Annihilators on co-commutator with generalized derivations on Lie ideals

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Abstract. Let $R$ be a prime ring of characteristic different from 2, $U$ the Utumi quotient ring of $R$, $C = Z(U)$ the extended centroid of $R$, $L$ a non-central Lie ideal of $R$, $H$ and $G$ non-zero generalized derivations of $R$. Suppose that there exists $0 \neq a \in R$ such that $a(H(u)u - uG(u)) = 0$, for all $u \in L$, then one of the following holds:

1. There exist $b', c' \in U$ such that $H(x) = b'x + xc'$, $G(x) = c'x$ with $ab' = 0$;
2. $R$ satisfies $s_4$ and there exist $b', c', q' \in U$ such that $H(x) = b'x + xc'$, $G(x) = c'x + xq'$, with $a(b' - q') = 0$.

1. Introduction

Let $R$ be a prime ring of characteristic different from 2 with center $Z(R)$ and extended centroid $C$. The standard polynomial of degree 4 is defined as $s_4(x_1, \ldots, x_4) = \sum_{\sigma \in S_4}(-1)^{\sigma}x_{\sigma(1)} \cdots x_{\sigma(4)}$, where $\sigma$ runs over $S_4$ the symmetric group of degree 4 and where $(-1)^{\sigma}$ is 1 or $-1$ according as $\sigma$ is an even or odd permutation.

A well known result of Posner [18] states that if $d$ is a derivation of $R$ such that $[d(x), x] \in Z(R)$, for any $x \in R$, then either $d = 0$ or $R$ is commutative. This theorem indicates that the global structure of a ring $R$ is often tightly connected to the behaviour of additive mappings defined on $R$. Following this line of investigation, several authors generalized the Posner’s Theorem. For instance in [2] Bresar proves that if $d$ and $\delta$ are derivations of $R$ such that

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Let there exist $R$ if

We denote by $\text{Der}(\cdot)$ every derivation following theorem: view might be interesting (see for example [13]). Here our purpose is to prove the on operator algebras. Therefore any investigation from the algebraic point of derivations are called inner. Generalized derivations have been primarily studied example is a map of the form $d$ derivation an additive map $\delta$ of the form $\delta = \frac{\partial}{\partial x}$ such that, for all $x, y \in R, G(xy) = G(x)y + xd(y)$. A significative example is a map of the form $G(x) = ax + xb$, for some $a, b \in R$; such generalized derivations are called inner. Generalized derivations have been primarily studied on operator algebras. Therefore any investigation from the algebraic point of view might be interesting (see for example [13]). Here our purpose is to prove the following theorem:

**Theorem 1.** Let $R$ be a prime ring of characteristic different from 2, $U$ the Utumi quotient ring of $R$, $C = Z(U)$ the extended centroid of $R$, $L$ a non-central Lie ideal of $R$, $H$ and $G$ non-zero generalized derivations of $R$. Suppose that there exists $0 \neq a \in R$ such that $a(\text{H}(u)u - u\text{G}(u)) = 0$, for all $u \in L$, then one of the following holds:

1. there exist $b', c' \in U$ such that $H(x) = b'x + xc', G(x) = c'x$ with $ab' = 0$;
2. $R$ satisfies $s_4$ and there exist $b', c', q' \in U$ such that $H(x) = b'x + xc'$, $G(x) = c'x + xq'$, with $a(b' - q') = 0$.

In all that follows let $R$ be a non-commutative prime ring of characteristic different from 2, $U$ its Utumi quotient ring and $C = Z(U)$ the center of $U$. We refer the reader to [1] for the definitions and the related properties of these objects. In particular we make use of the following well known facts:

**Fact 1.** If $I$ is a two-sided ideal of $R$, then $R, I$ and $U$ satisfies the same generalized polynomial identities with coefficients in $U$ ([4]).

**Fact 2.** Every derivation $d$ of $R$ can be uniquely extended to a derivation of $U$ (see Proposition 2.5.1 in [1]).

**Fact 3.** We denote by $\text{Der}(U)$ the set of all derivations on $U$. By a derivation word we mean an additive map $\Delta$ of the form $\Delta = d_1d_2\ldots d_m$, with each

$d(x)x - x\delta(x) \in Z(R)$, for all $x \in R$, then either $d = \delta = 0$ or $R$ is commutative. Later in [12] Lee and Wong consider the case when $d(x)x - x\delta(x) \in Z(R)$, for all $x$ in some non-central Lie ideal $L$ of $R$. They prove that either $d = \delta = 0$ or $R$ satisfies $s_4$, the standard identity of degree 4. Recently in [17] Niu and Wu study the left annihilator of the set $\{d(u)u - u\delta(u), u \in L\}$, where $d$ and $\delta$ are derivations of $R$ and $L$ is a non-central Lie ideal of $R$. In case the annihilator is not zero, the conclusion is that $R$ satisfies the standard identity $s_4$ and $d = -\delta$ are inner derivations. These facts in a prime ring are natural tests which evidence that the set $\{d(u)u - u\delta(u), u \in L\}$ is rather large in $R$.

Here we will consider the same situation in the case the derivations $d$ and $\delta$ are replaced respectively by the generalized derivations $H$ and $G$. More specifically an additive map $G : R \rightarrow R$ is said to be a generalized derivation if there is a derivation $d$ of $R$ such that, for all $x, y \in R, G(xy) = G(x)y + xd(y)$. A significative example is a map of the form $G(x) = ax + xb$, for some $a, b \in R$; such generalized derivations are called inner. Generalized derivations have been primarily studied on operator algebras. Therefore any investigation from the algebraic point of view might be interesting (see for example [13]). Here our purpose is to prove the following theorem:
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$d_i \in \text{Der}(U)$. Then a differential polynomial is a generalized polynomial, with coefficients in $U$, of the form $\Phi(\Delta_j x_i)$ involving non-commutative indeterminates $x_i$ on which the derivations words $\Delta_j$ act as unary operations. The differential polynomial $\Phi(\Delta_j x_i)$ is said to be a differential identity on a subset $T$ of $U$ if it vanishes for any assignment of values from $T$ to its indeterminates $x_i$.

Let $D_{\text{int}}$ be the $C$-subspace of $\text{Der}(U)$ consisting of all inner derivations on $U$ and let $d$ be a non-zero derivation on $R$. By Theorem 2 in [10] we have the following result (see also Theorem 1 in [14]): If $\Phi(x_1, \ldots, x_n, d x_1, \ldots, d x_n)$ is a differential identity on $R$, then one of the following holds:

1. either $d \in D_{\text{int}}$;
2. or $R$ satisfies the generalized polynomial identity $\Phi(x_1, \ldots, x_n, y_1, \ldots, y_n)$.

Fact 4. If $I$ is a two-sided ideal of $R$, then $R$, $I$ and $U$ satisfies the same differential identities ([14]).

We refer the reader to Chapter 7 in [1] for a complete and detailed description of the theory of generalized polynomial identities involving derivations.

Fact 5. Since we assume that $\text{char}(R) \neq 2$, then there exists a non-zero two-sided ideal $I$ of $R$ such that $0 \neq [I, R] \subseteq L$. In particular, if $R$ is a simple ring it follows that $[R, R] \subseteq L$.

This follows from pp. 4–5 in [9], Lemma 2 and Proposition 1 in [6], Theorem 4 in [11].

2. The case of inner generalized derivations

We dedicate this section to prove the theorem in case both the generalized derivations $H$ and $G$ are inner, that is there exist $b, c, p, q \in U$ such that $H(x) = bx + xc$ and $G(x) = px + xq$, for all $x \in R$.

In light of Fact 5, since we suppose $\text{char}(R) \neq 2$, there exists a non-central ideal $I$ of $R$ such that $[I, I] \subseteq L$. This implies that $a(b[r_1, r_2]^2 + [r_1, r_2]([c-p][r_1, r_2] - [r_1, r_2]^2 q) = 0$ for all $r_1, r_2 \in I$. Moreover by Fact 1, $I$ and $R$ satisfy the same generalized polynomial identities, thus $a(b[r_1, r_2]^2 + [r_1, r_2]([c-p][r_1, r_2] - [r_1, r_2]^2 q) = 0$ for all $r_1, r_2 \in R$. Hence in all that follows we assume that $R$ satisfies the following generalized polynomial identity

$$P(x_1, x_2) = a(b[x_1, x_2]^2 + [x_1, x_2]([c-p][x_1, x_2] - [x_1, x_2]^2 q).$$

$P(x_1, x_2)$ is a generalized polynomial in the free product $U * C C\{x_1, x_2\}$ of the $C$-algebra $U$ and the free $C$-algebra $C\{x_1, x_2\}$. 

We first prove the following:

**Proposition 1.** If \( a \in Z(R) \), then one of the following holds:
1. either there exists \( b' \in U \) such that \( H(x) = xb' \) and \( G(x) = b'x \), for all \( x \in R \);
2. or \( R \) satisfies \( s_4 \) and there exists \( \alpha \in C \) such that \( p = c - \alpha \) and \( q = b + \alpha \), that is \( H(x) = bx + xc \) and \( G(x) = cx + xb \).

**Proof.** Since \( a \in Z(R) \), then \( a \) is not a zero-divisor, then by main assumption it follows that \( (H(u)v - uG(v)) = 0 \), for all \( u \in [R, R] \). In this case, it is proved in [15] that either there exists \( b' \in U \) such that \( H(x) = xb' \) and \( G(x) = b'x \), for all \( x \in R \), or \( R \) satisfies \( s_4 \). In this last case \( R \) is PI-ring, moreover \( U \) satisfies the same generalized polynomial identities of \( R \). Therefore \( U \) is a central simple algebra of dimension at most 4 over its center, and it is known that in this case \([r, s]^2 \in Z(U) = C\) for all \( r, s \in U \). Moreover \( U \) satisfies

\[
b[x_1, x_2]^2 + [x_1, x_2]([c - p][x_1, x_2] - [x_1, x_2]^2)q.
\]

Since the polynomial \([x_1, x_2]^2\) is central valued in \( U \), then \( U \) satisfies

\[
(b - q)[x_1, x_2]^2 + [x_1, x_2][c - p][x_1, x_2]. \tag{1}
\]

Denote \( e_{ij} \) the usual matrix unit, with 1 in the \((i, j)\)-entry and zero elsewhere, and write \( w = (c - p) = \sum_r w_{rs} e_{rs} \), for suitable \( w_{rs} \in C \). Therefore for any \( i \neq j \), let \( r_1 = e_{ii}, r_2 = e_{ij} \) and \([r_1, r_2] = e_{ij} \). It follows by (1) that \( e_{ij}we_{ij} = 0 \) for all \( i \neq j \), that is \( w_{ij} = 0 \) and \( w \) is a diagonal matrix in \( M_2(C) \). Moreover, for all \( \varphi \in Aut_F(M_2(C)) \), \( U \) satisfies

\[
\varphi((b - q)[x_1, x_2]^2 + [x_1, x_2][c - p][x_1, x_2])
\]

which is

\[
(b - q)[x_1, x_2]^2 + [x_1, x_2]((c - p)[x_1, x_2])
\]

since the set of all the evaluation of \([x_1, x_2] \) is invariant under the action of any element of \( Aut_F(M_2(C)) \). By the above argument, \( \varphi(c - p) \) must be diagonal. In particular, let \( r \neq s \) and \( \varphi(x) = (1 + e_{rs})x(1 - e_{rs}) \), hence

\[
\varphi(c - p) = \sum_i w_{ii} e_{ii} + w_{ss} e_{rs} - w_{rr} e_{rs}
\]

which implies \( w_{rr} = w_{ss} \), for all \( r \neq s \). Thus \( c - p \) is a central matrix, namely \( c - p = \alpha \). By (1) we get that \( U \) satisfies \((b - q + \alpha)[x_1, x_2]^2\), and since \( 0 \neq [U, U]^2 \subseteq C \), we also have \( q - b = \alpha = c - p \). Thus we conclude that, in case \( R \) satisfies \( s_4 \), \( p = c - \alpha \) and \( q = b + \alpha \). \( \square \)
Proposition 2. If \( a \notin Z(R) \) then either \( P(x_1, x_2) \) is a non-trivial generalized polynomial identity for \( R \) or \( H(x) = b'x + xc', \ G(x) = c'x \) for some \( b', c' \in U \) satisfying \( ab' = 0 \).

PROOF. Suppose now that \( R \) does not satisfy any non-trivial generalized polynomial identity. Let \( T = U \ast_C C\{X\} \) be the free product over \( C \) of the \( C \)-algebra \( U \) and the free \( C \)-algebra \( C\{X\} \), with \( X \) the countable set consisting of non-commuting indeterminates \( x_1, x_2, \ldots, x_n, \ldots \).

For brevity we write \( P(X) \) instead of \( P(x_1, x_2) \) and \( f(X) \) instead of \( [x_1, x_2] \).

Now consider the generalized polynomial \( P(X) \in U \ast_C C\{X\} \). By our hypothesis, \( R \) satisfies the following generalized polynomial identity:

\[
P(X) = abf(X)^2 + a(f(X)) \cdot (c - p)f(X) - af(X)^2q = 0 \in T.
\]

Since \( R \) does not satisfy non-trivial GPIs, by [4], the coefficients \( \{ab, a\} \) must be linearly \( C \)-dependent. Therefore there exist \( \beta_1, \beta_2 \in C \) such that \( \beta_1(ab) + \beta_2a = 0 \), with \( \beta_1 \neq 0 \) since \( a \notin C \). Hence we may write \( ab = \lambda a \), for a suitable \( \lambda \in C \). In this situation \( R \) satisfies

\[
a(\lambda f(X)^2 + f(X)(c - p)f(X) - f(X)^2q)
\]

that is

\[
\lambda f(X)^2 + f(X)(c - p)f(X) - f(X)^2q = 0 \in T.
\]

Again since \( R \) does not satisfy any non-trivial generalized polynomial identity, \( \{1, q\} \) must be linearly \( C \)-dependent, that is \( q \in C \). This implies that \( G(x) = (p + q)x \) and also that \( R \) satisfies

\[
f(X)(\lambda + (c - p) - q)f(X)
\]

which implies \( \lambda + (c - p) - q = 0 \), that is \( H(x) = bx + x(p + q - \lambda) = (b - \lambda)x + x(p + q) \), and we obtain the required conclusion, for \( b' = b - \lambda \) and \( c' = p + q \). \( \square \)

Lemma 1. Let \( R = M_m(F) \) be the ring of all \( m \times m \) matrices over a field \( F \) of characteristic different from 2. If \( a \) is not central in \( R \) then there exists \( \alpha \in F \) such that \( p = c - \alpha \cdot I_m \), where \( I_m \) is the identity matrix of order \( m \), and one of the following holds:

1. \( q \in Z(R) \) and there exists \( \gamma \in F \) such that \( p + q = c + \gamma \cdot I_m \), that is \( H(x) = bx + xc, \ G(x) = (c + \gamma \cdot I_m)x; \) moreover \( a(b - \gamma \cdot I_m) = 0 \);

2. \( R \) satisfies \( s_4 \) and there exists \( q' \in R \) such that \( G(x) = cx + xq' \), with \( a(b - q') = 0 \).
**Proof.** Denote $a = \sum_{rs} e_{rs} a_{rs}$, $q = \sum_{rs} e_{rs} q_{rs}$, $c - p = w = \sum_{rs} e_{rs} w_{rs}$, for suitable $a_{rs}, q_{rs}, w_{rs} \in F$. By the main assumption, $R$ satisfies

$$a(b[x_1, x_2]^2 + [x_1, x_2](c - p)[x_1, x_2] - [x_1, x_2]^2 q).$$

(2)

Fix $[x_1, x_2] = e_{ij}$, for any $i \neq j$. In this case from (2) we have

$$ae_{ij}(c - p)e_{ij} = 0$$

(3)

that is

either $a_{ki} = 0 \ \forall k$ or $w_{ji} = 0$. (3’).

Here we first prove that $w$ is a diagonal matrix. In order to do this, we suppose that there exists some non-zero off-diagonal entry of $w$ and divide the proof into two cases:

**Case 1:** $m = 2$.

Suppose $w_{21} \neq 0$, then by (3) it follows $a_{11} = a_{21} = 0$. Of course, since we suppose $a \neq 0$, we must assume now $w_{12} = 0$.

Choose $[x_1, x_2] = [e_{12}, e_{21}] = e_{11} - e_{22}$ and by (2) we have

$$0 = Y = a(b(e_{11} - e_{22})^2 + (e_{11} - e_{22})(c - p)(e_{11} - e_{22}) - (e_{11} - e_{22})^2 q)$$

in particular the (1,1)-entry of the matrix $Y$ is $a_{12}(b_{21} - w_{21} - q_{21}) = 0$ and the (2,1)-one is $a_{22}(b_{21} - w_{21} - q_{21}) = 0$. Therefore, from $a \neq 0$ follows

$$b_{21} - w_{21} - q_{21} = 0.$$  

(4)

In the same way, for $[x_1, x_2] = [e_{12} - e_{21}, e_{22}] = e_{12} + e_{21}$ in (2) we have

$$0 = T = a(b(e_{12} + e_{21})^2 + (e_{12} + e_{21})(c - p)(e_{12} + e_{21}) - (e_{12} + e_{21})^2 q).$$

The (1,1)-entry of the matrix $T$ is $a_{12}(b_{21} - q_{21}) = 0$ and the (2,1)-one is $a_{22}(b_{21} - q_{21}) = 0$. Since $a \neq 0$ we get

$$b_{21} - q_{21} = 0.$$  

(5)

Thus by (5) and (4) we obtain the contradiction $w_{21} = 0$.

**Case 2:** $m \geq 3$.

Also in this case we suppose that there exists $w_{ji} \neq 0$ for some $i \neq j$, so that $a_{ki} = 0$ for all $k$, that is the $i$-th column of $a$ is zero.
Let now $q \neq i, j$ and fix $[x_1, x_2] = [e_{ij} + e_{qj}, e_{jj}] = e_{ij} + e_{qj}$. Then (2) implies $ae_{ij}w(e_{ij} + e_{qj}) = 0$ and since $a_{ki} = 0$ for all $k$, it follows that $ae_{qj}w(e_{ij} + e_{qj}) = 0$. Moreover, by (3), we get $ae_{qj}we_{ij} = 0$, which implies that $ae_{qj}we_{ij} = 0$. The assumption $w_{tj} \neq 0$ implies that $a_{kj} = 0$ for all $k$, that is $a$ has just one non-zero column, the $j$-th one: $a = \sum_r a_{rj}e_{rj}$.

Notice that if $w_{tj} \neq 0$ for some $t \neq j$, by the same argument we get that $a$ has just the $t$-th column non-zero, that is $a = 0$. Thus we may assume that $w_{tj} = 0$ for all $t \neq j$.

Let $t \neq i, j$ and denote by $\sigma_t$ and $\tau_t$ the following automorphisms of $R$:

$$\sigma_t(x) = (1 + e_{jt})x(1 - e_{jt}) = x + e_{jt}x - xe_{jt} - e_{jt}xe_{jt},$$

$$\tau_t(x) = (1 - e_{jt})x(1 + e_{jt}) = x - e_{jt}x + xe_{jt} - e_{jt}xe_{jt},$$

and say $\sigma_t(w) = \sum_{rs} \sigma_{rs}e_{rs}, \tau_t(w) = \sum_{rs} \tau_{rs}e_{rs}$ where $\sigma_{rs}, \tau_{rs} \in F$. We have

$$\sigma_{ji} = w_{ji} + w_{tj} \quad \text{and} \quad \tau_{ji} = w_{ji} - w_{tj}.$$ 

If there exists $t$ such that $\sigma_{ji} = w_{ji} + w_{tj} = 0$ or $\tau_{ji} = w_{ji} - w_{tj} = 0$ then $w_{tj} = -w_{ji} \neq 0$ or $w_{tj} = w_{ji} \neq 0$. Therefore $w_{tj} \neq 0$ and $w_{tj} \neq 0$, and so, by using (3), $a = 0$.

Hence assume that $\sigma_{ji} \neq 0$ and $\tau_{ji} \neq 0$, for all $t \neq i, j$, and recall that, for any $F$-automorphism $\varphi$ of $R$, the following holds

$$\varphi(a)(\varphi(b)[x_1, x_2]^2 + [x_1, x_2] \varphi(c - p)[x_1, x_2] - [x_1, x_2]^2 \varphi(q)).$$

Thus in this case by (3), for any $t \neq i, j$, the non-zero entries of the matrices $\sigma_t(a)$ and $\tau_t(a)$ are just in the $j$-th column. In particular, since

$$\sigma_t(a) = a + e_{jt}a - ae_{jt} - e_{jt}ae_{jt} = \sum_r a_{rj}e_{rj} - \sum_r a_{rj}e_{rt} + a_{tj}e_{jj} - a_{tj}e_{jt},$$

$$\tau_t(a) = a - e_{jt}a + ae_{jt} - e_{jt}ae_{jt} = \sum_r a_{rj}e_{rj} + \sum_r a_{rj}e_{rt} - a_{tj}e_{jj} - a_{tj}e_{jt},$$

then both the above matrices have zero in the $(j, t)$ entry that is

$$-a_{jj} - a_{tj} = 0 \quad \text{for} \quad \sigma_t(a)$$

$$a_{jj} - a_{tj} = 0 \quad \text{for} \quad \tau_t(a).$$

By char($R$) $\neq 2$ we obtain $a_{jj} = a_{tj} = 0$ for all $t \neq i$, that is $a = a_{ij}e_{ij}$.
Denote now by $\varphi$ and $\chi$ the following automorphisms of $R$:

$$
\varphi(x) = (1 + e_{ji})x(1 - e_{ji}) = x + e_{ji}x - xe_{ji} - e_{ji}xe_{ji},
$$
$$
\chi(x) = (1 - e_{ji})x(1 + e_{ji}) = x - e_{ji}x + xe_{ji} - e_{ji}xe_{ji},
$$
and say $\varphi(w) = \sum \varphi_{rs}e_{rs}$, $\chi(w) = \sum \chi_{rs}e_{rs}$ where $\varphi_{rs}, \chi_{rs} \in F$. Since, by (3'), $w_{ij} \neq 0$ implies $a = 0$, we assume that $w_{ij} = 0$. Then we have

$$
\varphi_{ji} = w_{ji} - w_{jj} + w_{ii} \quad \text{and} \quad \chi_{ji} = w_{ji} + w_{jj} - w_{ii}
$$

If $\varphi_{ji} = \chi_{ji} = 0$, then we get the contradiction $w_{ji} = 0$.

If at least one of $\varphi_{ji}$ and $\chi_{ji}$ is not zero, then, by (3), one of $\varphi(a)$ and $\chi(a)$ has zero in all the entries of the $i$-th column. In particular notice that

$$
\varphi(a) = a_{ij}e_{ij} - a_{ij}e_{ii} + a_{ij}e_{jj} - a_{ij}e_{ji},
$$
$$
\chi(a) = a_{ij}e_{ij} + a_{ij}e_{ii} - a_{ij}e_{jj} - a_{ij}e_{ji}
$$

which means that in any case the $(j, i)$-entry is $a_{ij} = 0$, a contradiction again.

All the previous arguments say that if $a$ is not zero, then $w$ must be a diagonal matrix, $w = \sum_i w_i e_{ii}$.

Moreover, for all $\lambda \in \text{Aut}_F(M_m(F))$, since $\lambda(a) \neq 0$ and $R$ satisfies

$$
\lambda(a)(\lambda(b)[x_1, x_2]^2 + [x_1, x_2]\lambda(c - p)[x_1, x_2] - [x_1, x_2]^2\lambda(q)),
$$

we also have that $\lambda(c - p)$ is diagonal. In particular, let $r \neq s$ and $\lambda(x) = (1 + e_{rs})x(1 - e_{rs})$, hence

$$
\lambda(c - p) = \sum_i w_i e_{ii} + w_s e_{rs} - w_r e_{rs}
$$

is diagonal implying $w_r = w_s = \alpha$, for all $r \neq s$. Thus $c - p$ is a central matrix, namely $c - p = \alpha \cdot I_m$. Therefore $R$ satisfies

$$
ab[x_1, x_2]^2 + a[x_1, x_2]^2(\alpha - q).
$$

Denote by $G$ the additive subgroup of $R$ generated by all the evaluations of the polynomial $[x_1, x_2]^2$. By [3], since $\text{char}(R) \neq 2$, either $[R, R] \subseteq G$ or $[x_1, x_2]^2$ is central valued on $R$ that is $R$ satisfies $s_4$.

In the first case $R$ satisfies

$$
ab[x_1, x_2] + a[x_1, x_2](\alpha - q).
$$
Let \( \alpha - q = u = \sum_{r,s} u_{rs} e_{rs} \), with \( u_{rs} \in F \). For \([x_1, x_2] = e_{ij} \), with any \( i \neq j \), it follows \( ab e_{ij} + ae_{ij}(\alpha - q) = 0 \). By right multiplying for any \( e_{qq} \), with \( q \neq j \), we have \( ae_{ij}(\alpha - q)e_{qq} = 0 \) that is

either \( a_{ki} = 0 \quad \forall k \) or \( u_{jq} = 0 \quad \forall q \neq j \).

In particular

either \( a_{ki} = 0 \quad \forall k \) or \( u_{ji} = 0 \quad (3'''). \)

Notice that \((3''')\) has the same flavour of \((3')\). By the same argument as above, in case \( a \neq 0 \) we have that \( u = \alpha - q \) is a central matrix, and so \( a(b + u)[r_1, r_2] = 0 \), for all \( r_1, r_2 \in R \). This implies \( a(b + u) = 0 \), which is the conclusion 1 of Lemma 1, for \( \gamma = -u \).

Consider finally the case when \([x_1, x_2]^2\) is central valued on \( R \). Here \( R \) satisfies \( a(b + \alpha - q)[x_1, x_2]^2 \), moreover there exists \( 0 \neq [r_1, r_2]^2 \in F \cdot I_m \), which implies \( a(b + \alpha - q) = 0 \), the conclusion 2 of Lemma 1, for \( q' = q - \alpha \). \( \square \)

**Lemma 2.** Let \( R \) be a prime ring of characteristic different from 2. If \( a \) is not central in \( R \) then \( c - p = \alpha \in C \) and one of the following holds:

1. \( q \in C \) and there exist \( \lambda \in C \), \( b' = b - \lambda \), \( c' = p + q \) such that \( H(x) = b'x + x c' \), \( G(x) = c'x \), with \( ab' = 0 \);
2. \( q \in C \) and there exists \( \gamma = q - \alpha \in C \) such that \( p + q = c + \gamma \), that is \( H(x) = bx + xc \), \( G(x) = (c + \gamma)x \), with \( a(b - \gamma) = 0 \);
3. \( R \) satisfies \( s_4 \) and there exists \( q' = q - \alpha \) such that \( G(x) = cx + x q' \), with \( a(b - q') = 0 \).

**Proof.** As above we denote for brevity \( P(x_1, x_2) \) by \( P(X) \) and \([x_1, x_2] \) by \( f(X) \) and consider the generalized polynomial

\[ P(X) = af(X)^2 + af(X)(c - p)f(X) - af(X)^2q. \]

Since \( U \) and \( R \) satisfy the same generalized polynomial identities with coefficients in \( U \) (see Fact 1), then \( P(X) \) is also a generalized identity for \( U \).

Suppose first that \( U \) does not satisfy any non-trivial generalized polynomial identity. Therefore by Proposition 2 we get conclusion 1.

Hence we may suppose now that \( U \) satisfies some non-trivial generalized polynomial identity. By [16] \( U \) is primitive having a non-zero socle \( \text{Soc}(U) \) with \( C \) as the associated division ring and by Jacobson’s Theorem (p. 75 in [8]) \( U \) is isomorphic to a dense ring of linear transformations of some vector space \( V \) over \( C \).
If $V$ is finite-dimensional over $C$, it follows that $R \subseteq U = M_k(C)$, for $k = \dim_C V$. In this case we get the required conclusions by Lemma 1.

Let $\dim_C V = \infty$. Denote $\text{End}_C V$ the ring of endomorphisms of $C V$ and recall that the range of a polynomial $f(X) \in C\{x_1, x_2\}$ is defined as follows

$$r(f; U) = \{f(x_1, x_2) \in \text{End}_C V : x_1, x_2 \in U\}.$$

In [19] (Lemma) it is proved that, if $U$ is a dense subring of $\text{End}_C V$ and $\dim_C V = \infty$, then $r(f; U)$ is a dense subset of $\text{End}_C V$ and this implies that $U$ satisfies the generalized polynomial identity

$$abx^2 + ax(c - p)x - ax^2q. \quad (6)$$

Suppose that there exists a minimal idempotent element $e$ of $\text{Soc}(U)$ such that $e(c - p)(1 - e) \neq 0$. Replace in (6) $x$ by $(1 - e)re$ for any $r \in U$, then it follows that $a(1 - e)re(c - p)(1 - e)re = 0$, which implies $a(1 - e) = 0$, since $e(c - p)(1 - e) \neq 0$. This means that $a = ae$.

On the other hand, if in (6) we replace $x$ by $ere$ for any $r \in U$, we get

$$ab(ere)^2 + aer(e - p)ere - a(ere)^2q = 0,$$

and by right multiplying by $(1 - e)$ one has $-ae(ere)^2q(1 - e) = 0$. Since $0 \neq a = ae$, we have $eq(1 - e) = 0$, that is $eq = eae$.

Finally replace in (6) $x$ by $x + y$. It follows that $U$ satisfies:

$$ab(xy) + ab(yx) + ax(c - p)y + ay(c - p)x - a(xy)q - a(yx)q$$

and for any $x = re$ and $y = (1 - e)s$, with $r, s \in U$, we get

$$ab(1 - e)sr + are(c - p)(1 - e)s + (1 - e)s(c - p)re - a(1 - e)sreq = 0.$$

By right multiplying by $(1 - e)$ and since $eq(1 - e) = 0$, we have $are(c - p)(1 - e)s(1 - e) = 0$, for all $r, s \in U$. By the primeness of $U$ and by the assumption that $e(c - p)(1 - e) \neq 0$, the contradiction $a = 0$ follows.

Therefore $e(c - p)(1 - e) = 0$, for any idempotent element $e \in \text{Soc}(U)$ of rank 1. Hence $[c - p, e] = 0$, for any idempotent of rank 1, and $[c - p, \text{Soc}(U)] = 0$, since $\text{Soc}(U)$ is generated by these idempotent elements. This argument gives $c - p \in C$, and as a consequence of (6), $U$ satisfies the generalized polynomial identity

$$abx^2 + ax^2(c - p - q). \quad (7)$$

As above, suppose that there exists a minimal idempotent element $e$ of $\text{Soc}(U)$ such that $(1 - e)(c - p - q)e \neq 0$. If we replace in (7) $x$ by $(1 - e)r(1 - e)$
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for any \( r \in U \) and multiply by \( e \) on the right, then we get

\[ a((1 - e)r(1 - e))^2(c - p - q)e = 0, \]  

that is \( a(1 - e) = 0 \), since \((1 - e)(c - p - q)e \neq 0\).

Now by (7), for \( x = t + y \), it follows that \( U \) satisfies

\[ abty + abyt + aty(c - p - q) + ay(c - p - q) = 0, \]

for all \( r_1, r_2 \in I \). Under these assumptions we have that:

**Theorem 2.** If \( R \) is a prime ring of characteristic different from \( 2 \), then one of the following holds:

1. There exist \( b', c' \in U \) such that \( H(x) = b'x + xc', G(x) = c'x \) with \( ab' = 0 \);
2. \( R \) satisfies \( s_4 \) and there exist \( b', c', q' \in U \) such that \( H(x) = b'x + xc', G(x) = c'x + xq' \), with \( a(b' - q') = 0 \).

**Proof.** By Theorem 3 in [13] every generalized derivation \( g \) on a dense right ideal of \( R \) can be uniquely extended to the Utumi quotient ring \( U \) of \( R \), and thus we can think of any generalized derivation of \( R \) to be defined on the whole \( U \) and to be of the form \( g(x) = bx + d(x) \) for some \( b \in U \) and \( d \) a derivation on \( U \). Thus we may assume that there exist \( b, p \in U \) and \( d, \delta \) derivations on \( U \) such that

\[ H(x) = bx + d(x) \quad \text{and} \quad G(x) = px + \delta(x). \]

Since \( I, R \) and \( U \) satisfy the same differential identities [14], then without loss of generality, in order to prove our results we may assume that

\[ a(H([r_1, r_2])[r_1, r_2] - [r_1, r_2]G([r_1, r_2])) = 0 \]

3. The general case

We consider now the more general situation and prove the main Theorem of the paper. As in Section 1, since we suppose char(\( R \)) \( \neq 2 \), by Fact 5 we may assume that there exists a non-zero ideal \( I \) of \( R \) such that

\[ a(H([r_1, r_2])[r_1, r_2] - [r_1, r_2]G([r_1, r_2])) = 0 \]

for all \( r_1, r_2 \in I \). Under these assumptions we have that:

**Theorem 2.** If \( R \) is a prime ring of characteristic different from 2, then one of the following holds:

1. There exist \( b', c' \in U \) such that \( H(x) = b'x + xc', G(x) = c'x \) with \( ab' = 0 \);
2. \( R \) satisfies \( s_4 \) and there exist \( b', c', q' \in U \) such that \( H(x) = b'x + xc', G(x) = c'x + xq' \), with \( a(b' - q') = 0 \).

**Proof.** By Theorem 3 in [13] every generalized derivation \( g \) on a dense right ideal of \( R \) can be uniquely extended to the Utumi quotient ring \( U \) of \( R \), and thus we can think of any generalized derivation of \( R \) to be defined on the whole \( U \) and to be of the form \( g(x) = bx + d(x) \) for some \( b \in U \) and \( d \) a derivation on \( U \). Thus we may assume that there exist \( b, p \in U \) and \( d, \delta \) derivations on \( U \) such that

\[ H(x) = bx + d(x) \quad \text{and} \quad G(x) = px + \delta(x). \]

Since \( I, R \) and \( U \) satisfy the same differential identities [14], then without loss of generality, in order to prove our results we may assume that

\[ a(H([r_1, r_2])[r_1, r_2] - [r_1, r_2]G([r_1, r_2])) = 0 \]
for all \( r_1, r_2 \in U \). Hence \( U \) satisfies
\[
a((b[x_1, x_2] + d([x_1, x_2]))[x_1, x_2] - [x_1, x_2](p[x_1, x_2] + \delta([x_1, x_2])))
\]
that is
\[
a((b[x_1, x_2] + [d(x_1), x_2] + [x_1, d(x_2)]))[x_1, x_2]
- [x_1, x_2](p[x_1, x_2] + [\delta(x_1), x_2] + [x_1, \delta(x_2)])) \quad (8)
\]
where \( d, \delta \) are derivations on \( U \). We divide the proof into 3 cases:

**Case 1:** Let \( d(x) = [c, x] \) and \( \delta(x) = [q, x] \) be both inner derivations in \( U \), so that \( H(x) = bx + [c, x] = (b + c)x + x(-c) \) and \( G(x) = px + [q, x] = (p + q)x + x(-q) \), for suitable elements \( c, q \in U \). In this case \( H \) and \( G \) are both inner generalized derivations in \( U \). We notice that, if \( a \in C \), then by Proposition 1 we have that either there exists \( b' \in U \) such that \( H(x) = xb' \) and \( G(x) = b'x \) for all \( x \in R \) (conclusion 1); or \( R \) satisfies \( s_4 \) and there exist \( b', c' \in U \) such that \( H(x) = b'x + xc', G(x) = c'x + xb' \) (which is a particular case of conclusion 2). In what follows we assume that \( a \notin C \).

Thus by Lemma 2 one of the following holds:

1. By conclusion 1 of Lemma 2 we get: \(-c - p - q = \alpha \in C \) and \( q \in C \),
   \[a(b + c - \lambda) = 0, \quad c' = p\] such that \( H(x) = (b + c - \lambda)x + xc' \) and \( G(x) = c'x \), which is the conclusion 1 of the Theorem.

2. By conclusion 2 of Lemma 2 it follows: \(-c - p - q = \alpha \in C \) and \( q \in C \),
   \[\gamma = -q - \alpha \in C, \quad p = -c + \gamma \] such that \( H(x) = (b + c)x + x(-c) \) and \( G(x) = (-c + \gamma)x \) with \( a(b + c - \gamma) = 0 \). By rewriting \( H(x) = (b + c - \gamma)x + x(\gamma - c) \), we obtain conclusion 1 of the Theorem.

3. By conclusion 3 of Lemma 2 it follows: \(-c - p - q = \alpha \in C, R \) satisfies \( s_4 \) and \( q' = -q - \alpha \) such that \( H(x) = (b + c)x + x(-c) \) and \( G(x) = -cx + xq' \) with \( a(b + c - q') = 0 \), which is the conclusion 2 of the Theorem.

**Case 2:** Assume now that both \( d \) and \( \delta \) are not inner derivations. Suppose first that \( d \) and \( \delta \) are linearly \( C \)-independent modulo \( X \)-inner derivations. In this case, by Kharchenko’s Theorem in [10] (see Fact 3), by (8) we have that \( U \) satisfies
\[
a((b[x_1, x_2] + [t_1, x_2] + [z_1, x_2])[x_1, x_2] - [x_1, x_2](b[x_1, x_2] + [z_1, x_2] + [x_1, z_2]))
\]
and in particular \( U \) satisfies the blended component
\[
a([[t_2], [x_1, z_2]]).
\]
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By Lemma 3 in [5], since we suppose \( a \neq 0 \), \( U \) must satisfy \([x_1, t_2], [x_1, z_2]\). In this case it is well known by Posner’s Theorem that there exists a suitable field \( F \) such that \( U \) and \( M_m(F) \), the ring of \( m \times m \) matrices over \( F \), satisfy the same polynomial identities. In particular, for \( m \geq 2 \), we get the contradiction that

\[
0 = [[e_{12}, e_{22}], [e_{12}, e_{21}]] = -2e_{12} \neq 0.
\]

Consider now the case when there exist \( \alpha, \beta \in C \) such that \( \alpha d + \beta \delta = ad(q) \), the inner derivation induced by some \( q \in U \). Of course both \( \alpha \) and \( \beta \) are not zero, since \( d \) and \( \delta \) are not inner derivations. So, if denote \( \lambda = -\alpha \beta^{-1} \) and \( \mu = \beta^{-1} \), it follows that \( \delta = \lambda d + \mu ad(q) \). Thus by (8) we have

\[
a(b[x_1, x_2] + [d(x_1), x_2] + [x_1, d(x_2)]|x_1, x_2]
- [x_1, x_2](p[x_1, x_2] + \lambda d(x_1), x_2) + \lambda[x_1, d(x_2)] + \mu[[q, x_1], x_2] + \mu[x_1, [q, x_2]] (9)
\]

From (9) and applying Kharchenko’s result, it follows that \( R \) satisfies

\[
a(b[x_1, x_2] + [t_1, x_2] + [x_1, t_2]|x_1, x_2]
- [x_1, x_2](p[x_1, x_2] + \lambda t_1, x_2) + \lambda[x_1, t_2] + \mu[[q, x_1], x_2] + \mu[x_1, [q, x_2]]
\]

and in particular \( R \) satisfies the blended component

\[
a(b[x_1, x_2] - \lambda[x_1, x_2][x_1, t_2]).
\]

As above by Lemma 3 in [5], since \( a \neq 0 \), \( R \) satisfies the polynomial identity \([x_1, t_2]|x_1, x_2] - \lambda x_1, x_2][x_1, t_2] \). Since \( R \) is a PI-ring, then there exists a field \( F \) such that \( R, U \) and \( M_m(F) \) satisfy the same polynomial identities. In particular \( M_m(F) \) satisfies

\[
[x_1, t_2]|x_1, x_2] - \lambda[x_1, x_2][x_1, t_2] (10)
\]

Consider \( m \geq 2 \). In (10) choose \( x_1 = e_{12}, x_2 = e_{21} \) and \( t_2 = e_{22} \). By calculations it follows \(- (1 + \lambda)e_{12} = 0 \), which means \( \lambda = -1 \).

On the other hand, for \( x_1 = e_{12} \) and \( x_2 = t_2 = e_{21} \), by (10) we have

\[
(1 - \lambda)(e_{11} + e_{22}) = 0,
\]

which implies \( \lambda = 1 \), that is a contradiction, since \( \text{char}(R) \neq 2 \).

Case 3: Finally assume that either \( d \) or \( \delta \) is an inner derivation on \( U \). Without loss of generality we may assume that \( d(x) = [c, x] \), for a suitable \( c \in U \) and let \( \delta \) be an outer derivation of \( U \). By (8) and Kharchenko’s result, we get that \( U \) satisfies

\[
a(b[x_1, x_2] + c[x_1, x_2] - [x_1, x_2]c[x_1, x_2] - [x_1, x_2]([p[x_1, x_2] + [z_1, x_2] + [x_1, z_2]])
\]
and in particular $U$ satisfies the component

$$a(-[x_1, x_2][x_1, z_2]).$$

As above, by Lemma 3 in [5] and since $a \neq 0$, it follows that $U$ satisfies the polynomial identity $[x_1, x_2][x_1, z_2]$. Let $M_m(F)$ be the ring of $m \times m$ matrices over a field $F$, which satisfies the same identities of $U$. This implies the following contradiction:

$$0 = [e_{12}, e_{22}][e_{12}, e_{21}] = -e_{12} \neq 0.$$

Notice that in the case $\delta$ is inner and $d$ is outer, we may obtain the same contradiction by using the same argument as above. \hfill \Box

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