Polynomial bases of split simple Lie algebras

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Abstract. We show that every simple Lie algebra $\mathfrak{g}$ of real rank at least two is isomorphic to a space of polynomials defined on the group $N = \exp(n)$, where $n$ is the nilpotent component of the Iwasawa decomposition of $\mathfrak{g}$. Using suitable coordinates on $N$, we then write a basis of this space of polynomials when $\mathfrak{g}$ is split.

1. Introduction

When $n \geq 3$, the action of the conformal group $O(1, 4)$ on $\mathbb{R}^3 \cup \{\infty\}$ may be characterized in differential geometric terms: Liouville proved in 1850 that a $C^4$ map between domains $\mathcal{U}$ and $\mathcal{V}$ in $\mathbb{R}^3$ whose differential is a multiple of an isometry at each point of $\mathcal{U}$ is the restriction to $\mathcal{U}$ of the action of some $g \in O(1, 4)$. This type of result has been extended to $\mathbb{R}^n$ with weaker smoothness assumptions and to more general spaces, see for instance [3]–[5], [7]–[10].

In [4], the authors consider the problem of characterizing the action of a semisimple Lie group $G$ on the homogeneous spaces $G/P$, where $P$ is a minimal parabolic subgroup. More precisely, they prove a Liouville type theorem for every semisimple Lie group $G$ with rank at least two. The proof of this theorem passes through a polynomial representation of simple real Lie algebras, that we intend to make explicit. In particular, it is possible to define an isomorphism $I$ between the Lie algebra of $G$ and a space of polynomials on $N$, the nilpotent component of the Iwasawa decomposition of $G$. The isomorphism induces a Lie algebra structure on this space of polynomials. We are interested in investigating the polynomial representation of the simple Lie algebras given by $I$.

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The paper is organized as follows. In Section 2 we fix the notations and recall a result of [4] that we are going to need. In particular, we give the definition of multicontact map and vector field, and recall (Theorem 1) that the space of multicontact vector fields on $N$ is isomorphic to the simple Lie algebra $\mathfrak{g}$ whose nilradical is $\mathfrak{n} = \text{Lie}(N)$. In Section 3 we discuss the isomorphism $I$ in some details. First we introduce the notion of homogeneous function and vector field and observe that the space of multicontact vector fields is generated as a vector space by its homogeneous parts. In fact, there is a one to one correspondence between suitable bases of $\mathfrak{g}$ and homogeneous generators of the multicontact vector fields. This correspondence allows us to define $I$. The idea is to fix a basis of each root space and therefore a basis of $\mathfrak{g}$. Hence $I$ is the linear map that assigns to each such basis element a suitable vector of polynomials. In Section 4 we restrict to the case of split simple Lie algebras $\mathfrak{g}$. In this case the image $I(X)$ is exactly one polynomial. In Lemma 2 we give a formula for computing $I(X)$, whenever $X$ lies in a root space or in the Cartan subspace. We then use this in Proposition 3 to find an explicit basis of the space of the polynomials in canonical coordinates. In the last section we consider the case where $\mathfrak{g}$ is $\mathfrak{sl}(3,\mathbb{R})$ and therefore $N$ is the Heisenberg group and apply Proposition 3 in order to write the polynomial basis of $\mathfrak{sl}(3,\mathbb{R})$.

2. Notations and preliminaries

We introduce some tools which come from the classical theory of semisimple Lie groups [1], [6], as well as some further properties proved in [4]. Let $\mathfrak{g}$ be a simple Lie algebra with Killing form $B$ and Cartan involution $\theta$. Then $B_\theta(X, Y) = -B(X, \theta Y)$ is an inner product on $\mathfrak{g}$. Let $\mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$, where $\mathfrak{k} = \{X \in \mathfrak{g} : \theta X = X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g} : \theta X = -X\}$. Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and denote by $\Sigma$ the set of restricted roots, $\Sigma$ is a subset of the dual $\mathfrak{a}'$ of $\mathfrak{a}$, which is endowed with an inner product $(\cdot, \cdot)$ induced by $B_\theta$. Choose an ordering $\succeq$ on $\mathfrak{a}'$. Call $\Sigma_+$ and $\Delta = \{\delta_1, \ldots, \delta_r\}$ the subsets for positive and simple positive restricted roots. We call rank of $\mathfrak{g}$ the cardinality of $\Delta$. Every positive root $\alpha$ can be written as $\alpha = \sum_{i=1}^{r} n_i \delta_i$ for uniquely defined non-negative integers $n_1, \ldots, n_r$. The positive integer $\text{ht}(\alpha) = \sum_{i=1}^{r} n_i$ is called the height of $\alpha$.

It is well-known that there is exactly one root $\omega$, called the highest root, that satisfies $\omega \succeq \alpha$ (strictly) for every other root $\alpha$. The root space decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$, where $\mathfrak{m} = \{X \in \mathfrak{t} : [X, H] = 0, H \in \mathfrak{a}\}$. The Iwasawa decomposition is $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{n} = \bigoplus_{\gamma \in \Sigma_+} \mathfrak{g}_\gamma$. We write $n_i = \bigoplus_{\text{ht}(\gamma) = i} \mathfrak{g}_\gamma$. 
Consider a diffeomorphism $f$ between open subsets $U$ and $V$ of $N$. For every positive root $\alpha$, the space $g_\alpha$ defines a subspace of the tangent space of $N$ at the identity and by left translation it defines a sub-bundle of the tangent bundle, for which we abuse the notation $g_\delta$. We say that $f$ is a multicontact mapping if its differential $f^*$ preserves $g_\delta$, for every simple root $\delta$. This is a generalized notion of contact mapping in the usual sense, because $n_1 = \oplus_{\delta \in \Delta} g_\delta$ and a basis of left invariant vector fields of $n_1$ generates via Lie bracket the whole algebra of left invariant vector fields. If $U = V$ we can compose two multicontact mappings, obtaining another multicontact map. We define a multicontact vector field as a vector field $V$ on $U$ whose local flow $\{\phi^V_t\}$ consists of multicontact maps. Such a vector field satisfies

$$[V, g_\delta] \subset g_\delta,$$

for every simple root $\delta$. The group $G$ acts on $G/P$. By means of the Bruhat decomposition, the action can be restricted to $N$. Let $\mathfrak{x}(N)$ denote the Lie algebra of vector fields on $N$. We define a representation of $g$ as vector fields on $N$

$$\tau : g \to \mathfrak{x}(N)$$

as

$$(\tau(X)f)(n) = \left. \frac{d}{dt} f([\exp(tX)n]) \right|_{t=0}.$$

Hence $[\exp(tX)n]$ is the $N$-component of the product $\exp(tX) \cdot n$ in the Bruhat decomposition of $G/P$ (see [4] for more details). The following theorem is proved in [4] and its proof contains the results we need.

**Theorem 1** ([4]). Suppose that $g$ has real rank at least two. Then every $C^1$ multicontact vector field is in fact smooth, and the Lie algebra of multicontact vector fields on $U$ consists of the restrictions of $\tau(g)$ to $U$.

### 3. The polynomial algebra $\mathcal{P}$

From now on we assume that $g$ has real rank at least two. For every $\alpha \in \Sigma_+$, denote by $m_\alpha$ the dimension of $g_\alpha$ and fix a basis $\{X_{\alpha,i} : \alpha \in \Sigma_+, i = 1, \ldots, m_\alpha\}$ of $n$ consisting of left-invariant vector fields on $N$. A smooth vector field $V$ on $U$ is

$$V = \sum_{\alpha \in \Sigma_+} \sum_{i=1}^{m_\alpha} v_{\alpha,i} X_{\alpha,i},$$

(1)
with smooth functions \(v_{\alpha,i}\). The proof of Theorem 1 points out that a multicontact vector field is determined by its component along the directions corresponding to the highest root, namely \(\{v_{\omega,i} : i = 1, \ldots, m_\omega\}\), the remaining components being obtained differentiating those. Further, the functions \(v_{\omega,i}\) are in fact polynomials in canonical coordinates.

We select an element \(H_0\) in the Cartan subspace \(a\) such that \(\delta(H_0) = -1\) for all simple roots \(\delta\). We say that a function \(v\) on \(N\) is homogeneous of degree \(r\) if it does not vanish identically and satisfies \(\tau(H_0)v = rv\). A vector field \(V\) is said to be homogeneous of degree \(s\) if it does not vanish identically and satisfies \([\tau(H_0), V] = sV\). Hence

\[
\deg(vV) = \deg(V) + \deg(v),
\]

\[
\deg(V(v)) = \deg(v) + \deg(V) \text{ (except when } V(v) = 0),
\]

\[
\deg([V,W]) = \deg(V) + \deg(V) \text{ (except when } V \text{ and } W \text{ commute}).
\]

The Lie algebra of multicontact vector fields is then generated by its homogeneous parts. More precisely, the set

\[
\{\tau(X_{\alpha,i}), \alpha \in \Sigma \cup \{0\}, i = 1, \ldots, m_\alpha\}
\]

defines a basis. Since \(\tau\) is a representation, we have

\[
[\tau(H_0), \tau(X_{\alpha,i})] = \tau([H_0, X_{\alpha,i}]) = \alpha(H_0)\tau(X_{\alpha,i}) = -\text{ht}(\alpha)\tau(X_{\alpha,i}).
\]

Let \(p\) be a \(\omega\)-component of \(\tau(X_{\alpha,i})\). Then the height of \(\alpha\) and the degree of \(p\) are related:

\[
-\text{ht}(\alpha) = \deg(\tau(X_{\alpha,i})) = \deg(pX_{\omega,j}) = \deg(p) + \deg(X_{\omega,j}) = \deg(p) - \text{ht}(\omega),
\]

whence

\[
\deg(p) = \text{ht}(\omega) - \text{ht}(\alpha).
\]

Define

\[
I : g \to P,
\]

by extending linearly the assignment \(I(X_{\alpha,i}) = (v_{\omega,1}, \ldots, v_{\omega,m_\omega})\), the vector of polynomials that corresponds to the coefficients of \(\tau(X_{\alpha,i})\) along \(\omega\). Here \(P\) is a vector space of polynomial vectors, namely the image of the above mapping inside the \(m_\omega\)-fold cartesian product of the algebra of polynomials in \(\dim(n)\) indeterminates over the reals. The map \(I\) is an isomorphism, that induces a Lie algebra structure on \(P\).
Since the homogeneity degree of the polynomials $I(X_{\alpha,i})$ depends only on the root space $\mathfrak{g}_\alpha$, all the components of a single basis vector have the same degree. Let $\alpha$ be a positive root, $X \in \mathfrak{g}_{\pm\alpha}$ or $X \in \mathfrak{m} \oplus \mathfrak{a}$, and let $p$ be a $\omega$-component of $\tau(X)$. The following diagram clarifies the various notions of degree:

<table>
<thead>
<tr>
<th>root space</th>
<th>$\deg(\tau(X))$</th>
<th>$\deg p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathfrak{g}_\alpha$</td>
<td>$-\text{ht}(\alpha)$</td>
<td>$\text{ht}(\omega) - \text{ht}(\alpha)$</td>
</tr>
<tr>
<td>$\mathfrak{m} \oplus \mathfrak{a}$</td>
<td>0</td>
<td>$\text{ht}(\omega)$</td>
</tr>
<tr>
<td>$\mathfrak{g}_{-\alpha}$</td>
<td>$\text{ht}(\alpha)$</td>
<td>$\text{ht}(\omega) + \text{ht}(\alpha)$</td>
</tr>
</tbody>
</table>

In particular each polynomial has degree between 0 and $2h$, where $h = \text{ht}(\omega)$.

4. The split case

We compute explicit formulas for a basis of $\mathcal{P}$. We restrict our discussion to the case of the split real form $\mathfrak{g}$ of a simple complex Lie algebra. The most relevant consequences of this assumption for our considerations are that $\mathfrak{m} = \{0\}$ and that each restricted root space has real dimension one. In particular, this implies that $I(\mathfrak{g}_\alpha)$ consists for all $\alpha$ of the real multiples of a single polynomial.

Our decomposition formulas are relative to a suitable decomposition of the restricted root system (see e.g. [2]), namely $\Sigma_+ = \Sigma_0 \oplus \Sigma_{1/2} \oplus \Sigma_1$, where

$\Sigma_0 = \{ \beta \in \Sigma_+ : (\omega, \beta) = 0 \}$,

$\Sigma_{1/2} = \left\{ \beta \in \Sigma_+ : (\omega, \beta) = \frac{1}{2}(\omega, \omega) \right\}$,

$\Sigma_1 = \{ \beta \in \Sigma_+ : (\omega, \beta) = (\omega, \omega) \} = \{ \omega \}$.

We shall write $\Delta_{1/2} = \Sigma_{1/2} \cap \Delta$ and $\Delta_0 = \Sigma_0 \cap \Delta$. According to the decomposition of $\Sigma_+$, we put

$\mathfrak{n} = \mathfrak{n}_0 \oplus \mathfrak{n}_{1/2} \oplus \mathfrak{n}_1$,

with obvious notations. Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$, and $(\alpha + \beta, \omega) = (\alpha, \omega) + (\beta, \omega)$, it follows that $\mathfrak{n}_0$ is a subalgebra and $\mathfrak{n}_{1/2} \oplus \mathfrak{n}_1$ is an ideal in $\mathfrak{n}$. The Cartan involution $\theta$ maps each root space $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$, so that $\overline{\mathfrak{n}} = \theta \mathfrak{n} = \oplus_{\gamma \in \Sigma_-} \mathfrak{g}_\gamma$, where $\Sigma_- = -\Sigma_+$. We write

$\overline{\mathfrak{p}} = \overline{\mathfrak{n}}_0 \oplus \overline{\mathfrak{n}}_{1/2} \oplus \overline{\mathfrak{n}}_1$,
so that
\[ g = n_1 \oplus n_{1/2} \oplus n_0 \oplus a \oplus \bar{n}_0 \oplus \bar{n}_{1/2} \oplus \bar{n}_1. \] (3)

By linearity of the scalar product, the following commutation rules hold
\[
\begin{align*}
[a, n_0] & \subset n_0 & [a, n_{1/2}] & \subset n_{1/2} & [a, n_1] & \subset n_1 \\
[n_0, \bar{n}_0] & \subset n_0 \oplus a \oplus \bar{n}_0 & [n_0, \bar{n}_{1/2}] & \subset \bar{n}_{1/2} & [n_0, \bar{n}_1] & = \{0\} \\
[n_{1/2}, \bar{n}_0] & \subset n_{1/2} & [n_{1/2}, \bar{n}_{1/2}] & \subset n_0 \oplus a \oplus \bar{n}_0 & [n_{1/2}, \bar{n}_1] & \subset \bar{n}_{1/2} \\
[n_1, \bar{n}_0] & = \{0\} & [n_1, \bar{n}_{1/2}] & \subset n_{1/2} & [n_1, \bar{n}_1] & \subset a
\end{align*}
\] (4)

We fix the following canonical coordinates on \( N \):
\[
n = n_1 n_{1/2} n_0 = \exp(zZ) \exp \left( \sum_{\alpha \in \Sigma_{1/2}} y_{\alpha} Y_{\alpha} \right) \exp \left( \sum_{\beta \in \Sigma_0} x_{\beta} X_{\beta} \right), \]
where \( \{X_{\beta}, \beta \in \Sigma_0\}, \{Y_{\alpha}, \alpha \in \Sigma_{1/2}\} \) and \( Z \) are a basis of \( n_0, n_{1/2} \) and \( n_1 \) respectively.

Set \( X \in g_{\alpha}, \alpha \in \Sigma \cup \{0\} \) and \( n \) in \( N \). By the Bruhat decomposition, for \( t \) small enough there exists \( b(t) \in P \) such that \( \exp(tX)nb(t) \in N \). Then consider the decomposition of \( n^{-1} \exp(tX)nb(t) \) with respect to the chosen coordinates, namely
\[
n^{-1} \exp(tX)nb(t) = n_1^{-1} (t)n_{1/2}^{-1}(t)n_0^{-1}(t).
\]

**Lemma 2.** With the notations as above, writing \( n = n_1 n_{1/2} n_0 \), we have
(i) there exists \( A \in n_1 \) and \( B \in n_{1/2} \oplus n_0 \oplus a \oplus \bar{n}_0 \) such that
\[
n_{1/2}^{-1} n_1^{-1} \exp(tX)n_1 n_{1/2} = \exp(tA) \exp(tB) \exp(o(t));
\]
(ii) \( \left. \frac{d}{dt} (n_1^X(t)) \right|_{t=0} = A; \)
(iii) If \( I \) is the isomorphism defined in (2), then \( I(X) = A \).

**Proof.** Write
\[
n^{-1} \exp(tX)n = n_0^{-1} n_{1/2}^{-1} n_1^{-1} \exp(tX)n_1 n_{1/2} n_0.
\]
Observe first that since \( n_1 = \exp(zZ) \),
\[
n_1^{-1} \exp(tX)n_1 = \exp(e^{-\text{ad}(zZ)t}X).
\]
Now, by (4)

\[ [Z, X] \in \begin{cases} n_1 & \text{if } X \in a \\ n_{1/2} & \text{if } X \in \pi_{1/2} \\ a & \text{if } X \in \pi_1, \end{cases} \]

and if \( X \) belongs to some other summand in the decomposition (3), then \([Z,X]=0\). Therefore by the Baker–Campbell–Hausdorff formula

\[
n^{-1}_1 \exp(tX)n_1 = \exp(tX + t(H_1 + A_{1/2} + A_1) + o(t)) = \exp(tA_1 + o(t))\exp(tX + t(H_1 + A_{1/2}) + o(t)),
\]

where \( H_1 \in a, A_{1/2} \in n_{1/2} \) and \( A_1 \in n_1 \).

Secondly, since \( n_{1/2} \) commutes with \( n_1 \), we consider

\[
n^{-1}_{1/2} \exp(tX + t(H_1 + A_{1/2}))n_{1/2} = \exp(e^{-\sum_\alpha \frac{\hbar}{2} \text{ad} Y_\alpha}(tX + tH_1 + tA_{1/2})).
\]

Since \( n_{1/2} \) is the exponential of some element in \( n_{1/2} \), in the above formula \( \alpha \in \Sigma_{1/2} \). Therefore, if the commutator \([Y_\alpha, X] \neq 0\), then by (4)

\[ [Y_\alpha, X] \in \begin{cases} \pi_{1/2} & \text{if } X \in \pi_1 \\ a & \text{if } X \in \pi_{1/2} \\ n_{1/2} & \text{if } X \in \pi_0 \oplus n_0 \oplus a \\ n_1 & \text{if } X \in n_{1/2}. \end{cases} \]

Moreover,

\[ [Y_\alpha, H_1] \in a, \quad [Y_\alpha, A_{1/2}] \in n_1. \]

Hence

\[
n^{-1}_{1/2} \exp(tX + t(H_1 + A_{1/2}))n_{1/2} = \exp(tX + t(B^{-1}_{1/2} + H_2 + B_{1/2} + B_1) + o(t)),
\]

\[
= \exp(tB_1 + o(t))\exp(tX + t(B^{-1}_{1/2} + H_2 + B_{1/2}) + o(t)),
\]

for some \( B^{-1}_{1/2} \in \pi_{1/2}, H_2 \in a, B_1/2 \in n_{1/2} \) and \( B_1 \in n_1 \). Also, observe that by the Baker–Campbell–Hausdorff formula

\[
\exp(tL + o(t)) = \exp(tL)\exp(o(t))
\]

for any \( L \in g \). Thus, by (6) and (4) we obtain that

\[
n^{-1}_{1/2}n^{-1}_1 \exp(tX)n_1n_{1/2} = \exp(tA)\exp(tB)\exp(o(t)),
\]
with

$$A = \begin{cases} A_1 + B_1 & \text{if } X \notin n_1 \\ A_1 + B_1 + X & \text{if } X \in n_1 \end{cases}$$

and

$$B = \begin{cases} B_{1/2} + H_2 + B_{1/2} & \text{if } X \in n_1 \\ B_{1/2} + H_2 + B_{1/2} + X & \text{if } X \notin n_1. \end{cases}$$

This proves (i).

Next, consider

$$n_0^{-1} \exp(tX + t(B_{1/2}^{-} + H_2 + B_{1/2}))n_0 = \exp(e^{-ad(\Sigma_1 X \beta \beta X \beta)}(tX + tB_{1/2}^{-} + tH_2 + tB_{1/2})).$$

If $[X_\beta, X] \neq 0$, then by (4)

$$[X_\beta, X] \in \begin{cases} \overline{n}_1/2 & \text{if } X \in \overline{n}_1/2 \\ n_0 \oplus \overline{n}_0 \oplus a & \text{if } X \in n_0 \oplus \overline{n}_0 \oplus a \\ n_1/2 & \text{if } X \in n_1/2. \end{cases}$$

Furthermore,

$$[X_\beta, B_{1/2}^{-}] \in \overline{n}_1/2, \quad [X_\beta, H_2] \in n_0, \quad [X_\beta, B_{1/2}] \in n_1/2.$$

Hence

$$n_0^{-1} \exp(tX + t(B_{1/2}^{-} + H_2 + B_{1/2}))n_0 = \exp(tX + t(C_1/2^{-} + C_0^{-} + H_3 + C_0^{+} + C_{1/2}) + o(t)), \quad (9)$$

for some $C_{1/2}^{-} \in \overline{n}_1/2$, $C_0^{-} \in \overline{n}_0$, $H_3 \in a$, $C_0^{+} \in n_0$ and $C_{1/2} \in n_1/2$.

Since $n_1$ commutes with $n$, using (6), (4), (8) and (9) we obtain

$$n^{-1} \exp(tX)n = \exp(tA_1 + tB_1 + o(t)) \times \exp(tX + t(C_{1/2}^{-} + C_0^{-} + H_3 + C_0^{+} + C_{1/2}) + o(t))$$

$$= \exp(tA_1 + tB_1 + o(t)) \exp(tX + tC_0^{+} tC_{1/2} + o(t)) \times \exp(t(C_{1/2}^{-} + C_0^{-} + H_3) + o(t))$$

$$= \exp(tA_1 + tB_1 + tk_1(X)) \exp(tC_{1/2} + tk_{1/2}(X)) \times \exp(tC_0 + tk_0(X)) \exp(tC_{1/2}^{-} + tC_0^{-} + th_3 + tk(X)) \exp(o(t))$$

$$= \exp(tA) \exp(tC) \exp(tD) \exp(tE) \exp(o(t)), \quad (10)$$
where

\[
k(X) = \begin{cases} 
X & \text{if } X \in \mathfrak{a} \oplus \overline{\Pi} \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
k_i(X) = \begin{cases} 
X & \text{if } X \in \mathfrak{n}_i, \; i = 0, 1/2, 1 \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
C = C_{1/2} + k_{1/2}(X), \quad D = C_0 + k_0(X), \quad E = C_{1/2}^- + C_0^- + H_3 + k(X).
\]

On the other hand, by hypothesis

\[
n^{-1} \exp(tX)n = n_1^X(t)n_{1/2}^X(t)n_0^X(t)b(t)^{-1}. \quad (11)
\]

Observe that since \(n^{-1} \exp(tX)n\) is the identity for \(t = 0\), then necessarily \(n_i^X(0) = e\) for every \(r = 1, 1/2, 0\), and \(b(0) = e\). Therefore, comparing (10) and (11),

\[
\frac{d}{dt} \left( \exp(tA) \exp(tC) \exp(tD) \exp(tE) \exp(o(t)) \right) \bigg|_{t=0} = \frac{d}{dt} \left( n_1^X(t)n_{1/2}^X(t)n_0^X(t)b(t)^{-1} \right) \bigg|_{t=0},
\]

whence

\[
A + C + D + E = \frac{d}{dt} \left( n_1^X(t) \right) \bigg|_{t=0} n_{1/2}^X(0)n_0^X(0)b(0)^{-1} + n_1^X(0) \frac{d}{dt} \left( n_{1/2}^X(t) \right) \bigg|_{t=0} n_0^X(0)b(0)^{-1} + n_1^X(0)n_{1/2}^X(0) \frac{d}{dt} \left( n_0^X(t) \right) \bigg|_{t=0} b(0)^{-1} + n_1^X(0)n_{1/2}^X(0)n_0^X(0) \frac{d}{dt} \left( b(t)^{-1} \right) \bigg|_{t=0}.
\]

This implies

\[
A = \frac{d}{dt} \left( n_1^X(t) \right) \bigg|_{t=0},
\]

because \(A\) and \(\frac{d}{dt} \left( n_1^X(t) \right) \bigg|_{t=0}\) are the only two terms in the above sum that lie along \(Z\). Thus also (ii) is proved.

In order to prove (iii), consider the multicontact vector field associated to \(X\):

\[
\tau(X)f(n) = \frac{d}{dt} f(\exp(tX)n) \bigg|_{t=0},
\]

where \([\exp(tX)n]\) is the \(N\)-component of \(\exp(tX)n\) in the Bruhat decomposition.
This is equivalent to saying that for \( t \) small enough there exists \( b(t) \in P \) such that \( [\exp(tX)n] = \exp(tX)nb(t) \in N \). Hence
\[
\tau(X)f(n) = \frac{d}{dt} f(\exp(tX)nb(t)) \bigg|_{t=0} = \frac{d}{dt} f(nn^{-1}\exp(tX)nb(t)) \bigg|_{t=0} = \frac{d}{dt} f(n_1n_{1/2}n_0n_1^X(t)n_{1/2}^X(t)n_1^0X(t)) \bigg|_{t=0}.
\]
Consider the left-invariant vector fields corresponding to the basis of \( \mathfrak{n} \) chosen in (5) and write \( \tau(X) \) accordingly. Then the image of \( X \) via the isomorphism \( I \) defined in (2) is \( p \), the coefficient along \( Z \) of \( \tau(X) \). We observed that \( n_1^X(0) = e \), for every \( r = 0, 1/2, 1 \). Therefore,
\[
p = \frac{d}{dt} (n_1^X(t)) \bigg|_{t=0}, \tag{12}
\]
and so \( p = A \).

We showed above that \( p(n) \) is obtained in two steps: first we compute the conjugation \( n_1^{-1}n_{1/2}^{-1}\exp(tX)n_1n_{1/2} \) and then write it in the form \( \exp(tA) \exp(tB + o(t)) \), where \( A \in \mathfrak{n}_1 \) and \( B \) has no components along \( \mathfrak{n}_1 \), according to the decomposition (3).

We shall obtain explicit formulas for the homogeneous polynomials corresponding to \( \mathfrak{g} \) using (12). We consider separately the cases with \( \alpha \) in \( \Sigma_0 \), \( \Sigma_{1/2} \), \( \Sigma_1 \), \{0\}, \( -\Sigma_1 \), \( -\Sigma_{1/2} \), \( -\Sigma_0 \). The resulting polynomials are a basis of the space \( \mathcal{P} \) and we collect them in the next proposition. We define on \( \Sigma_{1/2} \) the equivalence relation \( \sim \) given by
\[
\alpha \sim \beta \Leftrightarrow \alpha + \beta = \omega,
\]
and we choose one representative for each element of the quotient \( (\Sigma_{1/2}/\sim) \).
Denote the set of such representatives by \( \tilde{\Sigma}_{1/2} \).

**Proposition 3.** Denote \( p^\gamma = I(X_\alpha) \) for every \( X_\alpha \in \mathfrak{g}_\alpha \) and every non zero root \( \alpha \) and \( p^H = I(H) \) for every \( H \in \mathfrak{a} \). We write \( c_{\alpha,\beta} \) for the structure constants of \( [X_\alpha, X_\beta] \) and \( H_\gamma \) for the unique element in \( \mathfrak{a} \) for which \( \gamma(H_\gamma) = 1 \). Then the following formulas hold.
(i) If \( \gamma \in \Sigma_{1/2} \), then \( p^\gamma(n) = c_{\gamma,\omega-\gamma}y_{\omega-\gamma} \).
(ii) If \( H \in \mathfrak{a} \), then
\[
p^H(n) = \omega(H)z - \frac{1}{2} \sum_{\alpha \in \tilde{\Sigma}_{1/2}} y_\alpha y_{\omega-\alpha} ((\omega - \alpha)(H) - \alpha(H)) c_{\alpha,\omega-\alpha}.
\]
(iii) \( p^\nu(n) = 1 \).

(iv) If \( \nu \in \Sigma_0 \cup -\Sigma_0 \), then

\[
p^\nu(n) = \frac{1}{2} \sum_{\nu + \alpha_1 + \alpha_2 = \omega} c_{\alpha_1, \nu} c_{\alpha_2, \nu + \alpha_1} y_{\alpha_1} y_{\alpha_2},
\]

where \( \alpha_1 \) and \( \alpha_2 \) vary in \( \Sigma_{1/2} \).

(v) If \( \gamma \in \Sigma_{1/2} \), then

\[
p^{-\gamma}(n) = -\omega(H_\gamma) y_\gamma z \sum_{\alpha \in \Sigma_{1/2}} \alpha(H_\gamma) c_{\omega - \alpha} y_\alpha y_{\omega - \alpha}
- \frac{1}{6} \sum_{-\gamma + \alpha_1 + \alpha_2 + \alpha_3 = \omega} c_{\alpha_1, -\gamma} c_{\alpha_2, -\gamma + \alpha_1} c_{\alpha_3, -\gamma + \alpha_1 + \alpha_2} y_{\alpha_1} y_{\alpha_2} y_{\alpha_3},
\]

with \( \alpha_1, \alpha_2, \alpha_3 \in \Sigma_{1/2} \).

(vi) Finally,

\[
p^{-\omega}(n) = -\frac{1}{2} y_\omega^2 (H_\omega) + \frac{1}{2} \sum_{\alpha \in \Sigma_{1/2}} c_{\omega - \alpha} \alpha(H_\omega) y_\alpha y_{\omega - \alpha}
+ \frac{t}{24} \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \Sigma_{1/2}} c_{\alpha_1, -\omega} c_{\alpha_2, -\omega + \alpha_1} c_{\alpha_3, -\omega + \alpha_1 + \alpha_2} c_{\alpha_4, -\omega + \alpha_1 + \alpha_2 + \alpha_3} y_{\alpha_1} y_{\alpha_2} y_{\alpha_3} y_{\alpha_4},
\]

PROOF. (i) We will repeatedly use the following simple observation: if \( \alpha, \gamma \in \Sigma_{1/2} \) and \( \alpha + \gamma \in \Sigma \), then \( \gamma = \omega - \alpha \). Indeed \( (\alpha + \gamma, \omega) = (\alpha, \omega) + (\gamma, \omega) = (\omega, \omega) \). This implies that \( \alpha = \omega - \gamma \). Since \([Z, n] = 0\),

\[
n_{1/2}^{-1} n_1^{-1} \exp (tY_\gamma) n_{1/2} n_1 n_{1/2} = n_{1/2}^{-1} \exp \left( \sum_{n=0}^{+\infty} \frac{(ad z Z)^n}{n!} tY_\gamma \right) n_{1/2}
= n_{1/2}^{-1} \exp (tY_\gamma) n_{1/2}
= \exp \left( \sum_{n=0}^{+\infty} \frac{(ad (\sum_{\alpha \in \Sigma_{1/2}} y_\alpha Y_\alpha))^n}{n!} tY_\gamma \right)
= \exp (tY_\gamma - ty_{\omega - \gamma} [Y_{\omega - \gamma}, Y_\gamma])
= \exp (t c_{\gamma, \omega - \gamma} y_{\omega - \gamma} Z) \exp (tY_\gamma).
\]

By (12) and the remark thereafter, we have \( p^\gamma(n) = c_{\gamma, \omega - \gamma} y_{\omega - \gamma} \).
(ii) Since \([n_{1/2}, n_{1/2}] \subseteq n_1\), every bracket involving three or more vectors in \(n_{1/2}\) is zero. If \(H \in a\), then
\[
n_{1/2}^{-1} n_{1/2}^{-1} \exp (tH) n_{1/2} n_{1/2} = n_{1/2}^{-1} \exp (tH - tz[Z, H]) n_{1/2}
\]
\[
= \exp(t\alpha(H) zZ) \exp \left( t - \sum_{\alpha \in \Sigma_{1/2}} y_\alpha [Y_\alpha, H] + t/2 \sum_{\alpha + \beta = \omega} y_\alpha y_\beta [Y_\beta, [Y_\alpha, H]] \right)
\]
\[
= \exp \left( \alpha(H) z + \frac{1}{2} \sum_{\alpha + \beta = \omega} \alpha(H) c_{\alpha, \beta} y_\alpha y_\beta \right) tZ ...
\]
where the only relevant component is the linear term in \(t\) along \(Z\). Therefore
\[
p^H(n) = \omega(H) z - \frac{1}{2} \sum_{\alpha \in \Sigma_{1/2}} y_\alpha y_{\omega - \alpha} ((\omega - \alpha)(H) - \alpha(H)) c_{\alpha, \omega - \alpha},
\]
as required.

(iii) Since \([Z, n] = 0\), the conclusion is obvious.

(iv) If \(\alpha \in \Sigma_{1/2}\), then \((\nu + \alpha, \omega) = (\nu, \omega) + (\alpha, \omega) = \frac{1}{2} (\omega, \omega)\), whence \(\nu + \alpha \in \Sigma_{1/2}\), provided it is a root. Therefore
\[
n_{1/2}^{-1} n_{1/2}^{-1} \exp (tX_\nu) n_{1/2} n_{1/2} = n_{1/2}^{-1} \exp (tX_\nu) n_{1/2}
\]
\[
= \exp \left( tX_\nu - \sum_{\alpha \in \Sigma_{1/2}} y_\alpha [Y_\alpha, X_\nu] + \frac{t}{2} \sum_{\alpha_1, \alpha_2 \in \Sigma_{1/2}} y_{\alpha_1} y_{\alpha_2} [Y_{\alpha_2}, [Y_{\alpha_1}, X_\nu]] \right)
\]
\[
= \exp \left( \frac{t}{2} \sum_{\nu + \alpha_1 + \alpha_2 = \omega} c_{\alpha_1, \nu} c_{\alpha_2, \nu + \alpha_1} y_{\alpha_1} y_{\alpha_2} Z \right) \exp \left( tX_\nu - \sum_{\alpha \in A} c_{\alpha, \nu} y_\alpha Y_{\alpha + \nu} \right)
\]
So (12) gives \(p^\nu(n) = \frac{1}{2} \sum_{\nu + \alpha_1 + \alpha_2 = \omega} c_{\alpha_1, \nu} c_{\alpha_2, \nu + \alpha_1} y_{\alpha_1} y_{\alpha_2}\), where \(\alpha_1\) and \(\alpha_2\) are in \(\Sigma_{1/2}\).

(v) Take \(\gamma \in \Sigma_{1/2}\). Then
\[
n_{1/2}^{-1} n_{1/2}^{-1} \exp (tY_{-\gamma}) n_{1/2} n_{1/2} = n_{1/2}^{-1} \exp (tY_{-\gamma} - tc_{\omega, -\gamma} zY_{-\gamma}) n_{1/2}
\]
\[
= \exp \left( tY_{-\gamma} - tc_{\omega, -\gamma} zY_{-\gamma} - t \sum_{\alpha \in \Sigma_{1/2}} y_\alpha [Y_\alpha, Y_{-\gamma}] \right)
\]
\[
+ tz \sum_{\alpha \in \Sigma_{1/2}} c_{\omega, -\gamma} y_\alpha [Y_\alpha, Y_{-\gamma}] + \frac{t}{2} \sum_{\alpha_1, \alpha_2 \in \Sigma_{1/2}} y_{\alpha_1} y_{\alpha_2} [Y_{\alpha_2}, [Y_{\alpha_1}, Y_{-\gamma}]]
\]
\[
+ \frac{t}{6} \sum_{\alpha_1, \alpha_2, \alpha_3 \in \Sigma_{1/2}} y_{\alpha_1} y_{\alpha_2} y_{\alpha_3} [Y_{\alpha_3}, [Y_{\alpha_2}, [Y_{\alpha_1}, Y_{-\gamma}]]].
\]
Since \((-\gamma + \alpha, \omega) = -(\gamma, \omega) + (\alpha, \omega) = -\frac{1}{2}(\omega, \omega) + \frac{1}{2}(\omega, \omega) = 0\) for every \(\alpha \in \Sigma_{1/2}\), it follows that \(-\gamma + \alpha\) is either in \(\pm \Sigma_0\) or 0, or not a root. This implies that the bracket \([Y_{\alpha_1}, Y_{-\gamma}]\) is respectively in \(n_0, a\) or zero. Then (12) yields the desired expression for \(p^{-\gamma}(n)\), since the Jacobi identity implies that \(c_{\omega, -\gamma}c_{\gamma, \omega, -\gamma} = -\omega(H_\gamma)\).

(vi) Notice that in order to obtain \(\omega\) we must add to \(-\omega\) exactly four roots in \(\Sigma_{1/2}\). We have

\[
n^{-1}_1 n^{-1}_1 \exp tX_\omega n_1 n_{1/2} = n^{-1}_1 \exp \left( tX_\omega - tzH_\omega - \frac{t}{2} z^2 \omega(H_\omega)Z \right) n_{1/2} = \exp \left( -\frac{t}{2} z^2 \omega(H_\omega)Z \right) \exp \left( tX_\omega - tzH_\omega - \frac{t}{2} \sum_{\alpha \in \Sigma_{1/2}} c_{\alpha, -\omega} y_\alpha Y_{-\omega+\alpha} \right) - tz \sum_{\alpha \in \Sigma_{1/2}} \alpha(H_\omega)y_\alpha Y_\alpha + \frac{t}{2} \sum_{\alpha_1, \alpha_2 \in \Sigma_{1/2}} y_{\alpha_1} y_{\alpha_2} [Y_{\alpha_2}, [Y_{\alpha_1}, X_\omega]] + \frac{t}{2} \sum_{\alpha \in \Sigma_{1/2}} c_{-\alpha, \alpha}(H_\omega)y_\alpha y_\omega - \alpha Z - \frac{t}{6} \sum_{\alpha_1, \alpha_2, \alpha_3 \in \Sigma_{1/2}} y_{\alpha_1} y_{\alpha_2} y_{\alpha_3} [Y_{\alpha_3}, [Y_{\alpha_2}, [Y_{\alpha_1}, X_\omega]]] + \frac{t}{24} \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \Sigma_{1/2}} y_{\alpha_1} y_{\alpha_2} y_{\alpha_3} y_{\alpha_4} [Y_{\alpha_4}, [Y_{\alpha_3}, [Y_{\alpha_2}, [Y_{\alpha_1}, X_\omega]]]] \right) = \exp \left( \left( -\frac{t}{2} z^2 \omega(H_\omega) + \frac{t}{2} \sum_{\alpha \in \Sigma_{1/2}} c_{-\alpha, \alpha}(H_\omega)y_\alpha y_\omega - \alpha \right) + \frac{t}{24} \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \Sigma_{1/2}} c_{\alpha_1, -\omega} c_{\alpha_2, -\omega + \alpha_1} \times c_{\alpha_3, -\omega + \alpha_1 + \alpha_2} c_{\alpha_4, -\omega + \alpha_1 + \alpha_2 + \alpha_3} y_{\alpha_1} y_{\alpha_2} y_{\alpha_3} y_{\alpha_4} Z \right) \ldots \ .
\]

Therefore (vi) follows. \(\square\)

5. Example

We consider \(\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})\), the simple Lie algebra of real \(3 \times 3\) matrices with zero trace. Its Iwasawa nilpotent Lie algebra \(\mathfrak{n}\) is given by the matrices

\[
\nu(x, y, z) = \begin{bmatrix}
0 & x & z \\
0 & 0 & y \\
0 & 0 & 0
\end{bmatrix},
\]
for \( x, y \) and \( z \) in \( \mathbb{R} \). Notice that this is the Lie algebra of the three dimensional Heisenberg group. Take \( \alpha \) and \( \beta \) to be the simple roots relative to the standard Cartan subspace \( \mathfrak{a} \) of \( \mathfrak{sl}(3, \mathbb{R}) \) of diagonal matrices: \( \alpha(\text{diag}(a, b, c)) = (a - b) \) and \( \beta((\text{diag}(a, b, c)) = (b - c) \). Then

\[
\mathfrak{g}_\alpha = \{ \nu(x, 0, 0) : x \in \mathbb{R} \}, \\
\mathfrak{g}_\beta = \{ \nu(0, y, 0) : y \in \mathbb{R} \}, \\
\mathfrak{g}_{\alpha + \beta} = \{ \nu(0, 0, z) : z \in \mathbb{R} \},
\]

where \( \alpha + \beta \) is the highest root also denoted \( \omega \). The Lie algebra \( \mathfrak{g} \) decomposes as

\[
\mathfrak{g} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha + \beta} \oplus \mathfrak{a} \oplus \theta(\mathfrak{g}_\alpha) \oplus \theta(\mathfrak{g}_\beta) \oplus \theta(\mathfrak{g}_{\alpha + \beta}),
\]

where \( \theta \) is the Cartan involution. We choose the basis of \( \mathfrak{n} \) given by \( X = \nu(1, 0, 0) \), \( Y = \nu(0, 1, 0) \) and \( Z = \nu(0, 0, 1) \) and the basis of \( \mathfrak{a} \)

\[
H_\alpha = \text{diag} \left( \frac{1}{2}, -\frac{1}{2}, 0 \right) \\
H_\beta = \text{diag} \left( 0, \frac{1}{2}, -\frac{1}{2} \right).
\]

We can complete \( \{ X, Y, Z, H_\alpha, H_\beta \} \) to a basis of \( \mathfrak{sl}(3, \mathbb{R}) \) adding \( \theta(X) = -X^t \), \( \theta(Y) = -Y^t \) and \( \theta(Z) = -Z^t \), which are a basis of \( \mathfrak{g}_{-\alpha} \), \( \mathfrak{g}_{-\beta} \) and \( \mathfrak{g}_{-\alpha - \beta} \) respectively. In order to apply the formulas of Proposition 3 to the chosen basis of \( \mathfrak{g} \) we need the structure constants, that can be easily computed, and the vector \( H_\omega = H_\alpha + H_\beta = \text{diag}(1/2, 0, -1/2) \). The indeterminates of the polynomials are the canonical coordinates \( n = (x, y, z) = \exp(zZ)\exp(xX + yY) \). Hence, a straightforward calculation yields the following polynomials.

\[
\begin{align*}
p^\alpha(n) &= y, & p^{H_\alpha}(n) &= \frac{1}{2} z + \frac{3}{4} xy, & p^{-\alpha}(n) &= -\frac{1}{2} yz + \frac{1}{12} x^2 y, \\
p^\beta(n) &= -x, & p^{H_\beta}(n) &= \frac{1}{2} z - \frac{1}{4} xy, & p^{-\beta}(n) &= -\frac{1}{2} yz + \frac{5}{4} xy^2, \\
p^{\alpha + \beta}(n) &= 1, & p^{-\alpha - \beta}(n) &= -\frac{1}{2} z^2 - \frac{1}{6} x^2 y^2.
\end{align*}
\]

References

Polynomial bases of split simple Lie algebras


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