Parameter-independent structure in periodic orbits of an iterated function system on the real line

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Abstract. For the iterated function system on \( \mathbb{R} \) comprising the maps \( f(x) = ax + 1 \) and \( g(x) = bx \), with \( a > 0 \) and \( 0 < b < 1 \), we represent each \( n \)-cycle by the composition (or word) in \( f \) and \( g \) corresponding to the cycle’s point of least magnitude (or perigee). These representations are partitioned into equivalence classes using simple combinatorial criteria. Associated with each \( n \)-cycle are \( n \) polynomials in \( a \) and \( b \) whose values at a special value of \( a \) are partially ordered. An example is given showing that, for fixed \( b \), the perigee word of an \( n \)-cycle is a function of \( a \); but the ordering of the polynomial values enables us to prove that the maximal perigee word in each equivalence class is independent of the parameters \( a \) and \( b \).

1. Introduction

Let \( \Psi(a, b) \) be the iterated function system on \( \mathbb{R} \) comprising the maps

\begin{equation}
    f(x) = ax + 1 \quad \text{and} \quad g(x) = bx,
\end{equation}

where \( a > 0 \) and \( 0 < b < 1 \). The dynamics of this system for \( a > 1 \) are considered in \([2]\). Here we present some combinatorial properties of \( \Psi \)'s cycle structure which are independent of the parameters \( a \) and \( b \); in particular, these properties hold whether or not \( f \) is a contraction.

2000 Mathematics Subject Classification: 37E15, 39B12.

Key words and phrases: cycle point word, cycle word code, \( f \)-rank, \( g \)-rank, gaps vector, iterated function system, perigee, periodic orbit, subdiagonal, superdiagonal.

Jake Stroh read an early draft of this paper, Walt Tape and Marion Avrilyn Jones commented on parts of later drafts, and Mitch Roth and Marty Getz provided Mathematica® help. I am grateful for their assistance and for the anonymous referee's useful comments.
Given the maps \( f \) and \( g \) defined in (1), and a positive integer \( n \), choose functions \( t_i, 1 \leq i \leq n \), from the set \( \{f, g\} \), and compose them by right-to-left concatenation. We call \( w = t_n t_{n-1} \cdots t_1 \) the word for the cycle point \( x_1 \) satisfying

\[ t_n t_{n-1} \cdots t_1(x_1) = x_1. \]

Since the word \( g^n \) yields just the trivial cycle point \( x_1 = 0 \) for all \( n \in \mathbb{N} \) and all \( b \in (0, 1) \), we exclude words of this form in what follows. Let \( \Sigma^n \) be the set of \( n \)-letter words on the symbols \( f \) and \( g \) in which \( f \) appears at least once. The cyclic permutations, or rotations, of a word \( w \in \Sigma^n \) yield the set of cycle points \( \{x_1, x_2, \ldots, x_n\} \), in which the (not necessarily unique) point of least magnitude or perigee of the cycle [1] is generated by its corresponding perigee word. We use the following combinatorial properties of cycle words.

**Definition 1.** Given \( w \in \Sigma^n \).

(a) The \( f \)-rank of \( w \), denoted by \( r \), is its number of \( f \)s. Note \( r \geq 1 \).

(b) The density, denoted by \( \alpha \), is the ratio \((n - r)/r\) of the number of \( g \)s to \( f \)s in \( w \).

(c) The base markers in \( w \) are the \( f \)s indexed from left to right and from 1 to \( r \) in \( w \).

(d) For \( 2 \leq i \leq r \), the \( i \)th gap \( d_i \) is the number of \( g \)s between base markers \( f_{i-1} \) and \( f_i \), while \( d_1 \) is the number of \( g \)s to the left of \( f_1 \). The ordered \( r \)-tuple \( D(w) = (d_1, d_2, \ldots, d_r) \) is the gaps vector of \( w \).

(e) The \( g \)-rank of base marker \( f_i \) is the number of \( g \)s to its left in \( w \), and is denoted by \( q_i \). Equivalently, it is the sum of gaps \( d_1 + d_2 + \cdots + d_i \). The ordered \( r \)-tuple \( Q(w) = (q_1, q_2, \ldots, q_r) \) is the cycle word code, or more briefly the code of \( w \).

For example, \( ggffgffggff \in \Sigma^{10} \), for which \( D(w) = (2, 1, 0, 2, 0) \) and \( Q(w) = (2, 3, 3, 5, 5) \). Given \( a \) and \( b \), a word’s length and code determine its cycle point.

**Proposition 1.** For \( n \in \mathbb{N} \) and \( 1 \leq r \leq n \), let \( w \in \Sigma^n \) have code \((q_1, q_2, \ldots, q_r)\) and density \( \alpha \). Then

\[ x_1 = \frac{1}{1 - (ab^n)^r} \sum_{i=1}^{r} t^q_i a^{i-1} \tag{2} \]

is the unique point satisfying \( w(x_1) = x_1 \).
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Proof. We show that

\[ w(x) = a^r b^{n-r} x + \sum_{i=1}^{r} b^i a^{i-1}, \]  

(3)

from which the result follows. For \( m \geq 0 \) and \( n > m \), define \( \Sigma^n_m \) to be the set of words \( w \) that can be written \( w = t_n \cdots t_{m+2} f g^m \), where \( t_i \in \{f, g\} \), so that

\[ \Sigma^n = \bigcup_{m<n} \Sigma^n_m. \]

For each \( m \geq 0 \) we prove by induction on \( n \) that, for all \( n > m \),

\[ w \in \Sigma^n_m \implies w(x) = a^r b^{n-r} x + \sum_{i=1}^{r} b^i a^{i-1}. \]  

(4)

For the initial step, fix \( m \geq 0 \), use the base value \( n = m + 1 \), and let \( w \in \Sigma^{m+1}_m \).

Then \( w = f g^m \), \( r = 1 \), \( q_1 = 0 \), and

\[ f g^m(x) = ab^m x + 1 = a^r b^{(m+1)-r} x + \sum_{i=1}^{r} b^i a^{i-1}, \]  

(5)

as required. For the inductive step, let \( n > m \), and assume (4). Let \( w = t_{n+1} \cdots t_{m+2} f g^m \in \Sigma^{n+1}_m \), where \( Q(w) = (q_1, \ldots, q_r) \). Write \( w = t_{n+1} w' \), where \( w' \) has length \( n \). If \( t_{n+1} = f \), then \( q_1 = 0 \), \( Q(w') = (q_2, \ldots, q_r) \), and

\[ w(x) = a \left( a^{r-1} b^{n-(r-1)} x + \sum_{i=2}^{r} b^i a^{i-1} \right) + 1. \]  

(6)

If \( t_{n+1} = g \), on the other hand, then \( Q(w') = (q_1 - 1, q_2 - 1, \ldots, q_r - 1) \), and

\[ w(x) = b \left( a^r b^{n-r} x + \sum_{i=1}^{r} b^i a^{i-1} \right). \]  

(7)

The induction hypothesis is confirmed, since both (6) and (7) reduce to

\[ w(x) = a^r b^{(n+1)-r} x + \sum_{i=1}^{r} b^i a^{i-1}. \]

Note that, if \( a = b^{-\alpha} \), we have division by 0 in equation (2), and the cycle point does not exist; we return to this important fact in Section 7.
2. Representing cycles by perigee words

In general, the correspondence between a cycle point word in $\Sigma^n$ and its code is not a bijection. For instance, the words $ggfgf, ggfgf, ggfgfgg,$ and so on all have gaps vector $(2, 1)$ and code $(2, 3)$. We now define a subset $F^n$ of $\Sigma^n$ which contains all the perigee words, and in which every word is uniquely represented by its gaps vector, or, equivalently, by its code.

**Definition 2.** For each $n \in \mathbb{N}$ and for $1 \leq r \leq n$,

$$F^n = \{ w \in \Sigma^n \mid w = t_n t_{n-1} \cdots t_2 t_1 \text{ with } t_1 = f \},$$

$$F^n_r = \{ w \in F^n \mid w \text{ has } f\text{-rank } r \},$$

$$P^n_r = \{ w \in F^n_r \mid w \text{ is a perigee word} \}.$$

$P^n_r$ contains $P^n$ because, if $t_n \cdots t_2 g(x_1) = x_1$, then $|x_2| = |g(x_1)| < |x_1|$; hence $x_1$ cannot be the perigee, and no word ending in $g$ can be a perigee word. The sets $P^n_r$ are equivalence classes imposed on the set of $n$-length perigee words by the relation “possesses $r$ letters $f$” for $1 \leq r \leq n$. Table 1 shows the $P^n_r$ for $r = 1, \ldots, 7$ in $\Psi(\frac{5}{3}, \frac{1}{2})$. By construction, the perigees in this table increase in absolute value within each $P^n_r$.

Note that, for every $w \in F^n_r$, the base marker $f_r$ is always rightmost and its $g$-rank is always $n-r$, so all words in $F^n_r$ have $q_r = n-r = r\alpha$. Furthermore, the allowable rotations of any $w \in F^n_r$ put each $f$ in the rightmost position exactly once; consequently, $w$ admits of $r$ such rotations and $r$ (not necessarily distinct) cycle points. Finally, we have the useful property that any allowable rotation of a word in $F^n_r$ yields a corresponding cyclic permutation of its gaps vector. That is, for $w \in F^n_r$ and $1 \leq i \leq r$, $w$'s $i$th gap $d_i$ is the number of $g$s between base markers $f_{i-1}$ and $f_i$, indices taken modulo $r$.

While there is an obvious bijection between words in $F^n_r$ and their codes, we do not claim a bijection between words and cycle points. Distinct cycles need not be disjoint; for instance, in $\Psi(2, \frac{1}{2})$, the cycle words $ggggfgfggf$ and $ggfggfggf$ both yield the perigee $\frac{4}{7}$. However, disjoint cycles are not required here.

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2When the residues modulo $r$ are used as indices, we take them to be $\{1, \ldots, r\}$ rather than the usual $\{0, \ldots, r-1\}$. 
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<table>
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<tr>
<th>set</th>
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<th>code</th>
<th>perigee</th>
<th>decimal</th>
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<td>$33/263$</td>
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Table 1. The distinct perigee words of length 7, with their codes and perigee values, for $f := f(x) = \frac{5}{4}x + 1$ and $g := g(x) = \frac{5}{4}$.

3. Minimal and maximal perigees

Our main results show that the minimal and maximal perigee words in each equivalence class may be characterized purely combinatorially.

**Theorem 1.** In $\Psi(a, b)$, the minimal perigee word with density $\alpha$ in $P_r^n$ is $w_{\text{min}} = g^\alpha f^r$, whose code is the $r$-tuple

$$Q(w_{\text{min}}) = (ra, ra, \ldots, ra).$$

**Proof.** Let $g^\alpha f^r(y) = y$. Given any $w \neq g^\alpha f^r \in F_r^n$, with $Q(w) = (q_1, \ldots, q_r)$, there exists an integer $j, 1 \leq j \leq r$, for which

$$q_i < ra, \quad 1 \leq i \leq j,$$

$$q_i \leq ra, \quad j + 1 \leq i \leq r.$$
By Proposition 1 with $0 < b < 1$, the cycle point $x_1$ for $w$ satisfies

$$x_1 = \frac{1}{1 - (ab^a)^r} \sum_{i=1}^{r} b^{i-1} > \frac{1}{1 - (ab^a)^r} \sum_{i=1}^{r} b^{i-1} = y,$$

and the theorem follows. \qed

**Theorem 2** (Maximal Perigee Property). In $\Psi(a, b)$, the maximal perigee word $w_{\text{max}} \in P_r^n$ with density $\alpha$ has code

$$Q(w_{\text{max}}) = (\lceil \alpha \rceil, \lceil 2\alpha \rceil, \ldots, \lceil r\alpha \rceil),$$

where $\lceil \cdot \rceil$ is the ceiling function.

For example, $n = 7$ and $r = 4$ yield $\alpha = \frac{3}{4}$, and the maximal perigee word code in $P_7^4$ is

$$\left( \lceil \frac{3}{4} \rceil, \lceil 2 \cdot \frac{3}{4} \rceil, \lceil 3 \cdot \frac{3}{4} \rceil, \lceil 4 \cdot \frac{3}{4} \rceil \right) = (1, 2, 3, 3),$$

as in Table 1. (The result holds trivially when $\alpha = 0$, that is, when $r = n$.)

An inquiry into general $n$-cycles follows, culminating in a proof of Theorem 2. In Section 4 we show how to use a word’s code to calculate the code of any rotation. A lemma of Chisala (Section 5) implies that, among a word’s rotated codes, there are at least two “extremal” codes: one which is superdiagonal and one subdiagonal. In Section 6 we introduce the deviation vector and maximum deviation of a word, and prove that the deviation vectors of a word’s rotations become cyclic permutations of each other under a particular vertical translation; a special case involving subdiagonal words is crucial later.

Section 7 gives the name code function to the polynomial part of equation (2), along with an example showing that, for a given value of the parameter $a$ and for fixed $b$, the perigee of a cycle corresponds to the minimal code function. Although no cycle points exist when $a = b^{-\alpha}$, the code functions for a word and its rotations are well-defined for this value of $a$, and in Section 8 we show that the code function values at $a = b^{-\alpha}$ are partially ordered, with the smallest and largest values corresponding to super- and subdiagonal codes, respectively. Section 9 establishes upper and lower bounds for super- and subdiagonal code function values at $a = b^{-\alpha}$, respectively, over all words in $F_r^n$, and we prove that the only superdiagonal word in $F_r^n$ whose maximum deviation is less than 1 is the word whose code is $(\lceil \alpha \rceil, \lceil 2\alpha \rceil, \ldots, \lceil r\alpha \rceil)$. Finally, Section 10 employs these results to prove the Maximal Perigee Property.

We are less interested here with the cycle point values $x_i$ than with their combinatorial representations, largely because the interesting aspects of the representations occur when the parameter values preclude the existence of the points.
4. Rotations of codes

We begin by showing how codes change when the words they belong to are rotated. We use the following notation, where $i$ and $j \in \{1, \ldots, r\}$. Given $w \in F^n_r$ with $D(w) = (d_1, \ldots, d_r)$, $w_i$ is the rotation of $w$ whose gaps vector begins with $d_i$. (Hence $w_1 = w$.) $D(w_1)$ is abbreviated $D_1$, and likewise $Q_i = Q(w_i)$. $x_i$ is the cycle point for $w_i$; that is, $w_i(x_i) = x_i$. Lastly, we write $y \sim z$ if $y$ is a cyclic permutation of $z$. (For a given $w_1$, we have $w_i \sim w_j$ and $D_i \sim D_j$, but $Q_i \sim Q_j$ only for $w = f^n$).

Proposition 2. If $w_1 \in F^n_r$ has code $Q_1 = (q_1, q_2, \ldots, q_r)$, then the rotation $w_i$, $1 \leq i \leq r$, has code $Q_i = (q'_1, q'_2, \ldots, q'_r)$, where

$$q'_j = \begin{cases} q_{i+j-1} - q_{i-1}, & j \in \{1, \ldots, r - i + 1\} \\ q_r + q_{i+j-r-1} - q_{i-1}, & j \in \{r - i + 2, \ldots, r\} \end{cases},$$

and where we define $q_0 = d_0 = 0$.

Proof sketch. The gaps vector for $w_i$ is

$$D_i = (d_i, d_{i+1}, \ldots, d_r, d_1, \ldots, d_{i-1}),$$

so $D_i = (d'_1, d'_2, \ldots, d'_r)$ has

$$d'_j = \begin{cases} d_{i+j-1}, & j \in \{1, \ldots, r - i + 1\} \\ d_{i+j-r-1}, & j \in \{r - i + 2, \ldots, r\} \end{cases}.$$

The formulas (8) then follow directly. The definitions $q_0 = d_0 = 0$ preserve identity when $i = 1$. \qed

The transformation in Proposition 2 may also be expressed as

$$q_{j-i+1}' = q_j - q_{i-1}, \quad 1 \leq j \leq r,$$

where arithmetic is performed modulo $n - r$, and the indices are calculated modulo $r$. The formulas (8) will be used in the final proof of the Maximal Perigee Property (Section 10), while (9) will be applied in Section 6.

5. Chisala’s Lemma, and sub- and superdiagonal codes

We now invoke a modified lemma of CHISALA [3] to define two kinds of words which are “extremal,” in the sense that all the partial averages of the $g$-ranks are either no less than or no greater than the word’s density.
Lemma 1. Given a sequence $D = (d_1, \ldots, d_r)$ of real numbers and a sequence $M = (m_1, \ldots, m_r)$ of weights, let $A = \sum_{i=1}^{r} d_i m_i / \sum_{i=1}^{r} m_i$ be the weighted average.

(a) (Chisala 1994) There exists a cyclic permutation $\sigma$ on the indices such that for every $k \in \{1, \ldots, r\}$, the partial weighted averages $\sum_{i=1}^{k} d_{\sigma(i)} m_{\sigma(i)} / \sum_{i=1}^{k} m_{\sigma(i)}$ are bounded above by $A$.

(b) There exists a cyclic permutation $\tau$ on the indices such that for every $k \in \{1, \ldots, r\}$, the partial weighted averages $\sum_{i=1}^{k} d_{\tau(i)} m_{\tau(i)} / \sum_{i=1}^{k} m_{\tau(i)}$ are bounded below by $A$.

The proof of part (b) follows from part (a) by considering $(-d_1, \ldots, -d_r)$.

We call a word $w \in \mathbb{F}_n^r$, its gaps vector, and its code subdiagonal if the $r$-tuple

$$\left(\frac{q_1}{1}, \frac{q_2}{2}, \ldots, \frac{q_r}{r}\right)$$

satisfies part (a) of Chisala’s Lemma (with $A = \alpha$); or, equivalently, if $q_i \leq i\alpha$ for $1 \leq i \leq r$. Similarly, $w$ is superdiagonal if (10) satisfies part (b) of Chisala’s Lemma, or, equivalently, if $q_i \geq i\alpha$. Note that the word with code $(\alpha, 2\alpha, \ldots, r\alpha)$, where necessarily $\alpha \in \{0, 1, 2, \ldots\}$, is both sub- and superdiagonal.

For example, if $w_1 \in \mathbb{F}_5^3$ has gaps vector $D_1 = (1, 4, 2, 0, 3)$ and density $\alpha = 2$, then $w_3$ is a subdiagonal rotation with $D_3 = (2, 0, 3, 1, 4)$ and $Q_3 = (2, 2, 5, 6, 10)$, while $w_2$ is superdiagonal with $D_2 = (4, 2, 0, 3, 1)$ and $Q_2 = (4, 6, 6, 9, 10)$.

6. Deviations

We measure and compare a word’s rotations using the maximum signed difference $q_i - i\alpha$.

Definition 3. Let $w_1 \in \mathbb{F}_r^n$ have density $\alpha$ and code $Q_1 = (q_1, \ldots, q_r)$. The deviation vector of $w_1$ is

$$\Delta_1 = (q_1 - \alpha, q_2 - 2\alpha, \ldots, q_r - r\alpha),$$

and $q_j - j\alpha$ is the $j$th deviation, $1 \leq j \leq r$. The quantity

$$h(w_1) = \max_{1 \leq j \leq r} (q_j - j\alpha).$$

is the maximum deviation for $w_1$. For any $w_i \sim w_1$ we write $h_i = h(w_i)$.

Let $(y)^j$ be the $j$-tuple $(y, y, \ldots, y)$. Given a word and one of its rotations,
we now show that special vertical translations of their deviation vectors produce two new vectors which are again cyclic permutations of each other.

**Proposition 3.** Given \( w_1 \in F_r^n \), and adding termwise,

\[
\Delta_1 + (h_i)^r \sim \Delta_i + (h_1)^r
\]

for any \( w_i \sim w_1 \).

**Proof.** Suppose \( h_1 = q_k - k\alpha \) for some \( k \in \{1, \ldots, r\} \). Using equation (9), we find that \( h_i = (q_k - q_{i-1}) - (k - i + 1)\alpha \). Writing \((y_m)_{m=1}^r \) for the \( r \)-tuple \((y_1, y_2, \ldots, y_r)\), we then have

\[
\Delta_1 + (h_i)^r = (q_m - m\alpha)^{\frac{r}{m=1}} + (q_k - q_{i-1} - (k - i + 1)\alpha)^{\frac{r}{m=1}} = (q_m + q_k - q_{i-1} - (m + k - i + 1)\alpha)^{\frac{r}{m=1}},
\]

and

\[
\Delta_i + (h_1)^r = (q_m - q_{i-1} - (m - i + 1)\alpha)^{\frac{(i+r-1) \mod r}{m=i \mod r}} + (q_k - k\alpha)^{\frac{(i+r-1) \mod r}{m=i \mod r}} = (q_m + q_k - q_{i-1} - (m + k - i + 1)\alpha)^{\frac{(i+r-1) \mod r}{m=i \mod r}}.
\]

Equations (11) and (12) are identical, save for the limits on the index \( m \), which cycles once through the numbers 1, \ldots, \( r \) in both cases. \( \square \)

As an example, take \( w_1 \in F_5^{12} \) having \( D_1 = (1, 1, 3, 2, 0) \) and \( \alpha = \frac{\pi}{5} \). Then \( Q_1 = (1, 2, 5, 7, 7) \), \( \Delta_1 = \left( -\frac{2}{5}, -\frac{4}{5}, \frac{4}{5}, \frac{7}{5}, 0 \right) \), and \( h_1 = \frac{\pi}{5} \). We also have \( Q_3 = (3, 5, 5, 6, 7) \), \( \Delta_3 = \left( \frac{8}{5}, \frac{11}{5}, \frac{4}{5}, \frac{2}{5}, 0 \right) \), and \( h_3 = \frac{11}{5} \). We see that

\[
\Delta_1 + (h_3)^5 = \left( \frac{9}{5}, \frac{7}{5}, 3, \frac{18}{5}, \frac{11}{5} \right) \sim \left( 3, \frac{18}{5}, \frac{11}{5}, \frac{9}{5}, \frac{7}{5} \right) = \Delta_3 + (h_1)^5.
\]

For rotations of a given \( w_1 \), the largest maximum deviation occurs for a superdiagonal rotation, while the smallest is attained when every difference \( q_j - j\alpha \) is at most 0, namely when the rotation is subdiagonal. (Note that the only superdiagonal code with maximum deviation 0 is \( \left( \alpha, 2\alpha, \ldots, r\alpha \right) \), where necessarily \( \alpha \in \{0, 1, 2, \ldots \} \).) Proposition 3 for subdiagonal rotations merits special mention.

**Corollary 1.** Let \( w_m \) be a subdiagonal rotation of \( w_1 \in F_r^n \). Then \( \Delta_1 \sim \Delta_m + (h_1)^r \).

With this corollary we set the stage for making subdiagonal rotations the standard against which all other rotations are measured; this will be developed further in Section 8.
7. Code functions

We now apply these properties of codes and deviation vectors to the polynomial part of the rational function of Proposition 1. Here, \( a \) is a parameter to be varied through nonnegative values, and we make extensive use of the quantity

\[ \lambda = b^{-\alpha}. \]

Let \( S = (s_1, \ldots, s_j) \) be any sequence of nonnegative integers, and define \( u : S \times \mathbb{R} \to \mathbb{R} \) by

\[ u(S, x) = \sum_{i=1}^{j} b^{s_i} x^{i-1}. \]  

(13)

Definition 4. For \( w_1 \in F_{\alpha}^n \), let \( w_i \sim w_1 \) have code \( Q_i \). We call \( u(Q_i, a) \) (also written \( u_i(a) \) or \( u_i \)) the \( i \)th code function of \( w_1 \).

Code functions will be called sub- or superdiagonal in accordance with their corresponding cycle point words.

Figure 1 shows the code functions corresponding to the ten rotations of the word \( w_1 \in F_{10}^{30} \) whose gaps vector is \( D_1 = (4, 2, 5, 1, 1, 0, 1, 2, 3, 1) \) and whose density is \( \alpha = 2 \). Here \( b = \frac{1}{2} \). The code functions for the perigee words are drawn in bold lines, and the upper sections of the curves are shown with compressed vertical scale for clarity. The codes for \( w_1 \)'s rotations are listed in Table 2.

<table>
<thead>
<tr>
<th>( D_1 )</th>
<th>( Q_1 )</th>
<th>( Q_2 )</th>
<th>( Q_3 )</th>
<th>( Q_4 )</th>
<th>( Q_5 )</th>
<th>( Q_6 )</th>
<th>( Q_7 )</th>
<th>( Q_8 )</th>
<th>( Q_9 )</th>
<th>( Q_{10} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (4, 2, 5, 1, 1, 0, 1, 2, 3, 1) )</td>
<td>( (4, 6, 11, 12, 13, 13, 14, 16, 19, 20) )</td>
<td>( (2, 7, 8, 9, 10, 12, 15, 16, 20) )</td>
<td>( (5, 6, 7, 8, 10, 13, 14, 18, 20) )</td>
<td>( (1, 2, 2, 3, 5, 8, 9, 13, 15, 20) )</td>
<td>( (1, 1, 2, 4, 7, 8, 12, 14, 19, 20) )</td>
<td>( (0, 1, 3, 6, 7, 11, 13, 18, 19, 20) )</td>
<td>( (1, 3, 6, 7, 11, 13, 18, 19, 20) )</td>
<td>( (2, 5, 6, 10, 12, 17, 18, 19, 20) )</td>
<td>( (3, 4, 8, 10, 15, 16, 17, 18, 20) )</td>
<td>( (1, 5, 7, 12, 13, 14, 15, 17, 20) )</td>
</tr>
<tr>
<td>( h_i )</td>
<td>( \frac{151}{128} )</td>
<td>( \frac{151}{32} )</td>
<td>( \frac{151}{32} )</td>
<td>( \frac{151}{4} )</td>
<td>( \frac{1}{8} )</td>
<td>( \frac{151}{64} )</td>
<td>( \frac{151}{64} )</td>
<td>( \frac{151}{128} )</td>
<td>( \frac{151}{128} )</td>
<td>( \frac{151}{64} )</td>
</tr>
</tbody>
</table>

Table 2. Gaps vector \( D_1 \) and codes \( Q_i \) for the code functions of Figure 1, with maximum deviations \( h_i \) and the code function values at \( a = \lambda \).

We see in this example that, for fixed \( b \), the minimal code function (and thus the perigee word) depends on \( a \), and this is true in general. Proof of the following formal statement is left to the reader.
Figure 1. The code functions of the ten rotations of the word whose gaps vector is $D_1 = (4, 2, 5, 1, 1, 0, 1, 2, 3, 1)$, where $b = \frac{1}{2}$.

**Proposition 4.** In $\Psi(a_0, b)$ with $a_0 \in \mathbb{R}_+ \setminus \{\lambda\}$, $w_i$ is a perigee word precisely when $u_i(a_0)$ is minimal over all $i \in \{1, \ldots, r\}$.

8. Ordering of the code functions at $a = \lambda$

Although every rotation $w_i$ yields an undefined cycle point at $a = \lambda$, the code function values $u_i(\lambda)$ are finite and, it turns out, in a convenient order. As the reader may surmise from the last two columns of Table 2, there is an elegant relationship between $h_i$, $u_i(\lambda)$, and subdiagonal rotations.

**Theorem 3.** In $\Psi(a, b)$, let $w_1 \in F^n_r$ have code function $u_1$, maximum deviation $h_1$, and density $\alpha$. Let $w_m \sim w_1$ be subdiagonal with code function $u_m$. Then

$$u_1(\lambda) = b^{h_1} u_m(\lambda). \quad (14)$$

This formula says that, in a given $n$-cycle, a code function’s value at $a = \lambda$ is a multiple of any subdiagonal code function’s value at $\lambda$, where the multiplier
is a monotone function of the maximum deviation. (If \( w_1 \) is itself subdiagonal, then \( h_1 = 0 \), and (14) holds trivially.)

**Proof.** We employ the function \( u(S,x) \) from equation (13), using various values for \( x \) and sequences \( S \). Begin with

\[
u_1(\lambda) = u(Q_1, \lambda) = \sum_{i=1}^{r} b^{x_i} (b^{-\alpha})^{i-1} = b^\alpha \sum_{i=1}^{r} b^{x_i - i\alpha} = b^\alpha u(\Delta_1, 1).
\tag{15}
\]

Because \( w_m \) is subdiagonal, we know from Corollary 1 that \( \Delta_1 \) is simply a cyclic permutation of \( \Delta_m + (h_1)^r \). Thus \( u(\Delta_1, 1) = u(\Delta_m + (h_1)^r, 1) \). Substitution in (15) yields

\[
u(Q_1, \lambda) = b^\alpha u(\Delta_m + (h_1)^r, 1) = b^\alpha b^{h_1} u(\Delta_m, 1) = b^{h_1} u(Q_m, \lambda) = b^{h_1} u_m(\lambda).
\]

If \( h_i < h_1 \), and \( w_m \) is subdiagonal, then by Theorem 3 we have \( u_i(\lambda) = b^{h_1} u_m(\lambda) > b^{h_1} u_m(\lambda) = u(\lambda) \). Indeed, we can say

**Corollary 2.** Given \( w_i \sim w_1 \in F_r^n \).

(a) If \( h_i < h_1 \), then \( u_i(\lambda) > u_1(\lambda) \).

(b) If \( h_i = h_1 \), then \( u_i(\lambda) = u_1(\lambda) \).

It follows that the points \( u_i(\lambda) \) are partially ordered. Since subdiagonal rotations have the smallest maximum deviation \( (h_i = 0) \), and superdiagonal rotations have the largest, Corollary 2 implies that the subdiagonal code functions intersect the line \( a = \lambda \) at the highest point, while the superdiagonal code functions meet the line at the lowest point. This is illustrated in Figure 1, where \( u_4 \) is subdiagonal and \( u_1, u_8, \) and \( u_9 \) are superdiagonal. (This example was constructed to show that, additionally, the perigee word need not be superdiagonal in intervals not containing \( \lambda \), as shown by \( w_3 \) and \( w_7 \) on \((0, \gamma) \) and \((\mu, \infty) \), respectively.)

9. **Code function bounds**

Two final lemmas are needed to prove the Maximal Perigee Property; the first establishes upper and lower bounds, respectively, on super- and subdiagonal code functions at \( a = \lambda \) over all words in \( F_r^n \).

**Lemma 2.** Let \( w_1 \in F_r^n \) have density \( \alpha \) and sub- and superdiagonal rotations \( w_m \) and \( w_k \), respectively. Then

\[
u_k(\lambda) \leq rb^\alpha \leq u_m(\lambda).
\tag{16}
\]
Proof. If \( w_m \) is subdiagonal with code \((q_{m,1}, q_{m,2}, \ldots, q_{m,r})\), then \( q_{m,i} - ia \leq 0 \), and hence \( b^{q_{m,i} - ia} \geq 1 \) for \( i \in \{1, \ldots, r\} \). Thus
\[
 u_m(\lambda) = \sum_{i=1}^{r} b^{q_{m,i} - i\lambda} = b^{\alpha} \sum_{i=1}^{r} b^{q_{m,i} - i\alpha} \geq rb^{\alpha}.
\]
The derivation for the left-hand inequality in (16) is similar.

Lemma 3. The only superdiagonal word in \( F^n_r \) whose maximum deviation is less than 1 is the word whose code is \((\lceil \alpha \rceil, \lceil 2\alpha \rceil, \ldots, \lceil r\alpha \rceil)\).

Proof. For any \( w_1 \in F^n_r \), every \( g \)-rank \( q_i \) is an integer. If \( \alpha \notin \{0, 1, 2, \ldots\} \), then \( w_1 \)'s superdiagonality and the condition \( 0 < \max(q_i - i\alpha) < 1 \) imply that \( ia < q_i < ia + 1 \) for each \( i \in \{1, \ldots, r\} \), except for at least one \( i \) for which \( qa < qa + 1 \). But the only such integers are \( q_i = \lceil ia \rceil \). As noted at the end of Section 6, the only superdiagonal code of maximum deviation 0 has \( a \in \{0, 1, 2, \ldots\} \); we thus have \( ia = \lceil ia \rceil \) for each \( i \), and again the lemma holds.

10. Proof of the Maximal Perigee Property

To prove the Maximal Perigee Property, we show that, for the particular value \( a = \lambda \), the maximal perigee word \( w_{\text{max}} \in P^n_r \) has code \((\lceil \alpha \rceil, \lceil 2\alpha \rceil, \ldots, \lceil r\alpha \rceil)\). It will then follow that this \( w_{\text{max}} \) is unique and minimal over all \( a \in \mathbb{R}_+ \).

Since the case \( r = n \) is trivially true, assume \( \alpha \neq 0 \). By Corollary 2, the minimal code function among a given word’s rotations is superdiagonal at \( a = \lambda \). To obtain the largest such minimal function, we seek a superdiagonal \( w_1 \in F^n_r \) for which the nonnegative quantity
\[
r b^{\alpha} - u_1(\lambda)
\]
from Lemma 2 is minimized. Using Theorem 3, we may write this as
\[
0 \leq rb^{\alpha} - b^{h_1} u_m(\lambda),
\]
where \( w_m \) is a subdiagonal rotation of \( w_1 \). From Lemma 2 we also have \( u_m(\lambda) \geq rb^{\alpha} \). Therefore
\[
-b^{h_1} u_m(\lambda) \leq -b^{h_1} rb^{\alpha},
\]
and this, combined with equations (17) and (18), yields
\[
0 \leq rb^{\alpha} - u_1(\lambda) = rb^{\alpha} - b^{h_1} u_m(\lambda) \leq rb^{\alpha} - rb^{\alpha + h_1}.
\]
or, more simply,
\[ rb^{n+1} \leq u_1(\lambda) \leq rb^n. \]

The smallest possible \( h_1 \) minimizes the range of \( u_1(\lambda) \). Therefore the terms of the desired superdiagonal \( Q_1 \) are the \( r \) integers on or above, and closest to, the line \( y = \alpha x \); that is, \( Q_1 = ([\alpha], [2\alpha], \ldots, [r\alpha]) \). Furthermore, Lemma 3 implies that \( u_1 \) is unique; it is the only code function whose value at \( a = \lambda \) lies in the interval \((rb^{\alpha+1}, rb^{\alpha})\). We conclude that the Maximal Perigee Property holds at \( a = \lambda \); that is, at this one value of \( a \), \( w_1 = w_{\text{max}} \) and \( Q(w_1) = Q(w_{\text{max}}) = ([\alpha], [2\alpha], \ldots, [r\alpha]) \).

We now prove that this same \( u_1 \) is the maximal minimum code function for all \( a \in \mathbb{R}_+ \). Write \( Q_1 = (q_{1,1}, q_{1,2}, \ldots, q_{1,r}) \), where \( q_{1,j} = [j\alpha] \). We find the rotated code \( Q_i = (q_{i,1}, q_{i,2}, \ldots, q_{i,r}) \), \( i \in \{1, \ldots, r\} \), using Proposition 2:

\[
q_{i,j} = \begin{cases} 
[(i + j - 1)\alpha] - [(i - 1)\alpha], & j \in \{1, \ldots, r - i + 1\} \\
[r\alpha] + [(i + j - r - 1)\alpha] - [(i - 1)\alpha], & j \in \{r - i + 2, \ldots, r\}
\end{cases}
\]

But the latter case reduces as follows:

\[
[r\alpha] + [(i + j - r - 1)\alpha] - [(i - 1)\alpha] = n - r + [(i + j - 1)\alpha] - (n - r) - [(i - 1)\alpha] = [(i + j - 1)\alpha] - [(i - 1)\alpha],
\]

so in fact

\[
q_{i,j} = [(i + j - 1)\alpha] - [(i - 1)\alpha] \tag{19}
\]

for all \( j \in \{1, \ldots, r\} \). Because \([x] + [y] \geq [x + y]\) for nonnegative real numbers \(x\) and \(y\), equation (19) allows us to write

\[
[j\alpha] + [(i - 1)\alpha] \geq [(i + j - 1)\alpha] \\
[j\alpha] \geq [(i + j - 1)\alpha] - [(i - 1)\alpha]
\]

\[
q_{i,j} \geq q_{i,j} \tag{20}
\]

Observe, however, that equality cannot hold in (20) for all \( j \in \{1, \ldots, r\} \); if it did, we would have \( Q_1 \sim Q_i \), which is possible only when \( w_1 = f^n \) and \( \alpha = 0 \). Therefore, \( b^{n,j} \leq b^{n,j} \) for \( j \in \{1, \ldots, r\} \), except for at least one \( j' \) for which \( b^{n,j'} < b^{n,j'} \). Thus

\[
u_1(a) = \sum_{j=1}^{r} b^{n,j} a^{j-1} < \sum_{j=1}^{r} b^{n,j} a^{j-1} = u_i(a)
\]

for \( i \in \{2, \ldots, r\} \). By Proposition 4, it follows that \( x_1 \) is less than any other cycle point \( x_i \) for any nonnegative \( a \). We conclude that \( w_1 = w_{\text{max}} \) for all \( a \in \mathbb{R}_+ \), and the proof of the Maximal Perigee Property is complete.
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(Received June 26, 2008; revised August 19, 2009)