Real hypersurfaces in complex two-plane Grassmannians with parallel normal Jacobi operator

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Abstract. In this paper we give a non-existence theorem for Hopf hypersurfaces in complex two-plane Grassmannians \( G_2(C^{m+2}) \) with parallel normal Jacobi operator \( \tilde{R}_N \).

1. Introduction

In the geometry of real hypersurfaces in complex space forms or in quaternionic space forms there have been many characterizations of homogeneous hypersurfaces of type \((A_1), (A_2), (B), (C), (D) \) and \((E)\) in complex projective space \( P_m(C) \), of type \((A_0),(A_1), (A_2) \) and \((B)\) in complex hyperbolic space \( H_m(C) \) or of type \((A_1),(A_2) \) and \((B)\) in quaternionic projective space \( QP^m \), which are completely classified by Cecil and Ryan [6], Kimura [9], Kimura and Maeda [10], Berndt [2], Martinez and Pérez [11] respectively.

On the other hand, Jacobi fields along geodesics of a given Riemannian manifold \( (\tilde{M}, \tilde{g}) \) satisfy an well-known differential equation. This classical differential equation naturally inspires the so-called Jacobi operator. That is, if \( \tilde{R} \) is the curvature operator of \( \tilde{M} \), and \( X \) is any tangent vector field to \( \tilde{M} \), the Jacobi operator with respect to \( X \) at \( p \in \tilde{M} \), \( \tilde{R}_X \in \text{End}(T_p\tilde{M}) \), is defined by

\[
(\tilde{R}_X Y)(p) = (\tilde{R}(Y, X)X)(p)
\]

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for any $Y \in T_p M$, becomes a self adjoint endomorphism of the tangent bundle $TM$ of $M$. Clearly, each tangent vector field $X$ to $M$ provides a Jacobi operator with respect to $X$.

In a complex space form $M_n(c), c \neq 0$, Ki, Pérez, Santos and Suh [8] have investigated real hypersurfaces $M$ in $M_n(c)$ under the condition that $\nabla_\xi S = 0$ and $\nabla_\xi R_\xi = 0$, where $S$ and $R_\xi$ respectively denote the Ricci tensor and the structure Jacobi operator of $M$ in $M_n(c)$. The almost contact structure vector field $\xi$ are defined by $\xi = -JN$, where $N$ denotes a unit normal to $M$ and $J$ a Kaehler structure on $M_n(c)$. Moreover, Pérez, Santos and Suh [13] gave a complete classification of real hypersurfaces in complex projective space whose structure Jacobi operator $R_\xi$ is Lie $\xi$-parallel, that is, $\mathcal{L}_\xi R_\xi = 0$.

In a quaternionic projective space $QP^m$ Pérez and Suh [12] have classified real hypersurfaces in $QP^m$ with $\mathfrak{D}^\perp$-parallel curvature tensor $\nabla_\xi R = 0$, $i = 1, 2, 3$, where $R$ denotes the curvature tensor of $M$ in $QP^m$ and $\mathfrak{D}^\perp$ a distribution defined by $\mathfrak{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$. In such a case they are congruent to a tube of radius $\frac{\pi}{4}$ over a totally geodesic $QP^k$ in $QP^m, 2 \leq k \leq m - 2$.

The almost contact structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ are defined by $\xi_i = -J_iN, i = 1, 2, 3$, where $\{J_1, J_2, J_3\}$ denote a quaternionic Kähler structure of $QP^m$ and $N$ a unit normal field of $M$ in $QP^m$. In quaternionic space forms Berndt [2] has introduced the notion of normal Jacobi operator

$$\tilde{R}_N = \tilde{R}(X, N)N \in \text{End } T_x M, \quad x \in M$$

for real hypersurfaces $M$ in a quaternionic projective space $QP^m$ or in a quaternionic hyperbolic space $QH^m$, where $\tilde{R}$ denotes the curvature tensor of $QP^m$ and $QH^m$ respectively. He [2] has also shown that the curvature adaptedness, that is, the normal Jacobi operator $\tilde{R}_N$ commutes with the shape operator $A$, is equivalent to the fact that the distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant by the shape operator $A$ of $M$, where $T_x M = \mathfrak{D} \oplus \mathfrak{D}^\perp, x \in M$.

Now let us consider a complex two-plane Grassmannians $G_2(C^{m+2})$ which consists of all complex 2-dimensional linear subspaces in $C^{m+2}$. Then the situation for real hypersurfaces in $G_2(C^{m+1})$ related to the normal Jacobi operator $\tilde{R}_N$ is not so simple and will be quite different from the cases mentioned above. In a paper [7] due to Jeong, Suh and Pérez we have classified real hypersurfaces in $G_2(C^{m+2})$ with commuting normal Jacobi operator, that is, $\tilde{R}_N \circ \phi = \phi \circ \tilde{R}_N$ or $\tilde{R}_N \circ A = A \circ \tilde{R}_N$. The normal Jacobi operator $\tilde{R}_N$ commutes with the shape operator $A$ or the structure tensor $\phi$ of $M$ in $G_2(C^{m+2})$ means that the eigenspaces of the normal Jacobi operator is invariant by the shape operator $A$ or the structure tensor $\phi$.
In this paper we consider a real hypersurface $M$ in complex two-plane Grassmannians $G_2(C^{m+2})$ with parallel normal Jacobi operator, that is, $\nabla_X \bar{R}_N = 0$ for any tangent vector field $X$ on $M$, where $\nabla$, $\bar{R}$ and $N$ respectively denotes the induced Riemannian connection on $M$, the curvature tensor of the ambient space $G_2(C^{m+2})$ and a unit normal vector of $M$ in $G_2(C^{m+2})$. The normal Jacobi operator $\bar{R}_N$ is parallel on $M$ in $G_2(C^{m+2})$ means that the eigenspaces of the normal Jacobi operator $\bar{R}_N$ is parallel along any curve $\gamma$ in $M$. Here the eigenspaces of the normal Jacobi operator $\bar{R}_N$ are said to be parallel along $\gamma$ if they are invariant with respect to any parallel displacement along $\gamma$.

The curvature tensor $\bar{R}(X,Y)Z$ for any vector fields $X, Y$ and $Z$ on $G_2(C^{m+2})$ is explicitly defined in Section 2. Then the normal Jacobi operator $\bar{R}_N$ for the unit normal vector $N$ can be defined from the curvature tensor $\bar{R}(X,N)N$ by putting $Y = Z = N$.

The complex two-plane Grassmannians $G_2(C^{m+2})$ is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure $J$ and a quaternionic Kähler structure $J$ not containing $J$ (See Berndt [3]). So, in $G_2(C^{m+2})$ we have two natural geometric conditions for real hypersurfaces that $[\xi] = \text{Span}\{\xi\}$ or $\mathcal{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator. By using such conditions Berndt and Suh [4] have proved the following:

**Theorem 1.1.** Let $M$ be a connected real hypersurface in $G_2(C^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathcal{D}^\perp$ are invariant under the shape operator of $M$ if and only if

(A) $M$ is an open part of a tube around a totally geodesic $G_2(C^{m+1})$ in $G_2(C^{m+2})$, or

(B) $m$ is even, say $m = 2n$, and $M$ is an open part of a tube around a totally geodesic $QP^n$ in $G_2(C^{m+2})$.

The structure vector field $\xi$ of a real hypersurface $M$ in $G_2(C^{m+2})$ is said to be a Reeb vector field. If the Reeb vector field $\xi$ of a real hypersurface $M$ in $G_2(C^{m+2})$ is invariant by the shape operator, $M$ is said to be a Hopf hypersurface. In such a case the integral curves of the Reeb vector field $\xi$ are geodesics (See Berndt and Suh [5]). Moreover, the flow generated by the integral curves of the structure vector field $\xi$ for Hopf hypersurfaces in $G_2(C^{m+2})$ is said to be geodesic Reeb flow. Moreover, the corresponding principal curvature $\alpha$ is non-vanishing we say $M$ is with non-vanishing geodesic Reeb flow.

Now by putting a unit normal vector $N$ into the curvature tensor $\bar{R}$ of the ambient space $G_2(C^{m+2})$, we calculate the normal Jacobi operator $\bar{R}_N$ in such a
way that
\[
\bar{R}_N X = \bar{R}(X, N) N = X + 3\eta(X)\xi + 3\sum_{\nu=1}^3 \eta_\nu(X)\xi_\nu
\]
\[
- \sum_{\nu=1}^3 \{ \eta_\nu(\xi) J_\nu(\phi X + \eta(X) N) - \eta_\nu(\phi X)(\phi_\nu(\xi) + \eta_\nu(\xi)) \}
\]
\[
= X + 3\eta(X)\xi + 3\sum_{\nu=1}^3 \eta_\nu(X)\xi_\nu
\]
\[
- \sum_{\nu=1}^3 \{ \eta_\nu(\xi)(\phi_\nu(\phi X) - \eta(X)\xi_\nu) - \eta_\nu(\phi X)\phi_\nu(\xi) \}
\]
for any tangent vector field \(X\) on \(M\) in \(G_2(\mathbb{C}^{m+2})\).

We say that the normal Jacobi operator \(\bar{R}_N\) is parallel on \(M\) if the covariant derivative of the normal Jacobi operator \(\bar{R}_N\) identically vanishes, that is, \(\nabla_X \bar{R}_N = 0\) for any vector field \(X\) on \(M\). Related to such a parallel normal Jacobi operator \(\bar{R}_N\) of \(M\) in \(G_2(\mathbb{C}^{m+2})\), in Section 4 we prove an important theorem for hypersurfaces in \(G_2(\mathbb{C}^{m+2})\) as follows:

**Theorem 1.2.** Let \(M\) be a Hopf hypersurface in \(G_2(\mathbb{C}^{m+2})\), \(m \geq 3\), with parallel normal Jacobi operator. Then \(\xi\) belongs to either the distribution \(\mathcal{D}\) or the distribution \(\mathcal{D}^\perp\).

In Sections 5 and 6 we respectively prove a non-existence theorem for real hypersurfaces in \(G_2(\mathbb{C}^{m+2})\), \(m \geq 3\), when the Reeb vector \(\xi\) belongs to the distribution \(\mathcal{D}\) or the distribution \(\mathcal{D}^\perp\). Then we assert the following

**Theorem 1.3.** There do not exist any Hopf hypersurfaces in \(G_2(\mathbb{C}^{m+2})\), \(m \geq 3\), with parallel normal Jacobi operator.

## 2. Riemannian geometry of \(G_2(\mathbb{C}^{m+2})\)

In this section we summarize basic material about \(G_2(\mathbb{C}^{m+2})\), for details we refer to [3], [4] and [5].

By \(G_2(\mathbb{C}^{m+2})\) we denote the set of all complex two-dimensional linear subspaces in \(\mathbb{C}^{m+2}\). The special unitary group \(G = SU(m + 2)\) acts transitively on \(G_2(\mathbb{C}^{m+2})\) with stabilizer isomorphic to \(K = S(U(2) \times U(m)) \subset G\). Then \(G_2(\mathbb{C}^{m+2})\) can be identified with the homogeneous space \(G/K\), which we equip with the unique analytic structure for which the natural action of \(G\) on \(G_2(\mathbb{C}^{m+2})\)
becomes analytic. Denote by \( g \) and \( \mathfrak{f} \) the Lie algebra of \( G \) and \( K \), respectively, and by \( \mathfrak{m} \) the orthogonal complement of \( \mathfrak{f} \) in \( g \) with respect to the Cartan–Killing form \( B \) of \( g \). Then \( g = \mathfrak{f} \oplus \mathfrak{m} \) is an \( \text{Ad}(K) \)-invariant reductive decomposition of \( g \).

We put \( o = eK \) and identify \( T_o G_2^+(\mathbb{C}^m+2) \) with \( \mathfrak{m} \) in the usual manner. Since \( B \) is negative definite on \( g \), its negative restricted to \( \mathfrak{m} \times \mathfrak{m} \) yields a positive definite inner product on \( \mathfrak{m} \). By \( \text{Ad}(K) \)-invariance of \( B \) this inner product can be extended to a \( G \)-invariant Riemannian metric \( g \) on \( G_2^+(\mathbb{C}^m+2) \).

In this way \( G_2^+(\mathbb{C}^m+2) \) becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize \( g \) such that the maximal sectional curvature of \( (G_2^+(\mathbb{C}^m+2), g) \) is eight. Since \( G_2^+(\mathbb{C}^3) \) is isometric to the two-dimensional complex projective space \( \mathbb{C}P^2 \) with constant holomorphic sectional curvature eight we will assume \( m \geq 2 \) from now on. Note that the isomorphism \( \text{Spin}(6) \simeq SU(4) \) yields an isometry between \( G_2^+(\mathbb{C}^4) \) and the real Grassmann manifold \( G_2^+(\mathbb{R}^6) \) of oriented two-dimensional linear subspaces of \( \mathbb{R}^6 \).

The Lie algebra \( \mathfrak{f} \) has the direct sum decomposition \( \mathfrak{f} = su(m) \oplus su(2) \oplus \mathfrak{R} \), where \( \mathfrak{R} \) is the center of \( \mathfrak{f} \). Viewing \( \mathfrak{f} \) as the holonomy algebra of \( G_2(\mathbb{C}^{m+2}) \), the center \( \mathfrak{R} \) induces a Kähler structure \( J \) and the \( su(2) \)-part a quaternionic Kähler structure \( \mathfrak{J} \) on \( G_2(\mathbb{C}^{m+2}) \).

If \( J_1 \) is any almost Hermitian structure in \( \mathfrak{J} \), then \( JJ_1 = J_1 J \), and \( JJ_1 \) is a symmetric endomorphism with \((JJ_1)^2 = I\) and \( \text{tr}(JJ_1) = 0 \). This fact will be used frequently throughout this paper.

A canonical local basis \( J_1, J_2, J_3 \) of \( \mathfrak{J} \) consists of three local almost Hermitian structures \( J_\nu \) in \( \mathfrak{J} \) such that \( J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu \), where the index is taken modulo three. Since \( \mathfrak{J} \) is parallel with respect to the Riemannian connection \( \nabla \) of \((G_2(\mathbb{C}^{m+2}), g)\), there exist for any canonical local basis \( J_1, J_2, J_3 \) of \( \mathfrak{J} \) three local one-forms \( q_1, q_2, q_3 \) such that

\[
\nabla_X J_\nu = q_{\nu+2}(X) J_{\nu+1} - q_{\nu+1}(X) J_{\nu+2}
\]

for all vector fields \( X \) on \( G_2(\mathbb{C}^{m+2}) \).

The Riemannian curvature tensor \( \bar{R} \) of \( G_2^+(\mathbb{C}^m+2) \) is locally given by

\[
\bar{R}(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX
- g(JX,Z)JY - 2g(JX,Y)JZ
+ \sum_{\nu=1}^3 \{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\}
+ \sum_{\nu=1}^3 \{g(J_\nu Y,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},
\]
where $J_1, J_2, J_3$ is any canonical local basis of $\mathcal{J}$.

### 3. Some fundamental formulas

In this section we derive some basic formulae from the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (See [4], [5], [14], [15] and [16]).

Let $M$ be a real hypersurface of $G_2(\mathbb{C}^{m+2})$. The induced Riemannian metric on $M$ will also be denoted by $g$, and $\nabla$ denotes the Riemannian connection of $(M, g)$. Let $N$ be a local unit normal field of $M$ and $A$ the shape operator of $M$ with respect to $N$. The Kähler structure $J$ of $G_2(\mathbb{C}^{m+2})$ induces on $M$ an almost contact metric structure $(\phi, \xi, \eta, g)$. Furthermore, let $J_1, J_2, J_3$ be a canonical local basis of $\mathcal{J}$. Then each $J_\nu$ induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on $M$. Using the above expression for $\bar{R}$, the Codazzi equation becomes

\[
(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi + \sum_{\nu=1}^{3} \left\{ \eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu \right\} + \sum_{\nu=1}^{3} \left\{ \eta_\nu(\phi X)\phi_\nu Y - \eta_\nu(\phi Y)\phi_\nu X \right\} + \sum_{\nu=1}^{3} \left\{ \eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X) \right\} \xi_\nu.
\]

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

\[
\begin{align*}
\phi_{\nu+1} \xi_\nu &= -\xi_{\nu+2}, & \phi_\nu \xi_{\nu+1} &= \xi_{\nu+2}, \\
\phi_\nu \xi_\nu &= \phi_\nu \xi, & \eta_\nu(\phi X) &= \eta(\phi_\nu X), \\
\phi_\nu \phi_{\nu+1} X &= \phi_{\nu+2} X + \eta_{\nu+1}(X) \xi_\nu, \\
\phi_{\nu+1} \phi_\nu X &= -\phi_{\nu+2} X + \eta_\nu(X) \xi_{\nu+1}. & (3)
\end{align*}
\]

Now let us put

\[
JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \quad (4)
\]
for any tangent vector $X$ of a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$, where $N$ denotes a normal vector of $M$ in $G_2(\mathbb{C}^{m+2})$. Then from this and the formulas (1) and (3) we have that

$$(\nabla_X \phi) Y = \eta(Y) AX - g(AX, Y) \xi,$$  \hspace{0.5cm} (5)

$$\nabla_X \xi_{\nu} = q_{\nu+2}(X) \xi_{\nu+1} - q_{\nu+1}(X) \xi_{\nu+2} + \phi_{\nu} AX,$$  \hspace{0.5cm} (6)

$$(\nabla_X \phi_{\nu}) Y = -q_{\nu+1}(X) \phi_{\nu+2} Y + q_{\nu+2}(X) \phi_{\nu+1} Y + \eta_{\nu}(Y) AX - g(AX, Y) \xi_{\nu}.$$  \hspace{0.5cm} (7)

Moreover, from $J J_{\nu} = J_{\nu} J$, $\nu = 1, 2, 3$, it follows that

$$\phi \phi_{\nu} X = \phi_{\nu} \phi X + \eta_{\nu}(X) \xi - \eta(X) \xi_{\nu}.$$  \hspace{0.5cm} (8)

4. Parallel normal Jacobi operator

Now in this section we want to derive the normal Jacobi operator from the curvature tensor $\bar{R}(X, Y) Z$ of complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ given in (2).

Now let us consider a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ with parallel normal Jacobi operator $\bar{R}_N$, that is, $\nabla_X \bar{R}_N = 0$ for any vector field $X$ on $M$. Then first of all, we write the normal Jacobi operator $\bar{R}_N$, which is given by

$$\bar{R}_N(X) = \bar{R}(X, N) N = X + 3\eta(X) \xi + 3 \sum_{\nu=1}^{3} \eta_{\nu}(X) \xi_{\nu}$$

$$- \sum_{\nu=1}^{3} \left\{ \eta_{\nu}(\xi) J_{\nu}(\phi X + \eta(X) N) - \eta_{\nu}(\phi X)(\phi_{\nu} \xi + \eta_{\nu}(\xi) N) \right\}$$

$$= X + 3\eta(X) \xi + 3 \sum_{\nu=1}^{3} \eta_{\nu}(X) \xi_{\nu}$$

$$- \sum_{\nu=1}^{3} \left\{ \eta_{\nu}(\xi)(\phi_{\nu} \phi X - \eta(X) \xi_{\nu}) - \eta_{\nu}(\phi X) \phi_{\nu} \xi \right\},$$

where we have used the following

$$g(J_{\nu} J N, N) = -g(J N, J_{\nu} N) = -g(\xi, \xi_{\nu}) = -\eta_{\nu}(\xi),$$

$$g(J_{\nu} J X, N) = g(X, J J_{\nu} N) = -g(X, J \xi_{\nu}) = -g(X, \phi \xi_{\nu} + \eta(\xi_{\nu}) N) = -g(X, \phi \xi_{\nu})$$
and

\[ J_\nu J N = -J_\nu \xi = -\phi_\nu \xi - \eta_\nu(\xi) N. \]

Of course, by (8) we know that the normal Jacobi operator \( \bar{R}_N \) could be symmetric endomorphism of \( T_x M, \ x \in M. \)

Now let us consider a covariant derivative of the normal Jacobi operator \( \bar{R}_N \) along the direction \( X. \) Then it is given by

\[
(\nabla_X \bar{R}_N)Y = 3(\nabla_X \eta)(Y)\xi + 3\eta(Y)\nabla_X \xi + 3\sum_{\nu=1}^{3} (\nabla_X \eta_\nu)(Y)\xi_\nu
\]

\[
+ 3\sum_{\nu=1}^{3} \eta_\nu(Y)\nabla_X \xi_\nu - \sum_{\nu=1}^{3} \left[X(\eta_\nu(\xi))((\phi_\nu \phi Y - \eta(Y)\xi_\nu)
\right.
\]

\[
+ \eta_\nu(\xi)\left\{ (\nabla_X \phi_\nu)Y - (\nabla_X \eta)(Y)\xi_\nu - \eta(Y)\nabla_X \xi_\nu \right\}
\]

\[
- (\nabla_X \eta_\nu)(\phi Y)\phi_\nu \xi - \eta_\nu((\nabla_X \phi)\phi_\nu \xi - \eta_\nu(\phi Y)\nabla_X (\phi_\nu \xi)),
\]

where the formula \( X(\eta_\nu(\xi)) \) in the right side is given by

\[
X(\eta_\nu(\xi)) = g(\nabla_X \xi_\nu, \xi) + g(\xi_\nu, \nabla_X \xi)
\]

\[
= q_{\nu+2}(X)\eta_{\nu+1}(\xi) - q_{\nu+1}(X)\eta_{\nu+2}(\xi) + 2g(\phi_\nu AX, \xi).
\]

From this, together with the formulas given in Section 3, a real hypersurface \( M \) in \( G_2(\mathbb{C}^{n+2}) \) with parallel normal Jacobi operator, that is, \( \nabla_X \bar{R}_N = 0 \) for any vector field \( X \) on \( M, \) satisfies the following

\[
0 = 3g(\phi AX, Y)\xi + 3\eta(Y)\phi AX
\]

\[
+ 3\sum_{\nu=1}^{3} \left\{ q_{\nu+2}(X)\eta_{\nu+1}(Y) - q_{\nu+1}(X)\eta_{\nu+2}(Y) + g(\phi_\nu AX, Y) \right\} \xi_\nu
\]

\[
+ 3\sum_{\nu=1}^{3} \eta_\nu(Y) \left\{ q_{\nu+2}(X)\xi_\nu+1 - q_{\nu+1}(X)\xi_\nu+2 + \phi_\nu AX \right\}
\]

\[
- \sum_{\nu=1}^{3} \left\{ q_{\nu+2}(X)\eta_{\nu+1}(\xi) - q_{\nu+1}(X)\eta_{\nu+2}(\xi) + 2\eta_\nu(\phi AX) \right\} (\phi_\nu \phi Y - \eta(Y)\xi_\nu)
\]

\[
+ \eta_\nu(\xi) \left\{ -q_{\nu+1}(X)\phi_{\nu+2} \phi Y + q_{\nu+2}(X)\phi_{\nu+1} \phi Y + \eta_\nu(\phi Y) AX - g(AX, \phi Y)\xi_\nu
\]

\[
+ \eta(Y)\phi_\nu AX - g(AX, Y)\phi_\nu \xi - g(\phi AX, Y)\xi_\nu
\]

\[
- \eta(Y)(q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX) \right\}
\]

\[
- \left\{ q_{\nu+2}(X)\eta_{\nu+1}(\phi Y) - q_{\nu+1}(X)\eta_{\nu+2}(\phi Y) + g(\phi_\nu AX, \phi Y) \right\} \phi_\nu \xi
\]
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\[ - \{ \eta(Y)\eta_\nu(AX) - g(AX, Y)\eta_\nu(\xi) \} \phi_\nu \xi \
- \eta_\nu(\phi Y) \{ q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi \n+ \phi_\nu \phi AX - g(AX, \xi)\xi_\nu + \eta(\xi_\nu AX) \} \]. (9)

Put \( Y = \xi \) in (9), then it follows that

\[ 0 = 3\phi AX + 3 \sum_{\nu=1}^3 \{ q_{\nu+2}(X)\eta_{\nu+1}(\xi) - q_{\nu+1}(X)\eta_{\nu+2}(\xi) + g(\phi_\nu AX, \xi) \} \xi_\nu \n+ 3 \sum_{\nu=1}^3 \eta_\nu(\xi) \{ q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX \} \xi_\nu \n+ \sum_{\nu=1}^3 \eta_\nu(\xi) \{ q_{\nu+2}(X)\eta_{\nu+1}(\xi) - q_{\nu+1}(X)\eta_{\nu+2}(\xi) + 2\eta_\nu(\phi AX) \} \xi_\nu \n- \sum_{\nu=1}^3 \eta_\nu(\xi) \phi_\nu AX + \sum_{\nu=1}^3 \eta_\nu(\xi) \eta(AX) \phi_\nu \xi \n+ \sum_{\nu=1}^3 \{ \eta_\nu(AX) - \eta_\nu(\xi) \eta(AX) \} \phi_\nu \xi. \]

From this we have

\[ 0 = 3\phi AX + 4 \sum_{\nu=1}^3 \{ q_{\nu+2}(X)\eta_{\nu+1}(\xi) - q_{\nu+1}(X)\eta_{\nu+2}(\xi) \} \xi_\nu \n+ 4 \sum_{\nu=1}^3 \eta_\nu(\xi) \{ q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} \} + 5 \sum_{\nu=1}^3 \eta_\nu(\phi AX) \xi_\nu \n+ 3 \sum_{\nu=1}^3 \eta_\nu(\xi) \phi_\nu AX + \sum_{\nu=1}^3 \eta_\nu(AX) \phi_\nu \xi. \] (10)

On the other hand, we know that

\[ 4 \sum_{\nu=1}^3 \{ q_{\nu+2}(X)\eta_{\nu+1}(\xi) - q_{\nu+1}(X)\eta_{\nu+2}(\xi) \} \xi_\nu \n+ 4 \sum_{\nu=1}^3 \eta_\nu(\xi) \{ q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} \} = 0. \]
Then (10) reduces to

\[ 0 = 3\phi AX + 5\sum_{\nu=1}^{3} \eta_{\nu}(\phi AX)\xi_{\nu} + 3\sum_{\nu=1}^{3} \eta_{\nu}(\xi)\phi_{\nu} AX + \sum_{\nu=1}^{3} \eta_{\nu}(AX)\phi_{\nu}\xi. \]  

(11)

If we assume that \( M \) is a Hopf, then by putting \( X = \xi \) in (11) we have

\[ 4\alpha \sum_{\nu=1}^{3} \eta_{\nu}(\xi)\phi_{\nu}\xi = 0. \]

From this it follows that

\[ \alpha = 0 \quad \text{or} \quad \sum_{\nu=1}^{3} \eta_{\nu}(\xi)\phi_{\nu}\xi = 0. \]  

(12)

Now without loss of generality we may put the Reeb vector filed \( \xi \) in such a way that

\[ \xi = \eta(\xi_{0})\xi_{0} + \eta(\xi_{1})\xi_{1} \]

for some \( \xi_{0} \in \mathcal{D} \) and \( \xi_{1} \in \mathcal{D}^{\perp} \). Then the latter formula of (12) becomes

\[ 0 = \eta(\xi_{1})\phi_{1}\xi = \eta(\xi_{0})\eta(\xi_{1})\phi_{1}\xi_{0}. \]

This gives that \( \eta(\xi_{0}) = 0 \) or \( \eta(\xi_{1}) = 0 \), which means \( \xi \in \mathcal{D}^{\perp} \) or \( \xi \in \mathcal{D} \). Summing up above facts, we summarize such a situation as follows:

**Lemma 4.1.** Let \( M \) be a Hopf hypersurface in \( G_{2}(\mathbb{C}^{m+2}) \), \( m \geq 3 \), with parallel normal Jacobi operator. Then the Reeb vector \( \xi \) belongs to the distribution \( \mathcal{D} \) or the distribution \( \mathcal{D}^{\perp} \) unless the geodesic Reeb flow is non-vanishing.

When the geodesic Reeb flow is vanishing, that is \( \alpha = 0 \), we can differentiate \( A\xi = 0 \). Then by a theorem due to Berndt and Suh [5] we know that

\[ \sum_{\nu=1}^{3} \eta_{\nu}(\xi)\phi_{\nu}\xi = 0. \]

This also gives \( \xi \in \mathcal{D} \) or \( \xi \in \mathcal{D}^{\perp} \). From this, together with Lemma 4.1, we give a complete proof of Theorem 1.2 mentioned in the introduction.
5. Parallel normal Jacobi operator for $\xi \in \mathcal{D}$

In this section we want to prove the following proposition

**Proposition 5.1.** Let $M$ be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel normal Jacobi operator and $\xi \in \mathcal{D}$. Then $g(A\mathcal{D}, \mathcal{D}^\perp) = 0$.

**Proof.** By Lemma 4.1, let us consider the case that $\xi \in \mathcal{D}$ in (9). Then we have

$$0 = 3g(\phi AX, Y)\xi + 3\eta(Y)\phi AX + 3\sum_{\nu=1}^{3}\{q_{\nu+2}(X)\eta_{\nu+1}(Y) - q_{\nu+1}(X)\eta_{\nu+2}(Y) + g(\phi_\nu AX, Y)\}\xi_{\nu}$$

$$+ 3\sum_{\nu=1}^{3}\eta_{\nu}(Y)\{q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX\}$$

$$- \sum_{\nu=1}^{3}[2\eta_{\nu}(\phi AX)(\phi_\nu Y) - \eta(Y)\xi_{\nu}]$$

$$- \{q_{\nu+2}(X)\eta_{\nu+1}(\phi Y) - q_{\nu+1}(X)\eta_{\nu+2}(\phi Y) + g(\phi_\nu AX, \phi Y)\}\phi_\nu \xi$$

$$- \eta(Y)\eta_\nu(AX)\phi_\nu \xi - \eta_\nu(\phi Y)\{q_{\nu+2}(X)\phi_\nu+1 \xi - q_{\nu+1}(X)\phi_\nu+2 \xi$$

$$+ \phi_\nu AX - g(AX, \xi)\xi_{\nu}\}.$$  \hspace{1cm} (13)

Then, taking an inner product (13) with $\xi$, we have

$$0 = 3g(\phi AX, Y) + 3\sum_{\nu=1}^{3}\eta_{\nu}(Y)g(\phi_\nu AX, \xi)$$

$$- \sum_{\nu=1}^{3}[2\eta_{\nu}(\phi AX)g(\phi_\nu Y, \xi) - \eta_\nu(\phi Y)g(\phi_\nu AX, \xi)]$$

$$= 3g(\phi AX, Y) + 5\sum_{\nu=1}^{3}\eta_{\nu}(Y)g(\phi_\nu AX, \xi) + 3\sum_{\nu=1}^{3}\eta_{\nu}(\phi Y)g(\phi_\nu^2 AX, \xi_{\nu})$$

$$= 3g(\phi AX, Y) + 5\sum_{\nu=1}^{3}\eta_{\nu}(Y)g(\phi_\nu AX, \xi) - \sum_{\nu=1}^{3}\eta_{\nu}(\phi Y)\eta_\nu(AX).$$

From this, by putting $Y = \phi Z$ for any $Z \in \mathcal{D}$, it follows that for any $X \in \mathcal{D}^\perp$ and $\xi \in \mathcal{D}$

$$3g(AX, Z) = -5\sum_{\nu=1}^{3}\eta_{\nu}(\phi Z)g(\phi_\nu AX, \xi).$$  \hspace{1cm} (14)
Then by putting $Z = \phi \xi_i$ in (14), we have
\[ g(AX, \phi \xi_i) = 0 \]
for any $i = 1, 2, 3$. From this, together with (14), we assert that $g(AX, Z) = 0$ for any $X \in \mathfrak{D}^\perp$ and $Z \in \mathfrak{D}$. This completes the proof of our Proposition. \[ \square \]

Then by Proposition 5.1 and Theorem 1.1 in the introduction we assert the following

**Theorem 5.1.** Let $M$ be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with parallel normal Jacobi operator and $\xi \in \mathfrak{D}$. Then $M$ is a tube over a totally real and totally geodesic quaternionic projective space $\mathbb{Q}P^n$, $n = 2m$.

Now let us check whether a real hypersurface of type (B) in $G_2(\mathbb{C}^{m+2})$, that is, a tube over a totally real and totally geodesic $\mathbb{Q}P^n$, satisfy $\langle \nabla_X \bar{R}_N \rangle = 0$ or not? Corresponding to such a real hypersurface of type (B), we introduce a proposition in Berndt and Suh [4] as follows:

**Proposition 5.2.** Let $M$ be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A \mathfrak{D} \subset \mathfrak{D}$, $A \xi = \alpha \xi$, and $\xi$ is tangent to $\mathfrak{D}$. Then the quaternionic dimension $m$ of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and $M$ has five distinct constant principal curvatures
\[ \alpha = -2 \tan(2r), \quad \beta = 2 \cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r) \]
with some $r \in (0, \pi/4)$. The corresponding multiplicities are
\[ m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu) \]
and the corresponding eigenspaces are
\[ T_\alpha = \mathbb{R} \xi, \quad T_\beta = \mathbb{I} \mathbb{J} \xi, \quad T_\gamma = \mathbb{J} \xi, \quad T_\lambda, \quad T_\mu, \]
where
\[ T_\lambda \oplus T_\mu = (\mathbb{H} \mathbb{C} \xi)^\perp, \quad \mathbb{I} T_\lambda = T_\lambda, \quad \mathbb{J} T_\mu = T_\mu, \quad JT_\lambda = T_\mu. \]

Now let us suppose $M$ is of type (B) with parallel normal Jacobi operator $\bar{R}_N$ and $\xi \in \mathfrak{D}$. Then (11) for $\xi \in \mathfrak{D}$ gives
\[ 0 = 3 \phi AX + 5 \sum_{\nu=1}^{3} \eta_{\nu} (\phi AX) \xi_\nu + \sum_{\nu=1}^{3} \eta_{\nu} (AX) \phi_\nu \xi. \]
From this, by putting $X = \xi_\mu$ and using $A \phi_\nu \xi = 0$ we have
\[ 0 = 4 \beta \phi \xi_\mu. \]
Then it follows that $\beta = 0$. This makes a contradiction. Now, summarizing such a fact, we conclude the following
Theorem 5.2. There do not exist any Hopf hypersurfaces in \( G_2(\mathbb{C}^{m+2}) \), \( m \geq 3 \), with parallel normal Jacobi operator and \( \xi \in \mathfrak{D} \).

6. Parallel normal Jacobi operator for \( \xi \in \mathfrak{D}^\perp \)

In this section, we consider Hopf real hypersurfaces in \( G_2(\mathbb{C}^{m+2}) \), \( m \geq 3 \), with parallel normal Jacobi operator and \( \xi \in \mathfrak{D}^\perp \). Then (9) gives the following

\[
0 = 3g(\phi AX, Y)\xi + 3\eta(Y)\phi AX + 3\sum_{\nu=1}^{3} \{q_{\nu+2}(X)\eta_{\nu+1}(Y) - q_{\nu+1}(X)\eta_{\nu+2}(Y) + g(\phi_{\nu} AX, Y)\} \xi_{\nu} + 3\sum_{\nu=1}^{3} \eta_{\nu}(Y)\{q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_{\nu} AX\} - \left[ q_2(X)\phi_3 \phi Y + q_3(X)\phi_2 \phi Y - g(AX, \phi Y)\xi + \eta(Y)\phi_1 AX - g(\phi AX, Y)\xi - \eta(Y)(q_3(X)\xi_2 - q_2(X)\xi_3 + \phi_1 AX) - \{q_1(X)\eta_3(\phi Y) - q_3(X)\eta_1(\phi Y) + g(\phi_2 AX, \phi Y)\} \phi_2 \xi - \{q_2(X)\eta_1(\phi Y) - q_1(X)\eta_2(\phi Y) + g(\phi_3 AX, \phi Y)\} \phi_3 \xi + \eta(Y)\eta_3(AX)\xi_3 - \eta(Y)\eta_3(AX)\xi_2 - \eta_3(Y)\{q_1(X)\phi_3 \xi - q_3(X)\phi_1 \xi + \phi_2 \phi AX - g(AX, \xi)\xi_2\} + \eta_2(Y)\{q_2(X)\phi_1 \xi - q_1(X)\phi_2 \xi + \phi_3 \phi AX - g(AX, \xi)\xi_3\}\right].
\]

Then (15) can be rearranged as follows:

\[
0 = 3g(\phi AX, Y)\xi + 3\eta(Y)\phi AX + 3\sum_{\nu=1}^{3} \{q_{\nu+2}(X)\eta_{\nu+1}(Y) - q_{\nu+1}(X)\eta_{\nu+2}(Y) + g(\phi_{\nu} AX, Y)\} \xi_{\nu} + 3\sum_{\nu=1}^{3} \eta_{\nu}(Y)\{q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_{\nu} AX\} - \left[ -2\eta_2(AX)\phi_3 \phi Y + 2\eta_3(AX)\phi_2 \phi Y + g(\phi_2 AX, \phi Y)\xi_3 - g(\phi_3 AX, \phi Y)\xi_2 + 3\eta(Y)\eta_3(AX)\xi_3 - 3\eta(Y)\eta_3(AX)\xi_2 - \eta_3(Y)\{\phi_2 \phi AX - g(AX, \xi)\xi_2\} + \eta_2(Y)\{\phi_3 \phi AX - g(AX, \xi)\xi_3\}\right].
\]
From this, let us take an inner product with $\xi$, we have
\begin{align*}
0 &= 3g(\phi AX, Y) + 3\{q_3(X)\eta_2(Y) - q_2(X)\eta_3(Y) + g(\phi_1 AX, Y)\} \\
&\quad + 3\{\eta_3(Y)q_2(X) - \eta_2(Y)q_3(X) + \eta_2(Y)\eta_3(AX) - \eta_3(Y)\eta_2(AX)\} \\
&\quad - (2\eta_2(AX)\eta_3(Y) - 2\eta_3(AX)\eta_2(Y) + \eta_3(Y)\eta_2(AX) - \eta_2(Y)\eta_3(AX)) \\
&= 3g(\phi AX, Y) + 3g(\phi_1 AX, Y) + 6\eta_2(Y)\eta_3(AX) - 6\eta_3(Y)\eta_2(AX),
\end{align*}
where we have used the following formulas
\begin{align*}
\eta(\phi_3 Y) &= g(\phi_3 \xi, \phi Y) = g(\phi_4 Y, \xi) = -\eta_3(Y), \\
\eta(\phi_2 Y) &= g(\phi_2 \xi, \phi Y) = g(\phi_3 \xi, Y) = -\eta_2(Y), \\
g(\phi_2 \phi AX, \xi) &= g(\phi AX, \phi_2 \xi) = g(\phi AX, \xi_2) = -g(A X, Y), \\
g(\phi_3 \phi AX, \xi) &= g(\phi_3 AX, \phi_3 \xi) = g(AX, \phi_3 \xi) = -g(AX, \xi_3) = -\eta_3(AX).
\end{align*}
Then (16) can be reformed as follows:
\begin{equation}
0 = g(\phi AX, Y) + g(\phi_1 AX, Y) + 2\eta_2(Y)\eta_3(AX) - 2\eta_3(Y)\eta_2(AX).
\end{equation}
From this, by putting $Y = \xi_2$, we have
\begin{equation}
0 = g(\phi AX, \xi_2) + g(\phi_1 AX, \xi_2) + 2\eta_2(AX) = 2\eta_3(AX)
\end{equation}
for any vector field $X$ on $M$. Similarly, we are able to assert $\eta_2(AX) = 0$. From this, together with $M$ is Hopf, we assert the following

**Proposition 6.1.** Let $M$ be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel normal Jacobi operator and $\xi \in \mathfrak{D}^\perp$. Then $g(\mathfrak{A} \mathfrak{D}, \mathfrak{D}^\perp) = 0$.

From this proposition and together with Theorem 1.1 in the introduction we know that any real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel normal Jacobi operator $R_N$ are congruent to a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Now let us check whether real hypersurfaces of type (A) satisfy $\nabla_X R_N = 0$ or not? Then we recall a proposition given by Berndt and Suh [4] as follows:

**Proposition 6.2.** Let $M$ be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $\mathfrak{A} \mathfrak{D} \subset \mathfrak{D}$, $\mathfrak{A} \xi = \alpha \xi$, and $\xi$ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1 N$. Then $M$ has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures
\begin{align*}
\alpha &= \sqrt{8}\cot(\sqrt{8}r), \quad \beta = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0
\end{align*}
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with some r ∈ (0, π/√8). The corresponding multiplicities are

\[ m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m - 2 = m(\mu), \]

and the corresponding eigenspaces we have

\[ T_\alpha = \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1, \]
\[ T_\beta = \mathbb{C}^\perp \xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3, \]
\[ T_\lambda = \{X | X \perp H\xi, JX = J_1 X\}, \]
\[ T_\mu = \{X | X \perp H\xi, JX = -J_1 X\}, \]

where \( \mathbb{R}\xi, \mathbb{C}\xi \) and \( \mathbb{H}\xi \) respectively denotes real, complex and quaternionic span of the structure vector \( \xi \) and \( \mathbb{C}^\perp \xi \) denotes the orthogonal complement of \( \mathbb{C}\xi \) in \( \mathbb{H}\xi \).

Then, by putting \( X = \xi_3 \) in (17) and using Proposition 6.2 we have

\[ 2\eta_3 (A\xi_3) = 2\sqrt{2} \cot(\sqrt{2}r) = 0. \]

But \( r \in (0, \sqrt{2}) \), on which we know \( \cot(\sqrt{2}r) \neq 0 \). This makes a contradiction. Consequently, the normal Jacobi operator \( R_N \) of such a tube over a totally geodesic \( G_2(\mathbb{C}^{m+2}) \) can not be parallel. Summing up above facts we conclude the following

**Theorem 6.1.** There do not exist any Hopf hypersurfaces in \( G_2(\mathbb{C}^{m+2}) \), \( m \geq 3 \), with parallel normal Jacobi operator and \( \xi \in \mathbb{D}^\perp \).

Then by Theorem 1.2, together with Theorems 5.2 and 6.1 in Sections 5 and 6 respectively, we complete the proof of our Theorem 1.3 in the introduction.

**References**

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