On a characterization theorem on Abelian groups

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Abstract. Let $\xi_1, \xi_2, \ldots, \xi_n$, $n \geq 2$ be independent identically distributed random variables. It is well known that if $\bar{\xi} = \frac{1}{n} \sum_{j=1}^{n} \xi_j$ and $v = (\xi_1 - \bar{\xi}, \xi_2 - \bar{\xi}, \ldots, \xi_n - \bar{\xi})$ are independent, then all $\xi_j$ are Gaussian. We give a complete description of second countable locally compact Abelian groups for which a group analogue of this characterization theorem holds true.

1. Introduction

It is well known that if $\xi_1, \xi_2, \ldots, \xi_n$, $n \geq 2$ are independent identically distributed Gaussian random variables, then $\bar{\xi} = \frac{1}{n} \sum_{j=1}^{n} \xi_j$ and $v = (\xi_1 - \bar{\xi}, \xi_2 - \bar{\xi}, \ldots, \xi_n - \bar{\xi})$ are independent. Assume now that $\xi_1, \xi_2, \ldots, \xi_n$, $n \geq 2$ are independent identically distributed random variables such that $\bar{\xi}$ and $v$ are independent. Then $\bar{\xi}$ and $s^2 = \frac{1}{n} \sum_{j=1}^{n} (\xi_j - \bar{\xi})^2$ are also independent. This implies by Geary’s theorem ([6], [11], [10], [13]) that the random variables $\xi_j$ are Gaussian. Thus, a Gaussian measure on the real line is characterized by the independence of $\bar{\xi}$ and $v$.

The article deals with a generalization of this characterization theorem to the case when independent random variables take values in a locally compact Abelian group. Since an arbitrary Abelian group generally is not a group with unique division by $n$, instead of $\bar{\xi}$ and $v$ we consider $S = \sum_{j=1}^{n} \xi_j$ and $V = (n\xi_1 - S, \ldots, n\xi_n - S)$.

We will use in the article the standard results on structure of locally compact Abelian groups and the duality theory ([7]). Agree on notation. For an arbitrary

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locally compact Abelian group $X$ denote by $Y = X^*$ its character group, and by $(x, y)$ the value of a character $y \in Y$ at an element $x \in X$. If $H$ is a closed subgroup of the group $Y$, then denote by $A(X, H) = \{x \in X : (x, y) = 1 \text{ for all } y \in H\}$ its annihilator. Denote by $c_X$ the connected component of zero of the group $X$, and by $b_X$ the set of all compact elements of $X$. Let $n$ be a natural number. Put $X^{(n)} = \{x \in X : nx = 0\}$ and $X^{(n)} = \{x \in X : x = n\tilde{x}, \tilde{x} \in X\}$. Denote by $\mathbb{Z}$ the group of integers, by $\mathbb{R}$ the group of real numbers and by $T$ the circle group (the one-dimensional torus). Let $Y$ be an arbitrary Abelian group, $f(y)$ be a function on $Y$, $h$ be an element of $Y$. Denote by $\Delta_h$ the finite difference operator

$$\Delta_h f(y) = f(y + h) - f(y).$$

A function $f(y)$ on $Y$ is called a polynomial if for a nonnegative integer $m$ $f(y)$ satisfies the equation

$$\Delta^{m+1}_h f(y) = 0, \quad y, h \in Y.$$

The minimal $m$ for which this equality holds is called the degree of the polynomial $f(y)$.

We will assume in the article that $X$ is a second countable locally compact Abelian group. Denote by $M^1(X)$ the convolution semigroup of probability distributions on $X$. For $\mu \in M^1(X)$ denote by $\hat{\mu}(y) = \int_X (x, y) d\mu(x)$ its characteristic function. Note that if $\xi$ is a random variable with values in $X$ and distribution $\mu$, then the characteristic function of the distribution $\mu$ is the mathematical expectation

$$\hat{\mu}(y) = E[(\xi, y)].$$

Denote by $E_x$ the degenerate distribution concentrated at the point $x \in X$. The set of all degenerate distributions on the group $X$ denote by $D(X)$. For $\mu \in M^1(X)$ define the distribution $\bar{\mu} \in M^1(X)$ by the formula $\bar{\mu}(B) = \mu(-B)$ for any Borel set $B$. Denote by $\sigma(\mu)$ the support of a distribution $\mu$.

A probability measure $\gamma$ on the group $X$ is called Gaussian (in the sense of Parthasarathy) ([12, Ch. 4.6]), if its characteristic function can be represented in the form

$$\hat{\gamma}(y) = (x, y) \exp\{-\varphi(y)\}, \quad (1)$$

where $x \in X$, and $\varphi(y)$ is a continuous nonnegative function on $Y$ satisfying the equation

$$\varphi(u + v) + \varphi(u - v) = 2[\varphi(u) + \varphi(v)], \quad u, v \in Y. \quad (2)$$
Taking into account that in the article we will deal only with Gaussian measures in the sense of Parthasarathy we will name them Gaussian. Denote by $\Gamma(X)$ the set of Gaussian measures on the group $X$. Denote by $m_K$ the normalized Haar measure of a compact subgroup $K$ of the group $X$, and by $I(X)$ the set of shifts of such measures.

We will say that a distribution $\mu \in \Gamma_n(X)$ if there exist independent identically distributed random variables $\xi_j, j = 1, 2, \ldots, n, n \geq 2$ with values in the group $X$ and a distribution $\mu$ such that $S$ and $V$ are independent.

It is not difficult to verify that for a group $X$ the inclusion $\Gamma(X) \subset \Gamma_n(X)$ holds. In §2 we completely describe groups $X$ which have the following property: if $\mu \in \Gamma_n(X)$ and the characteristic function $\hat{\mu}(y)$ does not vanish, then $\mu \in \Gamma(X)$. We apply the results of §2 in §3 to give the complete description of locally compact Abelian groups $X$ for which any distribution $\mu \in \Gamma_n(X)$ is invariant with respect to a compact subgroup $K$ of the group $X$ and under the natural homomorphism $X \mapsto X/K$ induces on the factor-group $X/K$ a Gaussian measure. We can consider the obtained class of groups as the widest subclass of locally compact Abelian groups on which an analogue of the theorem about characterization of a Gaussian measure on the real line by the independence of $\bar{\xi} = \frac{1}{n} \sum_{j=1}^{n} \xi_j$ and $v = (\xi_1 - \bar{\xi}, \xi_2 - \bar{\xi}, \ldots, \xi_n - \bar{\xi})$ holds.

The above mentioned problems are reduced to solving of a functional equation in the class of continuous positive definite functions on the group $Y = X^*$.

2. The characteristic function $\hat{\mu}(y)$ does not vanish

**Lemma 1.** A distribution $\mu \in \Gamma_n(X)$ if and only if its characteristic function $\hat{\mu}(y)$ satisfies the equation

$$\prod_{j=1}^{n} \hat{\mu}(u + n v_j - (v_1 + \cdots + v_n))$$

$$= \hat{\mu}^n(u) \prod_{j=1}^{n} \hat{\mu}(n v_j - (v_1 + \cdots + v_n)), \quad u, v_1, \ldots, v_n \in Y. \quad (3)$$

**Proof.** Note that $S$ and $V$ are independent if and only if the equality

$$E[(S, u)(V, (v_1, \ldots, v_n))] = E[(S, u)|E[(V, (v_1, \ldots, v_n))]$$

holds for all $u, v_1, \ldots, v_n \in Y$. Taking into account that the random variables
\[ \xi_1, \ldots, \xi_n \text{ are independent, we transform the left-hand side of (4):} \]
\[ E[(S, u)(V, (v_1, \ldots, v_n))] = E[(\xi_1 + \cdots + \xi_n, u)((n_1 - (\xi_1 + \cdots + \xi_n), \ldots, n_n - (\xi_1 + \cdots + \xi_n)), (v_1, \ldots, v_n))] \]
\[ = E \left[ \prod_{j=1}^{n} (\xi_j, u + nv_j - (v_1 + \cdots + v_n)) \right] = \prod_{i=1}^{n} \hat{\mu}(u + nv_i - (v_1 + \cdots + v_n)). \]

Analogously we transform the right-hand side of (4):
\[ E[(S, u)]E[(V, (v_1, \ldots, v_n))] = E[(\xi_1 + \cdots + \xi_n, u)]E[((n_1 - (\xi_1 + \cdots + \xi_n), \ldots, n_n - (\xi_1 + \cdots + \xi_n)), (v_1, \ldots, v_n))] \]
\[ = \prod_{j=1}^{n} E[(\xi_j, u)]E \left[ \prod_{j=1}^{n} (\xi_j, nv_j - (v_1 + \cdots + v_n)) \right] \]
\[ = \prod_{j=1}^{n} E[(\xi_j, u)] \prod_{j=1}^{n} E[(\xi_j, nv_j - (v_1 + \cdots + v_n))] \]
\[ = \hat{\mu}^n(u) \prod_{i=1}^{n} \hat{\mu}(nv_i - (v_1 + \cdots + v_n)). \] \[ \square \]

Suppose that \( \gamma \in \Gamma(X) \) and the characteristic function \( \hat{\gamma}(y) \) has representation (1). By the function \( \varphi(y) \) we can construct a symmetric 2-additive function by the formula
\[ \psi(u, v) = \frac{1}{2}[\varphi(u + v) - \varphi(u) - \varphi(v)]. \]
Then \( \varphi(y) = \psi(y, y) \). Using this representation for the function \( \varphi(y) \) one can check directly that the characteristic function \( \hat{\gamma}(y) \) satisfies equation (3). Hence, by Lemma 1 the inclusion
\[ \Gamma(X) \subset \Gamma_n(X) \] holds. The main result of this section is the complete description of groups \( X \) which have the property: if \( \mu \in \Gamma_n(X) \) and the characteristic function \( \hat{\mu}(y) \) does not vanish, then \( \mu \in \Gamma(X) \). The following proposition is valid.

**Proposition 1.** Assume that \( \mu \in \Gamma_2(X) \) and the characteristic function \( \hat{\mu}(y) \) does not vanish. This implies that \( \mu \in \Gamma(X) \) if and only if the group \( X \) contains no subgroup topologically isomorphic to the two-dimensional torus \( \mathbb{T}^2 \). Assume that \( \mu \in \Gamma_n(X) \), where \( n \geq 3 \). This implies that \( \mu \in \Gamma(X) \) if and only if the group \( X \) contains no subgroup topologically isomorphic to the circle group \( \mathbb{T} \).
We need some lemmas to prove Proposition 1.

**Lemma 2.** Let $X = \mathbb{T}$ and $n \geq 3$. Then there exists a distribution $\mu \in \Gamma_n(X)$ such that the characteristic function $\hat{\mu}(y)$ does not vanish and $\mu \notin \Gamma(X)$.

**Proof.** Let $n = 3$. Consider on the group $\mathbb{Z}$ the function

$$l(k) = \begin{cases} 1 & \text{if } k \in \mathbb{Z}^{(3)} \\ \exp \left\{ \frac{2\pi i}{9} \right\} & \text{if } k \in 1 + \mathbb{Z}^{(3)} \\ \exp \left\{ -\frac{2\pi i}{9} \right\} & \text{if } k \in 2 + \mathbb{Z}^{(3)}. \end{cases} \tag{6}$$

Obviously, (6) implies that

$$l^3(k) = \exp \left\{ \frac{2\pi ki}{3} \right\}, \quad k \in \mathbb{Z}. \tag{7}$$

Taking into account (7) and the fact that $l(k + 3p) = l(k)$, $k, p \in \mathbb{Z}$, we can verify directly that the function $l(k)$ satisfies equation (3) for $n = 3$.

Take $\sigma > 0$ in such a way that the inequality

$$\sum_{k \in \mathbb{Z}, \ k \neq 0} \exp \{-\sigma k^2\} < 1 \tag{8}$$

holds. Put

$$\rho(t) = 1 + \sum_{k \in \mathbb{Z}, \ k \neq 0} l(k) \exp \{-\sigma k^2 - ikt\}, \quad t \in \mathbb{R}.$$ 

Since $l(-k) = l(k)$, $|l(k)| = 1$, $k \in \mathbb{Z}$, in view of (8) the inequality

$$\rho(t) > 0, \quad t \in \mathbb{R}$$

is valid. We also have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \rho(t) dt = 1.$$ 

Let $\mu$ be the distribution on the group $\mathbb{T}$ with density $r(e^{it}) = \rho(t)$ with respect to $m_\mathbb{T}$. By the construction the characteristic function of the distribution $\mu$ is of the form

$$\hat{\mu}(k) = l(k) \exp \{-\sigma k^2\}, \quad k \in \mathbb{Z}.$$ 

Since the function $l(k)$ satisfies equation (3) for $n = 3$, the characteristic function $\hat{\mu}(k)$ also satisfies equation (3) for $n = 3$. By Lemma 1 $\mu \in \Gamma_3(X)$. On the other hand, since the function $l(k)$ is not a character of the group $\mathbb{Z}$, we have $\mu \notin \Gamma(X)$. 

Let $n > 3$. If $n = 2p - 1$ we put

$$l(k) = \begin{cases} 
1 & \text{if } k \in q + \mathbb{Z}^{(n)}, \ q = 0, 1, \ldots, p - 2, p + 1, \ldots, n - 1, \\
\exp\left(\frac{2\pi i}{n}\right) & \text{if } k \in p - 1 + \mathbb{Z}^{(n)} \\
\exp\left(-\frac{2\pi i}{n}\right) & \text{if } k \in p + \mathbb{Z}^{(n)}. 
\end{cases}$$

If $n = 2p$ we put

$$l(k) = \begin{cases} 
1 & \text{if } k \notin p + \mathbb{Z}^{(n)} \\
-1 & \text{if } k \in p + \mathbb{Z}^{(n)}. 
\end{cases}$$

Next we argue as in the case when $n = 3$. □

Remark 1. Let $n \geq 3$. It is easily seen that the function $l(k)$ constructed in the proof of Lemma 2 is the characteristic function of a signed measure on the group $\mathbb{T}$ concentrated in the subgroup $\mathbb{Z}(n)$ (the multiplicative group of $n$th roots of unity).

Lemma 3. Assume that $\gamma \in \Gamma_n(X)$, and the characteristic function $\hat{\gamma}(y) > 0$ for $y \in Y$. Then $\gamma \in \Gamma(X)$, and the function $\hat{\gamma}(y)$ can be represented in the form (1), where $x = 0$.

Proof. Put $\psi(y) = -\log \hat{\gamma}(y)$. By Lemma 1 the characteristic function $\hat{\gamma}(y)$ satisfies equation (3). Taking the logarithm of both sides of (3), we get

$$\sum_{j=1}^{n} \psi(u + nv_j - (v_1 + \cdots + v_n)) = n\psi(y) + \sum_{j=1}^{n} \psi(nv_j - (v_1 + \cdots + v_n)), \quad u, v_j \in Y. \quad (9)$$

We use the finite difference method to solve equation (9). Let $h_1$ be an arbitrary element of the group $Y$. Substitute $u + h_1$ for $u$ and $v_j + h_1$ for $v_j$, $j = 1, 2, \ldots, n$ in equation (9). Subtracting equation (9) from the resulting equation we obtain

$$\sum_{j=1}^{n} \Delta_{h_1} \psi(u + nv_j - (v_1 + \cdots + v_n)) = n\Delta_{h_1} \psi(u), \quad u, h_1, v_j \in Y. \quad (10)$$

Let $h_2$ be an arbitrary element of the group $Y$. Substitute $u + h_2$ for $u$ and $v_1 + h_2$ for $v_1$ in equation (9). Subtracting equation (10) from the resulting equation we get

$$\Delta_{nh_2} \Delta_{h_1} \psi(u + nv_1 - (v_1 + \cdots + v_n)) = n\Delta_{h_2} \Delta_{h_1} \psi(u), \quad u, h_1, h_2, v_j \in Y. \quad (11)$$
Let \( h_3 \) be an arbitrary element of the group \( Y \). Substitute \( u + h_3 \) for \( u \) and \( v_2 + h_3 \) for \( v_2 \) in equation (9). Subtracting equation (11) from the resulting equation we find
\[
n \Delta_3 \Delta_2 \Delta_1 \psi(u) = 0, \quad u, h_1, h_2, h_3 \in Y.
\] (12)

Put in (12) \( h_1 = h_2 = h_3 = h \). We get
\[
\Delta^3 \psi(u) = 0, \quad u, h \in Y,
\] (13)
i.e. \( \psi(y) \) is a continuous polynomial of degree \( \leq 2 \). It is easy to see that each polynomial of degree \( \leq 2 \), in particular \( \psi(y) \), can be represented in the form
\[
\psi(y) = \varphi(y) + l(y) + c, \quad y \in Y,
\] (14)
where the function \( \varphi(y) \) satisfies equation (2), the function \( l(y) \) satisfies equation
\[
l(u + v) = l(u) + l(v), \quad u, v \in Y,
\] and \( c = \text{const} \). Since \( \hat{\gamma}(0) = 1 \), we can assume that \( c = 0 \). Since the function \( \hat{\gamma}(y) \) is real-valued, we have \( \hat{\gamma}(-y) = \overline{\hat{\gamma}(y)} = \hat{\gamma}(y) \). Hence, \( \psi(-y) = \psi(y) \). This implies that in (14) \( l(y) = 0, y \in Y \). So, \( \psi(y) = \varphi(y), y \in Y \). This proves the lemma.

A distribution \( \mu \in M^1(X) \) is called a Gaussian measure in the sense of Bernstein ([9, 5.3]) if \( \mu \) has the following property: if \( \xi_1 \) and \( \xi_2 \) are independent identically distributed random variables with values in \( X \) and distribution \( \mu \), then their sum and difference are independent. We denote by \( \Gamma_B(X) \) the set of Gaussian measures in the sense of Bernstein on the group \( X \).

**Lemma 4** ([9, 5.3]). A distribution \( \mu \in M^1(X) \) belongs to the class \( \Gamma_B(X) \) if and only if the characteristic function \( \hat{\mu}(y) \) satisfies the equation
\[
\hat{\mu}(u + v)\hat{\mu}(u - v) = \hat{\mu}^2(u)|\hat{\mu}(v)|^2, \quad u, v \in Y.
\] (15)

**Lemma 5.** \( \Gamma_2(X) = \Gamma_B(X) \).

**Proof.** Let \( \mu \in \Gamma_2(X) \). By Lemma 1 the characteristic function \( \hat{\mu}(y) \) satisfies equation (3) which takes the form
\[
\hat{\mu}(u + (v_1 - v_2))\hat{\mu}(u - (v_1 - v_2)) = \hat{\mu}^2(u)|\hat{\mu}(v_1 - v_2)|^2, \quad u, v_1, v_2 \in Y.
\] (16)
Substituting \( v_1 = v, v_2 = 0 \) into (16), we obtain that the characteristic function \( \hat{\mu}(y) \) satisfies equation (15). Hence, by Lemma 4 \( \mu \in \Gamma_B(X) \). Lemmas 1 and 4 also imply that if \( \mu \in \Gamma_B(X) \), then \( \mu \in \Gamma_2(X) \).
Proof of Proposition 1. Let $n = 2$. Applying Lemma 5 we reduce the proof of Proposition 1 to the proof of the corresponding statement for distributions from the class $\Gamma_B(X)$, but for such distributions this statement was proved in [2] (see also [5], Lemmas 9.6 and 9.7).

Let $n \geq 3$ and $\mu \in \Gamma_n(X)$. Put $\nu = \mu * \bar{\mu}$. It follows from Lemma 1 that $\nu \in \Gamma_n(X)$. Since $\hat{\nu}(y) = |\hat{\mu}(y)|^2 > 0$, by Lemma 3 $\nu \in \Gamma(X)$. If a group $X$ contains no subgroup topologically isomorphic to $\mathbb{T}$, then by Cramer’s theorem for locally compact Abelian groups ([1], see also [5, Theorem 4.6]), we get that $\mu \in \Gamma(X)$. Thus, the sufficiency when $n \geq 3$ is also proved. The necessity follows from Lemma 2. □

3. The main theorem

Put $I_n(X) = I(X) \cap \Gamma_n(X)$. It follows from (5) and Lemma 1 that the inclusion

$$\Gamma(X) * I_n(X) \subset \Gamma_n(X)$$

holds. The main problem solved in this section is the following: to describe all groups $X$ for which

$$\Gamma(X) * I_n(X) = \Gamma_n(X).$$

(17)

Observe that if $\mu \in \Gamma(X) * I(X)$, then $\mu$ is invariant with respect to a compact subgroup $K$ of the group $X$ and under the natural homomorphism $X \to X/K$ induces on the factor-group $X/K$ a Gaussian measure. We formulate now the main theorem.

Theorem 1. Equality (17) holds for a group $X$, when $n = 2$, if and only if the connected component of zero $c_X$ of $X$ contains no more than one element of order 2. Equality (17) holds for a group $X$, when $n \geq 3$, if and only if $c_X$ has the property

$$\{x \in c_X : nx = 0\} = \{0\}.$$  

(18)

To prove Theorem 1 we need some lemmas. First we will formulate as a lemma the following simple statement, but omit its proof.

Lemma 6. Let $n$ be a natural number, $K$ be a compact subgroup of a group $X$. Then the following statements are equivalent:

(i) $K^{(n)} = K$;

(ii) if $ny \in A(Y, K)$, then $y \in A(Y, K)$. 

We note that the characteristic function of the Haar measure $m_K$ is of the form
\[
\hat{m}_K(y) = \begin{cases} 
1 & \text{if } y \in A(Y, K) \\
0 & \text{if } y \notin A(Y, K).
\end{cases}
\] (19)

**Lemma 7.** Let $K$ be a compact subgroup of a group $X$. Then the following statements are equivalent:

(i) $m_K \in I_n(X)$;
(ii) $K^{(n)} = K$.

**Proof.** (i) $\Rightarrow$ (ii). By Lemma 1 the characteristic function $\hat{m}_K(y)$ satisfies equation (3). Substituting $v_1 = u$, $v_2 = \cdots = v_n = 0$ into (3), we get
\[
\hat{m}_K(nu) = \hat{m}_K(u)\hat{m}_K((n-1)u), \quad u \in Y.
\] (20)

It follows from (19) and (20) that if $ny \in A(Y, K)$, then $y \in A(Y, K)$. Hence, by Lemma 6 $K^{(n)} = K$.

(ii) $\Rightarrow$ (i). By Lemma 1 it suffices to check that the characteristic function $\hat{m}_K(y)$ satisfies equation (3), which takes the form
\[
\prod_{j=1}^{n} \hat{m}_K(u + nv_j - (v_1 + \cdots + v_n))
\]
\[
= \hat{m}_K(u) \prod_{j=1}^{n} \hat{m}_K(nv_j - (v_1 + \cdots + v_n)), \quad u, v_1, \ldots, v_n \in Y.
\] (21)

Both sides of equation (21) take the values either 0 or 1. If the right-hand side of (21) is equal to 1, then $u \in A(Y, K)$, $nv_j - (v_1 + \cdots + v_n) \in A(Y, K)$, $j = 1, 2, \ldots, n$. Hence, $u + nv_j - (v_1 + \cdots + v_n) \in A(Y, K)$, $j = 1, 2, \ldots, n$. Therefore the left-hand side of (21) is also equal to 1.

Let the left-hand side of (21) be equal to 1. This implies that
\[
u + nv_j - (v_1 + \cdots + v_n) \in A(Y, K), \quad j = 1, 2, \ldots, n.
\] (22)

It follows from (22) that
\[
\sum_{j=1}^{n} (u + nv_j - (v_1 + \cdots + v_n)) = nu \in A(Y, K).
\] (23)

By Lemma 6 we get from (23) that $u \in A(Y, K)$. Then (22) implies that $nv_j - (v_1 + \cdots + v_n) \in A(Y, K)$, $j = 1, 2, \ldots, n$. Hence, the right-hand side of (21) is also equal to 1. \qed
Lemma 8 (compare with [4, Lemma 10.4]). Let $X$ be a discrete torsion free group. Then $\Gamma_n(X) = D(X)$.

Proof. If $X$ is a discrete torsion free group, then $Y$ is a connected compact group ([7, (24.25)]). Assume first that $Y \not\cong T$. Then there exists a continuous monomorphism $p : \mathbb{R} \to Y$ such that its image $p(\mathbb{R})$ is everywhere dense in $Y$ ([7, (25.18)]). Let $\mu \in \Gamma_n(X)$. Consider the restriction of the characteristic function $\hat{\mu}(y)$ to $p(\mathbb{R})$. Put $f(t) = \hat{\mu}(p(t))$, $t \in \mathbb{R}$. By Lemma 1 the characteristic function $\hat{\mu}(y)$ satisfies equation (3). Therefore $f(t)$ is a characteristic function on $\mathbb{R}$ satisfying (3) too. Taking into account that $\Gamma_n(\mathbb{R}) = \Gamma(\mathbb{R})$, we have

$$f(t) = \exp\{-\sigma t^2 + i\beta t\}, \quad \sigma \geq 0, \beta \in \mathbb{R}.$$  

Since $p$ is a monomorphism and $\overline{p(\mathbb{R})} = Y$, for any neighborhood of zero $V$ of the group $Y$ we can choose a sequence of real numbers $t_n \to \infty$ such that $p(t_n) \in V$ for all $n$. If $\sigma > 0$, then

$$|f(t_n)| = |\hat{\mu}(p(t_n))| = \exp\{-\sigma t_n^2\} \to 0$$

when $t_n \to \infty$. Taking into account that $V$ is an arbitrary neighborhood of zero of the group $Y$ we come to contradiction with the continuity at zero of the function $\hat{\mu}(y)$. Hence, $\sigma = 0$, and this implies that $|\hat{\mu}(p(t))| = 1$ for $t \in \mathbb{R}$. Since $\overline{p(\mathbb{R})} = Y$, we have $|\hat{\mu}(y)| = 1$ for $y \in Y$. It follows from this that $\mu \in D(X)$.

If $Y \cong T$, then $X \cong Z$. Since $Z$ is a subgroup of $\mathbb{R}$, we have $\Gamma_n(Z) \subset \Gamma_n(\mathbb{R}) = \Gamma(\mathbb{R})$. Taking into account that any Gaussian measure on $\mathbb{R}$ supported in $Z$ is degenerated we completely prove the lemma. \hfill \Box

Corollary 1. Let $Y$ be a connected compact group, $f(y)$ be a characteristic function on $Y$ satisfying equation (3). Then $|f(y)| = 1$, $y \in Y$.

Lemma 9. Let $\mu \in \Gamma_n(X)$. Then $\mu$ is supported in a coset of a subgroup $G \cong R^m \times K$, where $m \geq 0$, and $K$ is a compact subgroup of $X$ such that $K^{(n)} = K$.

Proof. Note first that if $\lambda \in \Gamma_n(X)$, then for any $x \in X$ we have $\lambda \ast E_x \in \Gamma_n(X)$. Therefore if it is necessary we can substitute a distribution $\lambda \in \Gamma_n(X)$ by its shift, and the shift also belongs to the class $\Gamma_n(X)$.

Let $\lambda \in M^1(X)$ be an arbitrary distribution. Consider the set

$$E = \{y \in Y : |\hat{\lambda}(y)| = 1\}.$$
Then $E$ is a subgroup of $Y$, and there exists an element $x \in X$ such that $\lambda(y) = (x, y)$ for $y \in E$. It follows from what has been said that we can substitute the distribution $\mu$ by its shift and assume from the beginning that
\[ E = \{ y \in Y : |\hat{\mu}(y)| = 1 \} = \{ y \in Y : \hat{\mu}(y) = 1 \}. \tag{24} \]
It follows from (24) that $\sigma(\mu) \subset G$, where $G = A(X, E)$. Put $H = G^*$. The distribution $\mu$ considering as a distribution on $G$ has the property
\[ \{ y \in H : |\hat{\mu}(y)| = 1 \} = \{ 0 \}. \tag{25} \]

We will check that $G$ is the desired subgroup. It follows from the structure theorem for locally compact Abelian group that $H \cong R^m \times L$, where $m \geq 0$, and $L$ contains a compact open subgroup [7, (24.30)]. Consider the restriction of equation (3) to the connected component of zero $c_L$ of the group $L$. Since the group $c_L$ is compact, by Corollary 1 $|\hat{\mu}(y)| = 1$ for $y \in c_L$. Taking into account (25) this implies that $c_L = \{ 0 \}$, i.e. the group $L$ is totally disconnected. We will prove that the group $L$ is discrete.

By Lemma 1 the characteristic function $\hat{\mu}(y)$ satisfies equation (3). Substitute $v_1 = u, v_2 = \cdots = v_n = 0$ into (3). From the resulting equation we obtain
\[ |\hat{\mu}(nu)| = |\hat{\mu}(u)|^{2n-1} |\hat{\mu}((n-1)u)|, \quad u \in H. \]
This implies the inequality
\[ |\hat{\mu}(nu)| \leq |\hat{\mu}(u)|^{2n-1}, \quad u \in H. \tag{26} \]
It follows from (26) that for any natural $p$ the inequality
\[ |\hat{\mu}(npu)| \leq |\hat{\mu}(u)|^{(2n-1)p}, \quad u \in H \tag{27} \]
holds. Since $\hat{\mu}(0) = 1$ and the function $\hat{\mu}(y)$ is continuous, there exists a neighborhood of zero $V$ of the group $L$ such that $|\hat{\mu}(y)| > 0$ for $y \in V$. Inasmuch as the group $L$ is totally disconnected, for any neighborhood of zero of $L$, in particular for $V$, there exists a compact subgroup $W$ such that $W \subset V$ ([7, (7.7)]). Thus, we have
\[ |\hat{\mu}(y)| > 0, \quad y \in W. \tag{28} \]
Assume that at a point $y_0 \in W$ the inequality
\[ |\hat{\mu}(y_0)| < 1 \tag{29} \]
Let the connected component of zero for any compact subgroup \( W \). We have \( \mu(y) = 0 \), that implies \( \mu(y) = 0 \), that contradicts \( \mu(y) = 1 \) for \( y \in W \). Taking into account (25), we get that \( W = \{0\} \), i.e. the group \( L \) is discrete. Put \( K = L^* \). Since \( L \) is discrete, this implies that \( K \) is compact.

Take \( y_0 \in L(n) \), i.e. \( n y_0 = 0 \). It follows from (26) that \( |\mu(y_0)| = 1 \). Taking into account (25), we obtain that \( y_0 = 0 \). Hence, \( L(n) = \{0\} \). This implies that \( K(n) = K \).

**Lemma 10.** Let \( X = R^m \times K \), where \( m \geq 0 \) and \( K \) is a compact group such that \( K(n) = K \). Then \( X(n) \subset c_X \).

**Proof.** We have \( A(Y, Y(n)) = X(n) \), \( A(Y, b_Y) = c_X \) ([7, (24.17)]). The lemma will be proved if we check that \( Y(n) \subset b_Y \). Put \( L = K^* \). Obviously, \( b_Y \) is a discrete torsion group. Since \( K(n) = K \), we have \( L(n) = \{0\} \). Take \( y_0 \in b_Y \).

Let \( M \) be a finite cyclic subgroup generated by \( y_0 \). It follows from \( L(n) = \{0\} \) that the restriction of the mapping \( y \mapsto n y, y \in Y, \) to \( M \) is a monomorphism. Taking into account that the group \( M \) is finite, this mapping is an isomorphism. Hence, \( y_0 \in Y(n) \).

**Lemma 11.** Let \( X \) be a locally compact Abelian group. Then the following statements are equivalent:

(i) for any compact subgroup \( K \) of the group \( X \), satisfying the condition \( K(n) = K \), the equality \( (K^*)^n = K^* \) holds;

(ii) the connected component of zero \( c_X \) of the group \( X \) has the property (18);

(iii) for any compact subgroup \( K \) of the group \( X \), satisfying the condition \( K(n) = K \), the factor group \( X/K \) contains no subgroup topologically isomorphic to \( \mathbb{T} \).

**Proof.** The equivalence of (i) and (ii) was proved in [3] [see also [4, Lemma 11.15]]. To prove the equivalence of (i) and (iii) note first that \( K(n) = K \) if and only if \( K(p) = K \) for any prime divisor \( p \) of the number \( n \). Hence it suffices to prove the equivalence of (i) and (iii) assuming that \( n \) is a prime number.

Let us prove (iii) \( \Rightarrow \) (i). Let \( n \) be a prime number and \( K \) be a compact subgroup of \( X \) such that \( K(n) = K \). Put \( L = K^* \). Then \( L \) is a discrete group satisfying the condition \( L(n) = \{0\} \). It follows from this that if \( y_0 \) is an element of finite order in \( L \), then \( y_0 \in L(n) \). We will check that any element \( y_0 \) of infinite order in \( L \) also belongs to \( L(n) \). Thus, the implication (iii) \( \Rightarrow \) (i) will be proved.

Let \( y_0 \) be an element of infinite order in \( L \) such that \( y_0 \notin L(n) \). Consider the factor-group \( L/L(n) \). Since \( n \) is a prime number, all nonzero elements of the factor-group \( L/L(n) \) have order \( n \). It is obvious that \( [y_0] \neq 0 \). Therefore \( [y_0] \) has
order $n$ and hence, $ky_0 \notin L^{(n)}$, $k = 1, 2, \ldots, n - 1$. Denote by $M$ the subgroup of $L$ generated by $y_0$, i.e. $M = \{ y \in L : y = l y_0, l \in \mathbb{Z} \}$. Let $h \in L$ and $n h \in M$. Then $n h = l y_0, l \in \mathbb{Z}$. We have $l = q n + k$, where $q \in \mathbb{Z}, k \in \{ 0, 1, \ldots, n - 1 \}$. This implies that $ky_0 = n(h - q y_0) \in L^{(n)}$. Therefore $k = 0$, and hence $n h = q y_0$. Since $L_{(n)} = \{ 0 \}$, we get $h = q y_0 \in M$. Thus, the subgroup $M$ has the property: if $n h \in M$, then $h \in M$. By Lemma 6 it follows from this that for the annihilator $G = A(K, M)$ the equality $G^{(n)} = G$ holds. Note now that $(K/G)^* \cong M \cong \mathbb{Z}$. This implies that $K/G \cong \mathbb{T}$. Since $K/G$ is a subgroup of $X/G$, this contradicts to (iii).

The implication (i) $\Rightarrow$ (iii) for $n = 2$ was proved in [5, Lemma 7.7]. The proof in the general case is the same as in the case when $n = 2$. \hfill $\Box$

**Proof of Theorem 1.** Let $n = 2$. By Lemma 5 $\Gamma_2(X) = \Gamma_B(X)$ and the statement of the theorem in this case was proved in [2] (see also [4, Theorem 9.10]).

Assume that $n \geq 3$ and let us prove the sufficiency. Let $\mu \in \Gamma_n(X)$. By Lemma 1 the characteristic function $\hat{\mu}(y)$ satisfies equation (3). Substituting $v_1 = v, v_2 = -v, v_3 = \cdots = v_n = 0$ in (3), we get

$$\hat{\mu}(u + n v)\hat{\mu}(u - n v)\hat{\mu}^{n-2}(u) = \hat{\mu}^n(u)|\hat{\mu}(nv)|^2, \quad u, v \in Y. \quad (30)$$

By Lemma 9 we can assume that the group $X$ is of the form $X = R^m \times K$, where $m \geq 0$, and $K$ is a compact group such that $K^{(n)} = K$. Applying Lemma 10 we obtain $X_{(n)} \subset c X$. Then (18) implies that $X_{(n)} = \{ 0 \}$, and hence $A(Y, X_{(n)}) = Y^{(n)} = Y^{(n)} = Y$. Taking this into account, (30) implies that the characteristic function $\hat{\mu}(y)$ satisfies equation

$$\hat{\mu}(u + v)\hat{\mu}(u - v)\hat{\mu}^{n-2}(u) = \hat{\mu}^n(u)|\hat{\mu}(v)|^2, \quad u, v \in Y. \quad (31)$$

It follows from (31) that the set

$$B = \{ y \in Y : \hat{\mu}(y) \neq 0 \}$$

is an open subgroup of $Y$. Put $K = A(X, B)$. Since $B$ is an open subgroup, the group $K$ is compact. Inasmuch as the characteristic function $\hat{\mu}(y)$ satisfies equation (3), the function $\hat{\mu}(y)$ satisfies inequality (26), and it follows from (26) that the group $B$ has the property: if $n y \in B$, then $y \in B$. Applying Lemma 6 we get $K^{(n)} = K$. By Lemma 11 the factor-group $X/K$ contains no subgroup topologically isomorphic to $\mathbb{T}$. Note now that $(X/K)^* \cong B$ and consider the restriction of the characteristic function $\hat{\mu}(y)$ to $B$. By Lemma 1 this restriction
is the characteristic function of a distribution from \( \Gamma_n(X/K) \). Since the factor-group \( X/K \) contains no subgroup topologically isomorphic to \( T \), we can apply Proposition 1 to the factor-group \( X/K \). As a result we obtain the following representation for the characteristic function \( \hat{\mu}(y) \)

\[
\hat{\mu}(y) = \begin{cases} (x, y) \exp\{-\varphi(y)\} & \text{if } y \in B \\ 0 & \text{if } y \notin B. \end{cases}
\]

where \( x \in X \), and \( \varphi(y) \) is a continuous function on \( B \), satisfying equation (2).

It is well known that the function \( \varphi(y) \) can be extended from the subgroup \( B \) to \( Y \) retaining its properties ([9, Lemma 5.2.5]). Denote by \( \tilde{\varphi}(y) \) the extended function. Let \( \gamma \) be a Gaussian measure on \( X \) with the characteristic function \( \hat{\gamma}(y) = (x, y) \exp\{-\tilde{\varphi}(y)\}, \ y \in Y. \) (33)

Obviously, that \( \hat{\mu}(y) = \hat{\gamma}(y) \hat{m}_K(y) \). Hence, \( \mu = \gamma \ast m_K \).

Let us prove the necessity. Assume that (18) is not fulfilled. By Lemma 11 there exists a compact subgroup \( K \) of the group \( X \) such that \( K^n = K \) and the factor-group \( X/K \) contains a subgroup topologically isomorphic to \( T \). Therefore the distribution \( \mu \) on the group \( T \), constructed in Lemma 2, can be considered as a distribution on the factor-group \( X/K \). We retain the notation \( \mu \) for this distribution. Since \( (X/K)^* \cong A(Y, K) \), we may assume that the characteristic function \( \hat{\mu}(y) \) is defined on \( A(Y, K) \). Consider on the group \( Y \) the function

\[
h(y) = \begin{cases} \hat{\mu}(y) & \text{if } y \in A(Y, K) \\ 0 & \text{if } y \notin A(Y, K). \end{cases}
\]

Since the set \( A(Y, K) \) is a subgroup and \( \hat{\mu}(y) \) is a positive definite function, \( h(y) \) is also a positive definite function ([8, (32.43)]). Since \( K \) is a compact group, its annihilator \( A(Y, K) \) is an open subgroup, and hence the function \( h(y) \) is continuous. By Bochner’s theorem there exists a distribution \( \lambda \in M^1(X) \) such that \( \hat{\lambda}(y) = h(y) \). We will check that \( \lambda \in \Gamma_n(X) \). By Lemma 1 it suffices to verify that the function \( h(y) \) satisfies equation (3).

Take \( u \in A(Y, K) \). If \( nv_{j_0} - (v_1 + \cdots + v_n) \notin A(Y, K) \) holds at least for one \( j = j_0 \), then \( u + nv_{j_0} - (v_1 + \cdots + v_n) \notin A(Y, K) \), and both sides of equation (3) are equal to zero. Assume that \( nv_j - (v_1 + \cdots + v_n) \in A(Y, K) \) for all \( j = 1, 2, \ldots, n \). This implies that \( n(v_j - v_1) \in A(Y, K) \), and hence by Lemma 6 \( v_j - v_1 = h_j \in A(Y, K), \ j = 1, 2, \ldots, n. \) Substituting \( v_j = v_1 + h_j, \ j = 1, 2, \ldots, n \) in equation (3) and taking into account that the function \( \hat{\mu}(y) \) satisfies equation (3) on \( A(Y, K) \), we get the equality.
Take \( u \notin A(Y, K) \). Then the right-hand side of equation (3) is equal to zero. If in this case the left-hand side of equation (3) does not vanish, then the inclusions \( u + nv_j - (v_1 + \cdots + v_n) \in A(Y, K), j = 1, 2, \ldots, n \) are fulfilled. This implies that \( \sum_{j=1}^{n}(u + nv_j - (v_1 + \cdots + v_n)) = nu \in A(Y, K) \), and hence by Lemma 6 \( u \in A(Y, K) \), contrary to the choice of \( u \). So we proved that the function \( h(y) \) satisfies equation (3). Since \( \mu / \notin \Gamma(X/K) \), obviously that \( \lambda / \notin \Gamma(X) \). The theorem is completely proved. 

Remark 2. Let \( \mu \in \Gamma_n(\mathbb{R}), n \geq 2 \). Put \( \gamma = \mu * \bar{\mu} \in \Gamma_n(\mathbb{R}) \). Then \( \hat{\gamma}(y) = |\hat{\mu}(y)|^2 \geq 0 \). By Lemma 1 the characteristic function \( \hat{\gamma}(y) \) satisfies equation (3), and hence satisfies equation (30) too. Taking into account that \( \mathbb{R}^{(n)} = \mathbb{R} \), (30) implies (31), and hence the set \( B = \{ y \in \mathbb{R} : \hat{\gamma}(y) \neq 0 \} \) is an open subgroup of \( \mathbb{R} \).

So, \( B = \mathbb{R} \). Since \( \hat{\gamma}(y) > 0 \), \( y \in \mathbb{R} \), by Lemma 3 \( \gamma \in \Gamma(\mathbb{R}) \). This implies by Cramer’s theorem that \( \mu \in \Gamma(\mathbb{R}) \). Thus, we proved the equality \( \Gamma(\mathbb{R}) = \Gamma_n(\mathbb{R}), n \geq 2 \), which we used in the proof of Lemma 8, and this proof is independent from Geary’s theorem.

Remark 3. Comparing Proposition 1 and Theorem 1 we see that in both statements we have a particular case \( n = 2 \). If \( n \geq 3 \), then the description of the corresponding class of groups in Proposition 1 does not depend on \( n \), and does not depend on \( n \) in Theorem 1.

Remark 4. Observe that if \( \mu \) is an infinitely divisible distribution and \( \mu \in \Gamma_n(X) \), then \( \mu \in \Gamma(X) * I_n(X) \). Indeed, since \( \mu \) is an infinitely divisible distribution, the set \( B = \{ y \in Y : \hat{\mu}(y) \neq 0 \} \) is an open subgroup of \( Y \) ([12, p. 106]). Put \( \nu = \mu * \bar{\mu} \) and consider the restriction of the characteristic function \( \hat{\nu}(y) \) to \( B \). Since \( \hat{\nu}(y) = |\hat{\mu}(y)|^2 > 0 \) for \( y \in B \), by Lemma 3 \( \hat{\nu}(y) = \exp\{-\varphi(y)\}, y \in B \).

Note now that on an arbitrary locally compact Abelian group in the class of infinitely divisible distributions a Gaussian measure has only Gaussian factors ([5, Remark 4.8]). Thus, for the characteristic function \( \hat{\mu}(y) \) we get the representation similar to (32), and the desired statement follows from this.

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