Derivations in commutators with power central values in rings

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Abstract. Let $R$ be a prime ring of characteristic different from 2 and $I$ a nonzero ideal of $R$, $d$ a nonzero derivation of $R$ such that $[d(x)^k, x^n]$ is central, for all $x \in I$ where $k, n$ are fixed positive integers. Then $R$ satisfies $s_4$, the standard identity in 4 variables.

1. Introduction

Throughout this article, $R$ is always a prime ring with center $Z$. For any $x, y \in R$, we set $[x, y]_1 = [x, y] = xy - yx$ and $[x, y]_n = [[x, y]_{n-1}, y]$ where $n > 1$ is a positive integer. By $s_4$ we denote the standard identity in 4 variables. By $d$ we denote a nonzero derivation of $R$.

A well-known result proved by Posner [14] states that $R$ must be commutative if $[d(x), x] \in Z$ for all $x \in R$. In [12] Lee and Lee generalized Posner’s result by showing that if $\text{char}(R) \neq 2$ and $[d(x), x] \in Z$ for all $x$ in a noncentral Lie ideal of $R$, then $R$ is commutative. As to the case when $\text{char}(R) = 2$, Lanski obtained the same conclusion except when $R$ satisfies $s_4$ (see [9]). In [2] Carini and De Filippis studied the situation when $[d(x), x]^n \in Z$ for all $x$ in a noncommutative Lie ideal of $R$ with $\text{char}(R) \neq 2$. In [16] the second author and You removed the assumption of $\text{char}(R) \neq 2$. 

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In [6] Felzenszwalb proved that if $d(x^k) = 0$ for all $x \in R$, where $k$ is a fixed positive integer. A significant extension of [6] shows that $R$ is commutative if $[d(x^k), x^k]_n = 0$ for all $x$ in a nonzero left ideal of $R$ (see [11, Theorem 1]). In [15] Shue discussed the situation when $a[d(x^k), x^k]_n = 0$ for all $x$ in an one-sided ideal of $R$, where $0 \neq a \in R$.

The purpose of this paper is to investigate the situation when $[d(x^k), x^k]_n \in Z$ for all $x$ in a nonzero ideal of $R$. The main result is the following

**Theorem 1.** Let $R$ be a prime ring of characteristic different from 2 with its center $Z$, $I$ a nonzero ideal of $R$, and $d$ a nonzero derivation of $R$ such that $[d(x^k), x^k]_n \in Z$ for all $x \in I$ where $k$, $n$ are fixed positive integers. Then $R$ satisfies $s_4$, the standard identity in 4 variables.

The following counterexample shows that Theorem 1 is not valid on some one-sided ideals.

**Example 1.** Let $F$ be a field and $R = M_m(F)$, the ring of all $m \times m$ matrix algebra over $F$ with $m > 2$. Let $e_{ij}$ be the matrix unit with 1 in $(i, j)$-entry and zero elsewhere. It is easy to check that $([e_{11}, x^k]_2)_n = 0$ for all $x \in Re_{22}$ (or, $e_{22}R$), where $n > 1$.

2. **The proof of Theorem 1**

By $Q$ we denote the Martindale quotient ring of $R$ and $C$ the extended centroid $R$. The definitions and properties of these objects can be found in [1, Chapter 2].

We begin with the following easy result.

**Lemma 1.** Let $R = M_2(F)$, the ring of all $2 \times 2$ matrices over a field $F$ with char($F$) $\neq 2$. If $a$ is a nonzero element of $R$ such that $([a, x^k]_2)_n = 0$ for all $x \in R$, then $a \in F \cdot I_2$.

**Proof.** Let $a = \sum_{i,j} a_{ij}e_{ij}$ with $a_{ij} \in F$. We first claim that $a$ is a diagonal matrix. By assumption we get

$$0 = ([a, e_{11}]_2)_n = (a_{12}a_{21})^n e_{11} + (a_{12}a_{21})^n e_{22},$$

thus $a_{12}a_{21} = 0$. Without loss of generality we may assume that $a_{21} = 0$. Let $\varphi \in Aut_F(M_2(F))$ such that $\varphi(x) = (1 + e_{21})x(1 - e_{21})$. In particular, we have

$$\varphi(a) = (a_{11} - a_{12})e_{11} + a_{12}e_{12} + (a_{11} - a_{12} - a_{22})e_{21} + (a_{12} + a_{22})e_{22}. $$
Since \((\lbrack \varphi(a), x^k \rbrack)_2)^n = 0\) for all \(x \in R\), as above we can get that \(a_{12}(a_{11} - a_{12} - a_{22}) = 0\). That is, either \(a_{12} = 0\) or \(a_{11} - a_{12} - a_{22} = 0\). If \(a_{11} - a_{12} - a_{22} = 0\), then

\[
[a, e_{11} + e_{21}]_2 = -a_{12}e_{11} + a_{12}e_{12} - 3a_{12}e_{21} + a_{12}e_{22}.
\]

By assumption we get

\[
0 = ([a, e_{11} + e_{21}]_2)^n = (-2a_{12}^2)^n e_{11} + (-2a_{12}^2)^n e_{22}
\]

and so \(a_{12} = 0\), this implies that \(a\) is a diagonal matrix.

Write \(a = \sum_{i=1}^{2} a_i e_{ii}\), we see as above that \(\varphi(a) = \sum_{i=1}^{2} a_i e_{ii} + (a_{11} - a_{22}) e_{21}\) is also a diagonal matrix. Therefore \(a_{11} = a_{22}\) and so \(a \in F \cdot I_2\) as desired. \(\square\)

If \((\lbrack a, x^k \rbrack)_2)^n \in F \cdot I_2\) for all \(x \in M_2(F)\), one can not expect to obtain that \(a \in F \cdot I_2\). For example, it is easy to check that \((\lbrack e_{11}, x \rbrack)_2)^2 \in F \cdot I_2\) for all \(x \in M_2(F)\).

**Lemma 2.** Let \(R = M_m(F)\), the ring of all \(m \times m\) matrices over a field \(F\) with \(\text{char}(F) \neq 2\). If \(a\) is a noncentral element of \(R\) such that \((\lbrack a, x^k \rbrack)_2)^n \in F \cdot I_m\) for all \(x \in R\), then \(m \leq 2\).

**Proof.** Suppose on the contrary that \(m > 2\). Let \(a = \sum a_{ij} e_{ij}\) with \(a_{ij} \in F\). Write \(a = \begin{pmatrix} a_{11} & A \\ B & C \end{pmatrix}\), where \(A = (a_{12}, \ldots, a_{1m}), B = (a_{21}, \ldots, a_{m1})^T\), and \(C = (a_{ij})\) with \(2 \leq i, j \leq m\). Since \([a, e_{11}]_2 = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}\), by assumption we have

\[
([a, e_{11}]_2)^n = \begin{pmatrix} (AB)^n & 0 \\ 0 & (BA)^n \end{pmatrix} \in F \cdot I_m.
\]

Set \(\alpha = AB \in F\). Then \(\begin{pmatrix} \alpha^n & 0 \\ 0 & -\alpha^{-1} BA \end{pmatrix} \in F \cdot I_m\). If \(\alpha \neq 0\), then

\[
\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha^{-1} BA \end{pmatrix} \in F \cdot I_m.
\]

Thus, \(\alpha = a_{21}a_{12} = a_{31}a_{13}\) and \(a_{21}a_{13} = 0\). Thus \(\alpha = 0\), a contradiction. Hence \(AB = 0\).

Let \(\varphi_{ij}\) be an inner automorphism of \(R\) given by \(\varphi_{ij}(x) = (1 + e_{ij})x(1 - e_{ij})\) for all \(x \in R\). Write \(1 + e_{21} = \begin{pmatrix} 1 & 0 \\ E_2 & I_{m-1} \end{pmatrix}\), where \(E_2 = (1, 0, \ldots, 0)^T\) and \(I_{m-1}\) is the \((m-1)\)-identity matrix. So

\[
\varphi_{21}(a) = \begin{pmatrix} a_{11} - a_{12} & A \\ a_{11}E_2 - a_{12}E_2 + B - CE_2 & E_2A + C \end{pmatrix}.
\]
Since \( ([\varphi_{21}(a), x^k])^n \in F \cdot I_m \) for all \( x \in R \), as above we have
\[
A(a_{11}E_2 - a_{12}E_2 + B + CE_2) = 0.
\]
Recalling \( AB = 0 \) we get from the last relation that \( a_{11}a_{12} - a_{12}^2 - ACE_2 = 0 \).

Since \( [a, e_{11} + e_{21}] = \left( \begin{array}{cc} -a_{12} & A \\ D & E_2A \end{array} \right) \), where \( D = B + CE_2 - (a_{11} + 2a_{12})E_2 \), we get
\[
([a, e_{11} + e_{21}]^2 = \left( \begin{array}{cc}
-a_{12}^2 + AD & 0 \\
-a_{12}D + E_2AD & DA + a_{12}E_2A
\end{array} \right).
\]

Making use of both \( AB = 0 \) and \( a_{11}a_{12} - a_{12}^2 - ACE_2 = 0 \) we get \( AD = -3a_{12}^2 \).

Thus
\[
([a, e_{11} + e_{21}]^2 = \left( \begin{array}{cc}
-2a_{12}^2 & 0 \\
-a_{12}D - 3a_{12}E_2 & DA + a_{12}E_2A
\end{array} \right).
\]

Therefore
\[
([a, e_{11} + e_{21}]^2)^n = \left( \begin{array}{cc}
(-2a_{12}^2)^n & 0 \\
(DA + a_{12}E_2A)^n
\end{array} \right) \in F \cdot I_m
\]
where \( U \) is a \((m - 1) \times 1\) matrix. Since \( \text{rank}((DA + a_{12}E_2A)^n) \leq \text{rank}(A) \leq 1 \) and \( m > 2 \), we infer that \((-2a_{12}^2)^n = 0\) and so \( a_{12} = 0 \).

Now we claim that \( a \) is a diagonal matrix. Since \( ([\varphi_{j2}(a), x^k])^n \in F \cdot I_m \) for all \( x \in R \), where \( j > 2 \), as above we have that \( -a_{1j} = \varphi_{j1}(a)_{1j} = 0 \). So \( a_{1j} = 0 \) for \( j > 1 \). For \( 1 < j < t \leq m \), as above we get from \( ([\varphi_{j1}(a), x^k])^n \in F \cdot I_m \) for all \( x \in R \), that \( a_{jt} = \varphi_{j1}(a)_{1t} = 0 \). This shows that \( a \) must be lower triangular. Since the transpose of \( a \) satisfies the same condition, \( a \) is indeed diagonal.

We have showed that \( a = \sum_{i=1}^m a_{ii}e_{ii} \) with \( a_{ii} \in F \). For \( 1 \leq i \neq j \leq m \), as above we get that \( \varphi_{ij}(a) \) is a diagonal matrix. On the other hand \( \varphi(a) = a + (a_{jj} - a_{ii})e_{ij} \), we infer that \( a_{1j} = a_{ii} \) and so \( a \) is central in \( R \), which is a contradiction. The proof is thereby complete. \( \Box \)

The following result is a special case of Theorem 1, which is of independent interest.

**Lemma 3.** Let \( R \) be a prime ring with \( \text{char}(R) \neq 2 \) and \( I \) a nonzero ideal of \( R \), \( d \) a nonzero derivation of \( R \) such that \( [d(x^k), x^k]^n = 0 \) for all \( x \in I \) where \( k, n \) are fixed positive integers. Then \( R \) is commutative.

**Proof.** By assumption we see that \( I \) satisfies the differential identity
\[
\left[ \sum_{i=0}^{k-1} x^i d(x) x^{k-i-1}, x^k \right]^n = 0.
\]
If \( d \) is not \( Q \)-inner, by Kharchenko’s theorem [7], \( I \) satisfies the polynomial identity \( \left[ \sum_{i=0}^{k-1} x^i y^k x^{k-i-1}, x^k \right]^n = 0 \) and so for \( R \) too. It is well known that there exists a field \( F \) such that \( R \) and \( F_m \) satisfy the same polynomial identities [8, p. 57 and p. 89]. Suppose that \( m \geq 2 \). If we choose \( x = e_{11}, y = e_{12} + e_{21} \), then we get a contradiction as follows

\[
0 = \left[ \sum_{i=0}^{k-1} c_{i1} (e_{12} + e_{21}) e_{11}^{k-i-1}, e_{11} \right]^{2n} = [e_{12} + e_{21}, e_{11}]^{2n} = (-1)^n (e_{11} + e_{22}) \neq 0.
\]

Thus \( m = 1 \) and so \( R \) is commutative.

Assume next that \( d \) is \( Q \)-inner, that is, \( d(x) = [a, x] \) for all \( x \in R \), where \( a \) is a noncentral element in \( Q \). By assumption we get \( ([a, x^k]^n = 0 \) for all \( x \in I \). By a theorem of Chuang [4, Theorem 2], \( ([a, x^k]^n = 0 \) for all \( x \in Q \). In case \( C \) is infinite, we have \( ([a, x^k]^n = 0 \) for all \( x \in Q \otimes_C C \), where \( C \) is the algebraic closure of \( C \). Since both \( Q \) and \( Q \otimes_C C \) are centrally closed [5, Theorems 2.5 and 3.5], we may replace \( R \) by \( Q \otimes_C C \) according as \( C \) is finite or infinite. Thus we may assume that \( R \) is centrally closed over \( C \) which is either finite or algebraically closed and \( ([a, x^k]^n = 0 \) for all \( x \in R \). By Martindale’s theorem [13], \( R \) is a primitive ring and so isomorphic to a dense subring of linear transformations on a vector space \( V \) over \( C \).

If \( V \) is infinite dimensional over \( C \), for any given \( v \in V \) we claim that \( v \) and \( va \) are \( C \)-dependent. Suppose on the contrary that \( v \) and \( va \) are \( C \)-independent. We choose \( v_1, \ldots, v_{2k-1} \) such that \( v, va, v_1, \ldots, v_{2k-1} \) are \( C \)-independent. By the density of \( R \) on \( CV \), there exists \( x \in R \) such that \( vx = 0, vax = v_1, v_i x = v_{i+1}, v_{2k-1} x = v \), where \( i = 1, \ldots, 2k - 2 \). Thus

\[
v[a, x^k]_2 = vax v_{2k} = v_1 x^{2k-1} = \cdots = v_{2k-1} x = v
\]

and so \( 0 = v ([a, x^k]^n = v \), a contradiction. Therefore \( v \) and \( va \) are \( C \)-dependent for any \( v \in V \). A standard argument shows that \( a \in C \), a contradiction. So \( V \) must be of finite dimension. That is, \( R \cong M_s(C) \) for some \( s \). In view of both Lemma 1 and Lemma 2 we get that \( a \in C \), a contradiction. The proof is now complete.

The proof of Theorem 1. Suppose on the contrary that \( \text{dim}_C RC > 4 \). By assumption we have \([d(x^k), x^k]^n \in Z \) for all \( x \in I \), that is, \( I \) satisfies the following differential identity

\[
\left[ \sum_{i=0}^{k-1} x^i d(x) x^{k-i-1}, x^k \right]^n, y = 0.
\]

(1)
If \([d(x^k), x^k]^n = 0\) for all \(x \in I\), the result follows from Lemma 3. Otherwise, there exists \(r \in I\) such that \([d(r^k), r^k]^n \neq 0\). Thus \(I\) satisfies the central differential identity \([d(x^k), x^k]^n\). By [3, Theorem 1] we get that \(R\) is a PI-prime ring and so is \(Q\).

If \(d\) is not \(Q\)-inner, applying Kharchenko’s theorem to (1) we get that \(I\) satisfies the polynomial identity \(\left[\sum_{i=0}^{k-1} x^i y x^{k-i-1}, x^k\right]^n = 0\) for all \(x, y \in I\). It is well known that there exists a field \(F\) such that \(R\) and \(F\) satisfy the same polynomial identities. Thus \([\sum_{i=0}^{k-1} x^i y x^{k-i-1}, x^k]^n \in F \cdot I_m\). Note that \(m > 2\). If we choose \(x = e_{11}, y = e_{12} + e_{21}\), then
\[
\left[\sum_{i=0}^{k-1} e_{11}^{i}(e_{12} + e_{21}) e_{11}^{k-i-1}, e_{11}^{k}\right]^{2n} = (-1)^n(e_{11} + e_{22}) \in F \cdot I_m.
\]
This is a contradiction.

We next assume that \(d\) is an \(Q\)-inner derivation induced by a noncentral element \(b \in Q\). It follows from (1) that
\[
\left[\sum_{i=0}^{k-1} x^i [b, x] x^{k-i-1}, x^k\right]^n = 0 \quad \text{for all} \quad x, y \in I.
\]
In view of [4, Theorem 2] we have
\[
\left[\sum_{i=0}^{k-1} x^i [b, x] x^{k-i-1}, x^k\right]^n = 0 \quad \text{for all} \quad x, y \in Q.
\]
Since there exists \(r \in R\) such that \([d(r^k), r^k]^n \neq 0\), we see that (3) is a non trivial generalized polynomial identity on \(Q\). By Martindale’s theorem [13] \(Q\) is primitive ring. Since \(Q\) is a PI-ring, by the famous Kaplanksy’s theorem [1, Theorem 6.1.10] we see that \(Q\) is a finite dimensional central simple algebra over \(C\). It follows from [10, Lemma 2] that there exists a suitable field \(F\) of \(\text{char}(F) \neq 2\) such that \(Q \subseteq M_m(F)\) and moreover \(M_m(F)\) satisfies the same generalized polynomial identity (3). Then Lemma 2 tells us that \(m \leq 2\), which is a contradiction. The proof is thereby complete.

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References
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