ satisfying equation (4) and the estimation

$$
\|f(x) - g(x)\| \leq \sum_{k=0}^{\infty} 4^k \varphi(2^{-k-1}x, 2^{-k-1}x), \quad x \in S^*.
$$

**Proof.** Similarly as in the proof of Theorem 1 we obtain

$$
\|g(2x) - 4g(x)\| \leq \varphi(x, x), \quad x \in S^*.
$$

Putting $\frac{x}{4}$ instead of $x$ we get

$$
\|g(x) - 4g\left(\frac{x}{2}\right)\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}\right), \quad x \in S^*,
$$

whence

$$
\left\|4^{k+1}g\left(\frac{x}{2^{k+1}}\right) - 4^k g\left(\frac{x}{2^k}\right)\right\| \leq 4^k \varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\right), \quad x \in S^*.
$$

The rest of the proof runs “mutatis mutandis” as in the proof of Theorem 1. □

**Remark.** Let $\theta \geq 0$ and let $\omega(x, y) = \theta(\|x\|^\alpha + \|y\|^\alpha)$ or $\omega(x, y) = \theta\|x\|^\beta\|y\|^\beta$, Theorems 1 and 2 can be applied particularly for these functions with $\alpha \neq 2$ and $\beta \neq 1$.

**References**


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