Initial value problems of $p$-Laplacian with a strong singular indefinite weight

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Abstract. In this paper, we present some existence and uniqueness theorems for initial value problems of $p$-Laplacian with a strong singular indefinite weight which are related to singular $p$-Laplace eigenvalue problems. Our results improve and generalize some recent results.

1. Introduction

In this paper, we establish some existence and uniqueness results for the following initial value problem of $p$-Laplacian with a strong singular indefinite weight:

\[
\begin{aligned}
\varphi_p(u'(t))' + h(t)f(u(t)) &= 0, \quad \text{a.e. } t \in (0, 1), \\
u(t_0) &= 0, \quad u'(t_0) = a, \\
&\quad t_0 \in [0, 1], \quad a \in \mathbb{R},
\end{aligned}
\]

(IVP$_{t_0}$)

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $f \in C([0, 1], \mathbb{R})$, $h \in C((0, 1), [0, \infty))$ may be singular at $t = 0$ and/or $t = 1$. Problems (IVP$_{t_0}$) is related to the singular boundary value problem

\[
\begin{aligned}
\varphi_p(u'(t))' + h(t)f(u(t)) &= 0, \quad \text{a.e. } t \in (0, 1), \\
u(0) = u(1) &= 0.
\end{aligned}
\]

(BVP)

In particular, it is helpful to find sign-changing solutions for problem (BVP) (see e.g. [6], [8]).

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The studies on the singular initial and boundary value problems with sign-changing nonlinearity are much recent. For the studies on the first-order cases, one may refer to Agarwal, O’Regan [1], [2] and Agarwal, O’Regan, Lakshmikantham and Leela [3]. For second-order initial value problems, by using the Gronwall inequality, Yang [9] proved the existence and uniqueness of a solution of the following initial value problem:

\[
\begin{cases}
\varphi_p(u'(t))' + f(t, u(t)) = 0, \\
u(0) = 0, \quad u'(0) = a > 0,
\end{cases}
\]

where \(|f(t, u)| \leq ch(t)|u|^{p-1}\) with \(h \in L^q(0, 1), q > 1\).

For \(h \in L^1(0, 1)\), Zhang [10] showed the existence and uniqueness of a solution for (IVP\(t_0\)) with \(f(u) = \varphi_p(u), t_0 = 0\) and \(a = 1\) by transforming to a system and applying Sturmian comparison. One should also notice that García-Huidobro, Manásevich and Ótani [5] gave an existence and uniqueness result for initial value problem (IVP\(t_0\)) with \(t \in \mathbb{R}, h \in L^1_{loc}([0, \infty)), f(u) = \varphi_p(u)\).

For \(h \in A\) with \(A\) defined by the set

\[
\left\{ h \in C((0, 1), [0, \infty)) : \int_0^1 s^\alpha (1-s)^\beta h(s)ds < \infty \text{ for some } \alpha, \beta \in (0, p-1) \right\},
\]

some existence and uniqueness results were proved by Lee and Sim [7] for three special cases of initial problem (IVP\(t_0\)): \(t_0 = 0, a = 1; t_0 = 1, a = -1\) and \(a = 0\).

Denote

\[
\mathcal{B} = \left\{ h \in C((0, 1), [0, \infty)) : \int_0^1 (s(1-s))^{p-1}h(s)ds < \infty \right\}.
\]

It is clear that \(L^1(0, 1) \subset A \subset \mathcal{B}\). For more properties of the classes of singular indefinite weights \(A\) and \(\mathcal{B}\), one may refer to [4], [6]. For \(h \in \mathcal{B}, Kajikiya, Lee\) and Sim [6] obtained some existence and uniqueness results for (IVP\(t_0\)) under assumption that \(f(t, u(t)) = \lambda \varphi_p(u(t))\) with \(\lambda\) a positive real parameter. Moreover, these results were applied in the study of some eigenvalue problems for \(p\)-Laplacian.

The aim of this paper is to present some existence and uniqueness results for problem (IVP\(t_0\)) with \(h \in \mathcal{B}, t_0 \in [0, 1], a \in \mathbb{R}\) and \(f \in C(\mathbb{R}, \mathbb{R})\) by using Schauder’s fixed point theorem (see Theorem 2.1, 2.2 and Corollary 2.3). Our results improve and generalize some results in [6], [7] (see Remark 2.4).
2. Statements of the main results

Recall that a function $u$ is said to be a solution of (IVP) if $u \in C^1(0,1) \cap C[0,1]$ and $\varphi_p(u')$ is absolutely continuous in any compact subinterval of $(0,1)$ and $u$ satisfies (IVP).

Let us give the following assumptions on $f$:

(H₁) $\exists C > 0$ such that $|f(u)| \leq C|\varphi_p(u)|$ for $u \in \mathbb{R}$.

(H₂) $\exists C > 0$ such that $|f(u) - f(v)| \leq C|\varphi_p(u) - \varphi_p(v)|$ for $u, v \in \mathbb{R}$.

(H₂⁺) $\forall \varGamma > 0, \exists C > 0$ such that $|f(u) - f(v)| \leq C|\varphi_p(u) - \varphi_p(v)|$ for $u, v \in [0, \varGamma]$.

(H₂⁻) $\forall \varGamma > 0, \exists C > 0$ such that $|f(u) - f(v)| \leq C|\varphi_p(u) - \varphi_p(v)|$ for $u, v \in [-\varGamma, 0]$.

The main results of this paper are as follows.

**Theorem 2.1.** Assume $h \in B$ and (H₁). Then problem (IVP) has at least one solution. Especially, if $a = 0$, problem (IVP) has only a trivial solution.

**Theorem 2.2.** Assume $h \in B$. The following statements are true.

(i) Suppose that (H₂⁺) holds. If $a > 0$, $t_0 \in [0,1)$ or $a < 0$, $t_0 \in (0,1]$, then problem (IVP) has at most one solution.

(ii) Suppose that (H₂⁻) holds. If $a < 0$, $t_0 \in [0,1)$ or $a > 0$, $t_0 \in (0,1]$, then problem (IVP) has at most one solution.

It is clear that (H₂) implies (H₁), (H₂⁺) and (H₂⁻). Then by Theorem 2.1 and 2.2, we can get the following result immediately.

**Corollary 2.3.** Assume $h \in B$ and (H₂). Then problem (IVP) has a unique solution.

**Remark 2.4.** (a) Theorem 2.1 generalizes Theorem 1.1 and 1.3 in [7] by extending the class of singular indefinite weights from $h \in \mathcal{A}$ to $h \in B$.

(b) Theorem 2.2 (i) improves Theorem 1.2 in [7]. In fact, it is clear that assumption (H₂⁺) is weaker than the assumption

(P) $\exists C > 0$ such that $|f(u) - f(v)| \leq C|\varphi_p(u) - \varphi_p(v)|$ for $u, v \in [0,\infty)$.

For example, $f(u) = (\varphi_p(u))^2$ does not satisfy (P), but satisfies (H₂⁺) and (H₂⁻) with $C = 2\varphi_p(\varGamma)$. Then Theorem 2.2 (i) is the same as Theorem 1.2 in [7] if replace “$h \in B$ and (H₂⁺)” by “$h \in \mathcal{A}$ and (P)” and consider the two special cases $t_0 = 0$, $a = 1$ and $t_0 = 1$, $a = -1$.

(c) Corollary 2.3 generalizes Theorem 2.2 and 2.3 in [6] by considering the general function $f \in C(\mathbb{R}, \mathbb{R})$ instead of $\lambda \varphi_p(u)$ with $\lambda$ a positive real parameter.
3. Proofs of the main results

In the sequel, we always assume that \( t_0 \in [0,1) \) and \( h \in B \). For the case when \( t_0 \in (0,1] \), we can analyze exactly the same way and we omit the details.

Let us start with some lemmas which will be used in the proofs of our main results.

**Lemma 3.1.** Assume \( h \in B \) and \((H_1)\). If \( a = 0 \), problem \((IVP_{t_0})\) has only a trivial solution.

**Proof.** Clearly 0 is a solution of \((IVP_{t_0})\) if \( a = 0 \). Let \( u \) be a solution of \((IVP_{t_0})\). It is enough to prove that \( u = 0 \) for \( t \in [0,1] \). There are two cases to be considered: \( t_0 \in (0,1) \) and \( t_0 = 0 \).

**Case 1.** \( t_0 \in (0,1) \). For any \( t_1 \in (t_0,1) \), we have \( u \in C^1[t_0,t_1] \) and \( h \in C[t_0,t_1] \). For \( t \in [t_0,t_1] \), it follows from \((IVP_{t_0})\) that

\[
|u(t)| = \int_{t_0}^{t} \varphi_p^{-1} \left( \int_{t_0}^{\tau} h(\tau)f(u(\tau))d\tau \right) d\tau \leq \varphi_p^{-1} \left( \int_{t_0}^{t} h(s)|f(u(s))|ds \right).
\]

Then, by \((H_1)\) we have, for \( t \in [t_0,t_1] \),

\[
\varphi_p(|u(t)|) \leq \int_{t_0}^{t} h(s)|f(u(s))|ds \leq C \int_{t_0}^{t} h(s)\varphi_p(|u(s)|)ds.
\]

By the Gronwall inequality we have \( \varphi_p(|u(t)|) = 0 \) for \( t \in [t_0,t_1] \), that is \( u(t) = 0 \), \( t \in [t_0,t_1] \). This implies that \( u(t) = 0 \) for \( t \in [0,1] \) since \( t_1 \) is arbitrary in \((0,1)\) and \( u \) is continuous in \([0,1]\). Similarly, we can prove that \( u(t) = 0 \) for \( t \in [0,t_0] \) and then \( u(t) = 0 \) for all \( t \in [0,1] \).

**Case 2.** \( t_0 = 0 \). For \( t_1 \in (0,1) \), we have \( u \in C^1[0,t_1] \). Let \( v(0) = 0 \) and \( v(t) = u(t)/t \) for \( t \in (0,t_1] \). Then \( v \in C[0,t_1] \). For \( t \in (0,t_1] \), from \((IVP_{t_0})\) we have

\[
|v(t)| = \frac{1}{t} \int_{0}^{t} \varphi_p^{-1} \left( \int_{0}^{\tau} h(\tau)f(u(\tau))d\tau \right) d\tau \leq \varphi_p^{-1} \left( \int_{0}^{t} h(s)|f(u(s))|ds \right).
\]

Thus for \( t \in [0,t_1] \), by \((H_1)\) we get

\[
\varphi_p(|v(t)|) \leq \int_{0}^{t} h(s)|f(u(s))|ds \leq \int_{0}^{t} h(s)C|\varphi_p(u(s))|ds
\]

\[
= C \int_{0}^{t} h(s)s^{p-1}\varphi_p(|u(s)/s|)ds = C \int_{0}^{t} h(s)s^{p-1}\varphi_p(|v(s)|)ds.
\]

By the Gronwall inequality we have \( \varphi_p(|v(t)|) = 0 \) for \( t \in [0,t_1] \). That is \( u(t) = 0 \), \( t \in [0,t_1] \). Therefore, it follows from the arbitrariness of \( t_1 \) in \((0,1)\) and the continuity of \( u \) that \( u(t) = 0 \) for \( t \in [0,1] \). This completes the proof. \( \square \)
By the Mean Value Theorem and a fundamental calculation, it is easy to get the following inequality which will be used later:

\[ |\varphi_p(x) - \varphi_p(y)| \leq (p - 1)z^{p-2}|x - y|, \quad \forall x, y \in \mathbb{R}, \tag{3.1} \]

where \( z = \max\{|x|, |y|\} \).

Now we assume that \( a \neq 0 \). Let \( K > |a| \) be a constant. Since \( h \in \mathcal{B} \), there exists \( \beta \in (t_0, 1) \) such that

\[
\int_{t_0}^{\beta} h(s)(s - t_0)^{p-1}ds \leq \min \left\{ \frac{1 - \varphi_p(|a|/K)}{C}, \frac{\varphi_p(|a|/K)}{2C} \right\}, \tag{3.2}
\]

where \( C \) is the same constant as in (H1).

Define \( G : C^1_0[t_0, \beta] \to C^1_0[t_0, \beta] \) by

\[
G(u)(t) = \int_{t_0}^{t} \varphi_p^{-1} \left( \varphi_p(a) - \int_{t_0}^{\tau} h(\tau)f(u(\tau))d\tau \right) ds, \quad \text{for } t \in [t_0, \beta].
\]

For any \( u \in C^1_0[t_0, \beta] \), we have \( |u(t)| \leq ||u||_1(t - t_0) \) for \( t \in [t_0, \beta] \). Then by (H1) we have, for \( t \in [t_0, \beta] \),

\[
\left| \int_{t_0}^{t} h(\tau)f(u(\tau))d\tau \right| \leq \int_{t_0}^{t} h(\tau)|C||u(\tau)|^{p-1}d\tau \\
\leq C||u||_1^{p-1} \int_{t_0}^{t} h(\tau)(\tau - t_0)^{p-1}d\tau < \infty. \tag{3.3}
\]

So \( G \) is well defined. In addition, noticing that

\[
(G(u))'(t) = \varphi_p^{-1} \left( \varphi_p(a) - \int_{t_0}^{t} h(\tau)f(u(\tau))d\tau \right),
\]

(3.3) implies that \( G \) is bounded, i.e., send bounded subsets of \( C^1_0[t_0, \beta] \) into bounded subsets of \( C^1_0[t_0, \beta] \). Furthermore, it is easy to see that \( u(t) \) is a local solution of problem (IVP\(_{t_0}\)) for \( t \in [t_0, \beta] \) if and only if \( u \) is a fixed point of \( G \) in \( C^1_0[t_0, \beta] \).
Lemma 3.2. Assume $h \in B$ and $(H_1)$. Then $G(M) \subset M$ and $G : M \to M$ is continuous.

Proof. By $(H_1)$ and (3.2), for $u \in M, t \in [t_0, \beta]$, 

$$\| (G(u))' (t) \| = \left| \varphi_p^{-1} \left( \varphi_p (a) - \int_{t_0}^{t} h(\tau) f(u(\tau)) d\tau \right) \right|$$

$$\leq \varphi_p^{-1} \left( \varphi_p (|a|) + \int_{t_0}^{t} h(\tau) C |\varphi_p (u(\tau))| d\tau \right)$$

$$\leq \varphi_p^{-1} \left( \varphi_p (|a|) + C \|u\|_1^{p-1} \int_{t_0}^{t} h(\tau) (\tau - t_0)^{p-1} d\tau \right)$$

$$\leq \varphi_p^{-1} \left( \varphi_p (|a|) + CK^{p-1} \frac{1 - \varphi_p (|a|/K)}{C} \right) = K.$$ 

That is $\|G(u)\|_1 \leq K$, and then $G(M) \subset M$. 

Now we prove the continuity of $G$. For $u \in M, t \in [t_0, \beta]$, by $(H_1)$ and (3.2) we have 

$$\left| \int_{t_0}^{t} h(\tau) f(u(\tau)) d\tau \right| \leq \int_{t_0}^{t} h(\tau) C |\varphi_p (u(\tau))| d\tau \leq C \|u\|_1^{p-1} \int_{t_0}^{t} h(\tau) (\tau - t_0)^{p-1} d\tau$$

$$\leq CK^{p-1} \frac{\varphi_p (|a|/K)}{2C} = \varphi_p (|a|)/2.$$ 

Then 

$$\left| \varphi_p (a) - \int_{t_0}^{t} h(\tau) f(u(\tau)) d\tau \right| \leq 3 \varphi_p (|a|)/2 \quad \text{for } t \in [t_0, \beta].$$

Let $q = p/(p - 1)$, then $\varphi_p^{-1} = \varphi_q$. Denote 

$$C_1 = \frac{1}{2(q - 1)(3 \varphi_p (|a|)/2)^{q-2}}.$$ 

Given $\varepsilon > 0$, there exists $\eta \in (t_0, \beta)$ such that 

$$\int_{t_0}^{\eta} h(s) (s - t_0)^{p-1} ds \leq \frac{C_1 \varepsilon}{2CK^{p-1}}$$

since $h \in B$. Then for $u \in M$, by $(H_1)$, 

$$\int_{t_0}^{\eta} h(s) |f(u(s))| ds \leq \int_{t_0}^{\eta} h(\tau) C |\varphi_p (u(\tau))| d\tau \leq C \|u\|_1^{p-1} \int_{t_0}^{\eta} h(s) (s - t_0)^{p-1} ds$$

$$\leq CK^{p-1} \frac{C_1 \varepsilon}{2CK^{p-1}} = \frac{C_1 \varepsilon}{2}. \quad (3.6)$$
Let \( \{ u_n \} \subset M \) such that \( u_n \to u_0 \) in \( M \) as \( n \to \infty \). Then for \( t \in [t_0, \beta] \), by (3.1), (3.4) and (3.5) we get

\[
\|(G(u_n))'(t) - (G(u_0))'(t)\| \leq \frac{1}{2C_1} \left( \int_{t_0}^\beta \left( \int_{t_0}^\eta h(\tau) f(u_n(\tau)) d\tau \right) + \int_{t_0}^\beta h(\tau) f(u_0(\tau)) d\tau \right) \leq \frac{1}{2C_1} \left( C_1 \varepsilon + 2C_2 \int_{t_0}^\beta h(\tau) d\tau \right) = \varepsilon/2.
\]

This implies that \( G : M \to M \) is continuous.

Case 2. Suppose \( \int_\eta^\beta h(s) ds \neq 0 \). Since \( f \) is uniformly continuous in \([-K, K]\), there exists \( \rho > 0 \) such that \( u, v \in [K, K], |u - v| < \rho \) implies

\[
|f(u) - f(v)| \leq C_1 \varepsilon \left( \int_\eta^\beta h(s) ds \right)^{-1}.
\]

Meanwhile, there exists \( N > 0 \) such that \( |u_n(t) - u_0(t)| < \rho \) for \( t \in [t_0, \beta], n > N \). Thus

\[
|f(u_n(t)) - f(u_0(t))| \leq C_1 \varepsilon \left( \int_\eta^\beta h(s) ds \right)^{-1} \quad \text{for} \quad n > N, \ t \in [t_0, \beta].
\]

Now, for \( t \in [t_0, \beta] \) and \( n > N \), by (3.6) – (3.8) we have

\[
\|(G(u_n))'(t) - (G(u_0))'(t)\| \leq \frac{1}{2C_1} \left( \int_{t_0}^\eta h(\tau) \left( |f(u_n(\tau))| + |f(u_0(\tau))| \right) d\tau + \int_{t_0}^\beta h(\tau) \left( |f(u_n(\tau))| + |f(u_0(\tau))| \right) d\tau \right) \leq \frac{1}{2C_1} \left( C_1 \varepsilon + \int_\eta^\beta h(\tau) C_1 \varepsilon \left( \int_\eta^\beta h(s) ds \right)^{-1} d\tau \right) = \varepsilon,
\]

which implies that \( G : M \to M \) is continuous. The proof is complete. \( \Box \)
Lemma 3.3. Assume \( h \in B \) and (H\(_1\)). Then \( G : M \to M \) is compact.

**Proof.** Suppose \( \{u_n\} \subset M \) is bounded, then \( \{u_n\} \) and \( \{G(u_n)\} \) are bounded in \( M \). By the Arzela Ascoli Theorem, \( \{u_n\} \) and \( \{G(u_n)\} \) has a subsequence (denote again by \( \{u_n\} \) and \( \{G(u_n)\} \), respectively) converging to some \( u \) and \( v \) in \( C_0[t_0, \beta] \), respectively. By (H\(_1\)) we have

\[
|h(t)f(u_n(t))| \leq Ch(t)|\varphi_p(u_n(t))| \leq Ch(t)\varphi_p(\|u_n\|_1(t - t_0))
\]

\[
\leq CK^{p-1}h(t)(t - t_0)^{p-1}.
\]

for \( t \in [t_0, \beta] \). So by the Lebesgue dominated convergence theorem, we have

\[
G(u_n)(t) = \int_{t_0}^{t} \varphi_p^{-1}\left( \varphi_p(a) - \int_{t_0}^{s} h(\tau)f(u_n(\tau))d\tau \right) ds \to \int_{0}^{t} \varphi_p^{-1}\left( \varphi_p(a) - \int_{t_0}^{s} h(\tau)f(u(\tau))d\tau \right) ds = v(t),
\]

uniformly in \( t \in [t_0, \beta] \), and

\[
(G(u_n))'(t) = \varphi_p^{-1}\left( \varphi_p(a) - \int_{t_0}^{t} h(\tau)f(u_n(\tau))d\tau \right) \to \varphi_p^{-1}\left( \varphi_p(a) - \int_{t_0}^{t} h(\tau)f(u(\tau))d\tau \right),
\]

uniformly in \( t \in [t_0, \beta] \). So \( v \in C^1_0[0, 1] \) and

\[
v'(t) = \varphi_p^{-1}\left( \varphi_p(a) - \int_{t_0}^{t} h(\tau)f(u(\tau))d\tau \right).
\]

Clearly, \( \|v\|_1 \leq K \). Therefore, \( G : M \to M \) is compact. The proof is complete. \( \square \)

Now we are in a position to give the proofs of the main results.

**Proof of Theorem 2.1.** Lemma 3.1 leads to the conclusion for the special case \( a = 0 \). Now we assume that \( a \neq 0 \). It follows from Lemma 3.2 and 3.3 that \( G : M \to M \) is completely continuous. Then by Schauder’s Fixed Point Theorem, \( G \) has a fixed point in \( M \). That is problem (IVP\(_{t_0}\)) has a local solution \( u \in M \).

Now we prove the global existence of solutions of problem (IVP\(_{t_0}\)). Let \([t_0, T)\) be the right maximal interval of existence for solution \( u \). It is enough to show
that $T = 1$. Suppose on the contrary that $T < 1$. Then by $(H_1)$, for $t \in [t_0, T)$ we have
\[
|\varphi_p(u'(t))| = \left| \varphi_p(a) - \int_{t_0}^{t} h(\tau)f(u(\tau))d\tau \right| \leq \varphi_p(|a|) + C \int_{t_0}^{t} h(\tau)|u(\tau)|^{p-1}d\tau \\
\leq \varphi_p(|a|) + C \int_{t_0}^{t} h(\tau)(\tau - t_0)^{p-1} \max_{s \in [t_0, \tau]} |u'(s)|^{p-1}d\tau,
\]
so we have
\[
\max_{r \in [t_0, t]} |u'(r)|^{p-1} \leq \varphi_p(|a|) + C \int_{t_0}^{t} (\tau - t_0)^{p-1} \max_{s \in [t_0, \tau]} |u'(s)|^{p-1}d\tau.
\]
By the Gronwall inequality, we obtain
\[
\max_{r \in [t_0, t]} |u'(r)|^{p-1} \leq \varphi_p(|a|) \exp \left( C \int_{t_0}^{t} (\tau - t_0)^{p-1}d\tau \right) \\
\leq \varphi_p(|a|) \exp \left( C \int_{t_0}^{t} h(\tau)(\tau - t_0)^{p-1}d\tau \right),
\]
which implies that $u'$ is bounded in $[t_0, T)$, and consequently $u$ is bounded in $[t_0, T)$. This contradicts the fact that $[0, T)$ with $T < 1$ is the maximal existence interval for solution $u$. The proof is complete. \hfill \Box

**Proof of Theorem 2.2.** We prove statement (i). By a similar argument we can prove statement (ii) and we omit the details. Moreover, we only prove statement (i) for the case $a > 0$ and $t_0 \in [0, 1)$. The case $a < 0$ and $t_0 \in (0, 1]$ can be proved similarly and we also omit the details.

Suppose $u, v$ are two solutions of problem $(IVP_{t_0})$. It suffices to prove that $u(t) = v(t)$ for $t \in [t_0, \beta]$ with some $\beta \in (t_0, 1)$ which will be determined later.

Let $K > a$ such that $\max\{|u'(t)|, |v'(t)|\} \leq K$ for all $t \in [t_0, (1 + t_0)/2]$. Since $a > 0$, we can choose $\beta_1 \in (t_0, (1 + t_0)/2)$ such that $u(t), v(t) > 0$ for all $t \in (t_0, \beta_1)$. By $(H_{2+}),$ there exists some $C > 0$ such that
\[
|f(u)| \leq C\varphi_p(u), \quad |f(u) - f(v)| \leq C|\varphi_p(u) - \varphi_p(v)| \quad \text{for } u, v \in [0, K]. \quad (3.9)
\]
Then
\[
|f(u(t))| \leq C\varphi_p(u(t)), \quad |f(v(t))| \leq C\varphi_p(v(t)) \quad \text{and} \quad (3.10)
\]
\[
|f(u(t)) - f(v(t))| \leq C|\varphi_p(u(t)) - \varphi_p(v(t))| \quad \text{for } t \in [t_0, \beta_1]. \quad (3.11)
\]
Since $h \in B$, we can choose $\beta \in (t_0, \beta_1)$ such that
\[
\int_{t_0}^{t_0 + \beta} h(s)(s - t_0)^{p-1}ds < \min \left\{ \frac{\varphi_p(a/K)}{2C}, \frac{1}{(3\varphi_p(a)/2)^{q-2}K^{p-2}C} \right\}. \tag{3.12}
\]
where $q = p/(1 - p)$, $K$ and $C$ are the same constants as in (3.9). It is obvious that
\[
u, v \in M = \{ w \in C_0^1[t_0, \beta] : \|w\|_1 \leq K \}.
\]
Then by (3.10), (3.12) and the same way to get (3.4), we can obtain that (3.4) also holds for these $u, v$. That is
\[
\left| \varphi_p(a) - \int_{t_0}^{t} h(\tau)f(x(\tau))d\tau \right| \leq 3\varphi_p(a)/2 \quad \text{for } t \in [t_0, \beta], \tag{3.13}
\]
where $x = u, v$. Meanwhile, by the Mean Value Theorem, for $t \in [t_0, \beta]$, there exist some $\theta_1, \theta_2, \theta_3 \in (t_0, t)$ such that
\[
\left| \frac{u(t)}{t - t_0} \right| = |u'(\theta_1)| \leq \|u\|_1 \leq K, \tag{3.14}
\]
\[
\left| \frac{v(t)}{t - t_0} \right| = |v'(\theta_2)| \leq \|v\|_1 \leq K, \tag{3.15}
\]
\[
\left| \frac{u(t) - v(t)}{t - t_0} \right| = |u'(\theta_3) - v'(\theta_3)| \leq \|u - v\|_1. \tag{3.16}
\]
Notice that $\varphi_p^{-1} = \varphi_q$ and $(p - 1)(q - 1) = 1$. If $\|u - v\|_1 > 0$, then by (3.1), (3.11)–(3.16), for $t \in [t_0, \beta]$,
\[
|u'(t) - v'(t)|
\leq |\varphi_p^{-1}(\varphi_p(a) - \int_{t_0}^{t} h(\tau)f(u(\tau))d\tau) - \varphi_p^{-1}(\varphi_p(a) - \int_{t_0}^{t} h(\tau)f(v(\tau))d\tau)|
\leq (q - 1)(3\varphi_p(a)/2)^{q-2} \int_{t_0}^{t} h(\tau)|f(u(\tau)) - f(v(\tau))|d\tau
\leq (q - 1)(3\varphi_p(a)/2)^{q-2}C \int_{t_0}^{t} h(\tau)|\varphi_p(u(\tau)) - \varphi_p(v(\tau))|d\tau
\leq (q - 1)(3\varphi_p(a)/2)^{q-2}C \int_{t_0}^{t} h(\tau)(\tau - t_0)^{p-1} \left| \varphi_p\left( \frac{u(\tau)}{\tau - t_0} \right) - \varphi_p\left( \frac{v(\tau)}{\tau - t_0} \right) \right| d\tau
\leq (q - 1)(3\varphi_p(a)/2)^{q-2}C \int_{t_0}^{t} h(\tau)(\tau - t_0)^{p-1}(p - 1)K^{p-2} \left| \frac{u(\tau) - v(\tau)}{\tau - t_0} \right| d\tau
Initial value problems of p-Laplacian with a strong singular...

\[ \leq \left(3\varphi_p(a)/2\right)^{q-2}K^{p-2}C \int_{t_0}^\beta h(\tau)(\tau - t_0)^{p-1}d\tau \|u - v\|_1 < \|u - v\|_1, \]

which is impossible, and then \(\|u - v\|_1 = 0\), i.e. \(u(t) = v(t)\) for \(t \in [t_0, \beta]\). This completes the proof. \(\square\)

References


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