Some rigidity results for Dirac-harmonic maps
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Abstract. Let \((\phi, \psi)\) be a Dirac-harmonic maps from a Riemannian manifold into another Riemannian manifold. We call \((\phi, \psi)\) trivial if \(\phi\) is harmonic. By using Bochner-type formula and extending Chen–Jost–Li–Wang’ result, we give some sufficient conditions for a Dirac-harmonic map \((\phi, \psi)\) to be trivial. We also give a structure theorem of Dirac-harmonic maps from a Riemann surface generalizing result previously only known in the case when source manifold is a two sphere.

1. Introduction

Dirac-harmonic maps are a generalization and combination of harmonic maps and harmonic spinors while preserving the essential properties of the former. They arise from the supersymmetric nonlinear sigma model of quantum field theory [6].

Obviously, there are two types of basic examples, a harmonic map together with a vanishing spinor and a constant map together with a harmonic spinor. In [4], the authors constructed Dirac-harmonic maps \((\phi, \psi)\) from \(S^2\) to \(S^2\), where \(\phi\) is a harmonic map (or equivalently, a (possible branched) conformal map), \(\psi\) could be written in the form

\[
\psi = \sum_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi^*(\epsilon_\alpha)
\]

where \(\epsilon_\alpha (\alpha = 1, 2)\) is a local orthonormal basis of \(S^2\), and \(\Psi\) is a twistor spinor.

In the spirit of CHEN–JOST–LI–WANG, recently, JOST–MO–ZHU constructed explicit examples of Dirac-harmonic maps \((\phi, \psi)\) from an Euclidean space to a hyperbolic space which are non-trivial in the sense that \(\phi\) is not harmonic [4],

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In their examples, $\phi : \mathbb{R}^n \to H^{n+1}$ is an isometric immersion where $n \geq 3$ and $\psi$ could be written in the form

$$\psi^T = \sum \alpha \epsilon_\alpha \cdot \Psi \otimes \phi^* (\epsilon_\alpha)$$

where $(\ldots)^T$ denotes the orthogonal projection into the subbundle $\Sigma \mathbb{R}^n \otimes T\mathbb{R}^n$ and $\epsilon_\alpha (\alpha = 1, 2)$ is an orthonormal basis of $\mathbb{R}^n$. A natural question then is whether there exist non-trivial Dirac-harmonic maps for hypersphere in a hyperbolic space in this form.

In this paper, we will first give the following negative answer.

**Theorem 1.1.** Let $M^n$ be a compact positive scalar curved spinor manifold immersed in a non-positively constantly curved manifold $N$. Then there is no non-vanishing harmonic spinor $\psi$ along this immersion with $\psi^T = \sum \alpha \epsilon_\alpha \cdot \Psi \otimes \phi^* (\epsilon_\alpha)$, therefore, there is no non-trivial Dirac-harmonic map $(\phi, \psi)$ from $M$ into $N$ with $\psi^T = \sum \alpha \epsilon_\alpha \cdot \Psi \otimes \phi^* (\epsilon_\alpha)$.

For definitions of harmonic spinor and Dirac-harmonic map see Section 2 and Section 3. In the two-dimensional case we have the following:

**Proposition 1.2.** Let $\phi : M \hookrightarrow N$ be a surface in a Riemannian manifold of constant curvature $c$ with flat normal bundle. Then there is no non-trivial Dirac-harmonic map $(\phi, \psi)$ from $M$ into $N$ with $\psi^T = \sum \alpha \epsilon_\alpha \cdot \Psi \otimes \phi^* (\epsilon_\alpha)$.

Then we generalize Chen–Jost–Li–Wang’ construction [4] as following:

**Proposition 1.3.** Let $M$ be a Riemann surface and $N$ a Riemannian manifold. Let $\psi_{\phi, \Psi}$ be defined by $\psi_{\phi, \Psi} = \sum \alpha \epsilon_\alpha \cdot \Psi \otimes \phi^* (\epsilon_\alpha)$ from a nonconstant conformal map $\phi : M \to N$ and a spinor $\Psi \in \Gamma(\Sigma M)$. If $(\phi, \psi_{\phi, \Psi})$ is a Dirac-harmonic map then $\phi$ is a branched minimal immersion and $\Psi$ is a twistor spinor.

See Section 2 for definition of twistor spinor. Using Proposition 1.3, we obtain the following structure theorem of Dirac-harmonic maps from Riemann surfaces:

**Theorem 1.4.** Let $(\phi, \psi)$ is a non-constant Dirac-harmonic map from a compact Riemann surface $M_g$ of genus $g$ to the sphere $M_0$ with $|\deg \phi| > g - 1$. Then $\phi$ is $\pm$ holomorphic, and $\psi$ could be written in the form

$$\psi = \sum \alpha \epsilon_\alpha \cdot \Psi \otimes \phi^* (\epsilon_\alpha)$$

where $\epsilon_\alpha (\alpha = 1, 2)$ is a local orthonormal basis of $M_g$, and $\Psi$ is a twistor spinor.

It is worth mentioning the condition that $\deg \phi > g - 1$ is sharp. If $g \geq 1$ and $0 \leq d \leq g - 1$, Lemaire has constructed Riemann surface $M_g$ and harmonic non-$\pm$holomorphic maps $\phi : M_g \to M_0$ of degree $d$. Thus $(\phi, 0)$ is a Dirac-harmonic map from $M_g$ into $M_0$. 

2. Dirac-harmonic maps

In this section, we recall the basic definitions and introduce our notation. Let \((N, h)\) be a Riemannian manifold of dimension \(n'\). This will be our target manifold. Likewise, let \((M, g)\), our domain manifold, be an \(n\)-dimensional Riemannian manifold with fixed spin structure. By \(\Sigma M\), we denote its spinor bundle, on which we have a Hermitian metric \(\langle \cdot, \cdot \rangle\) induced by the Riemannian metric \(g(\cdot, \cdot)\) of \(M\). Let \(\phi\) be a smooth map from \((M, g)\) to \((N, h)\) and \(\phi^{-1}TN\) the pull-back bundle of \(TN\) by \(\phi\). On the twisted bundle \(\Sigma M \otimes \phi^{-1}TN\) there is a metric (still denoted by \(\langle \cdot, \cdot \rangle\)) induced from the metrics on \(\Sigma M\) and \(\phi^{-1}TN\). There is also a natural connection \(\tilde{\nabla}\) on \(\Sigma M \otimes \phi^{-1}TN\) induced from those on \(\Sigma M\) and \(\phi^{-1}TN\) (which in turn come from the Levi–Civita connections of \((M, g)\) and \((N, h)\), resp.).

We have the Clifford product \(X \cdot \Phi\) of \(X \in \Gamma(TM)\), \(\Phi \in \Gamma(\Sigma M)\). This Clifford product satisfies the skew-symmetry relation

\[
\langle X \cdot \Phi, \Psi \rangle = -\langle \Phi, X \cdot \Psi \rangle
\] (2.1)
as well as the Clifford relations

\[
X \cdot Y \cdot \Phi + Y \cdot X \cdot \Phi = -2g(X, Y)\Phi
\]

for \(X, Y \in \Gamma(TM)\), \(\Phi, \Psi \in \Gamma(\Sigma M)\).

We are now prepared to introduce an operator that couples the geometries of \(M\) and \(N\) via the map \(\phi\). Let \(\psi\) be a section of the bundle \(\Sigma M \otimes \phi^{-1}TN\). The Dirac operator along the map \(\phi\) is defined as

\[
\mathcal{D}\psi := \epsilon_\alpha \cdot \tilde{\nabla}_{\epsilon_\alpha} \psi
\]

where \(\epsilon_\alpha\) is a local orthonormal basis of \(M\). For background material about the spinor bundle and the Dirac operator, we refer to [9], [12].

We consider the space

\[
\chi := \{ (\phi, \psi) | \phi \in C^\infty(M, N) \text{ and } \psi \in C^\infty(\Sigma M \otimes \phi^{-1}TN) \}
\]

of mappings and sections along those mappings. On \(\chi\), we have the functional

\[
L(\phi, \psi) := \frac{1}{2} \int_M \left[ |d\phi|^2 + \langle \psi, \mathcal{D}\psi \rangle \right] \ast 1_M.
\]

This functional couples the two fields \(\phi\) and \(\psi\) because the operator \(\mathcal{D}\) depends on the map \(\phi\). The Euler–Lagrange equations of \(L(\phi, \psi)\) then also couple the two fields; they are:

\[
\tau(\phi) = \mathcal{R}(\phi, \psi)
\] (2.2)
\[ \mathcal{D}\psi = 0 \tag{2.3} \]

where \( \tau(\phi) \) is the tension field of the map \( \phi \) (the natural version of the Laplace operator for maps between manifolds) and the curvature term \( \mathcal{R}(\phi, \psi) \) is defined by

\[ \mathcal{R}(\phi, \psi) = \frac{1}{2} R_{ijkl}^d (\psi^k, \nabla \psi^j \cdot \psi^i) \frac{\partial}{\partial y^j}, \]

where

\[ \psi = \psi^i \otimes \frac{\partial}{\partial y^i}, \quad (d\phi)^a = \nabla \phi^a \otimes \frac{\partial}{\partial y^a}, \]

\[ R^{\phi^{-1}TN} \left( \frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^l} \right) \frac{\partial}{\partial y^i} = R_{ijkl}^d \frac{\partial}{\partial y^j} \]

where \( : TM \otimes \phi^{-1}TN \to TM \otimes \phi^{-1}TN \) is the standard (“musical”) isomorphism obtained from the Riemannian metric \( g \).

Solutions \((\phi, \psi)\) to (2.2) and (2.3) are called Dirac-harmonic maps from \( M \) into \( N \) [4].

We now start with some differential geometric identities: Let \( \epsilon_a \) be a local orthonormal basis of \( M \). By using the Clifford relations we have

\[ \epsilon_\alpha \cdot \epsilon_\beta \cdot \Psi = (-1)^{\delta_{\alpha\beta}} \epsilon_\beta \cdot \epsilon_\alpha \cdot \Psi = \begin{cases} -\Psi, & \alpha = \beta \\ -\epsilon_\beta \cdot \epsilon_\alpha \cdot \Psi, & \alpha \neq \beta \end{cases} \tag{2.4} \]

for \( \Psi \in \Gamma(\Sigma M) \).

**Lemma 2.1.** \( \mathcal{R}(\phi, \psi) \in \Gamma(\phi^{-1}TN) \); in particular, it is real.

**Proof.** For any (not necessarily orthonormal) frame \( \{\epsilon_i\} \) on \( \phi^{-1}TN \), we put

\[ \psi = \psi^a \otimes \epsilon_a, \tag{2.5} \]

\[ (d\phi)^a = \nabla \phi^a \otimes \epsilon_a, \quad R^{\phi^{-1}TN}(\epsilon_a, \epsilon_b)\epsilon_c = R_{abc}^d \epsilon_d \tag{2.6} \]

where \( : T^*M \otimes \phi^{-1}TN \to TM \otimes \phi^{-1}TN \) is the musical isomorphism as before.

Take

\[ \epsilon_a = u^i_a \frac{\partial}{\partial y^i}, \]

then

\[ \psi^i = u^i_a \psi^a, \quad \nabla \psi^i = u^i_a \nabla \psi^a, \quad u^i_a u^j_b u^l_c R_{ijkl}^d = R_{abc}^d u^d_i. \]
A simple calculation gives following
\[ R^i_{jkl}(\psi^k, \nabla\phi^j \cdot \psi^l) \frac{\partial}{\partial y^i} = R^{abcd}(\phi(x))(\psi^c, \nabla\phi^b \cdot \psi^d)\epsilon_{a}(\phi(x)). \] (2.7)

It follows that the definition of \( R(\phi, \psi) \) is independent of the choice of frame. It is then well-defined vector field on \( \phi^{-1}TN \). On the other hand, from the skew-symmetry of \( R_{ijkl} \) with respect to the induces \( k \) and \( l \), we have
\[
\frac{1}{2} R^i_{jkl}(\psi^k, \nabla\phi^j \cdot \psi^l) = \frac{1}{2} R^i_{jkl}(\nabla\phi^j \cdot \psi^k, \psi^l) = \frac{1}{2} R^i_{jkl}(\psi^k, \nabla\phi^j \cdot \psi^l).
\]

It follows that \( R(\phi, \psi) \in \Gamma(\phi^{-1}TN) \). \( \square \)

A spinor (field) \( \Psi \in \Gamma(\Sigma M) \) is called a \textit{twistor spinor} if \( \Psi \) belongs to the kernel of the twistor operator, equivalently,
\[
\nabla_X \Psi + \frac{1}{n} X \cdot \partial \Psi = 0, \quad \forall X \in \Gamma(TM)
\]
where we recall that \( n \) is the dimension of the Riemannian manifold \( M \), \( \Sigma M \) is the associated spinor bundle of \( M \) and \( \partial \) is the usual Dirac operator (cf. [1], [8], [11]).

In fact the concept of a twistor spinor (in particular, a Killing spinor) is motivated by theories from physics, like general relativity, 11-dimensional (resp. 10-dimensional) supergravity theory, supersymmetry (see, for example [2], [3], [5]).

We establish the following Lemma 2.2 required in the proof of Proposition 1.3.

**Lemma 2.2.** Let \( \Phi \in \Gamma(\Sigma M) \). Then \( \Phi \) is a twistor spinor if and only if
\[
\epsilon_1 \cdot \nabla_{\epsilon_1} \Phi = \cdots = \epsilon_n \cdot \nabla_{\epsilon_n} \Phi
\]
for some orthonormal frame field \( \epsilon_\alpha \) of \( M \).

**Proof.** Let us assume that (2.8) holds for some orthonormal frame field of \( \epsilon_\alpha \). Hence we may set
\[
\Psi := \epsilon_\alpha \cdot \nabla_{\epsilon_\alpha} \Phi.
\]
Then
\[
\partial \Phi = \sum_{\alpha} \epsilon_\alpha \cdot \nabla_{\epsilon_\alpha} \Phi = \sum_{\alpha} \Psi = n \Psi
\]
where \( n = \text{dim} \, M \). Together with (2.4) and (2.9) we have
\[
\nabla_{\epsilon_\beta} \Phi = -\epsilon_\beta \cdot \epsilon_\beta \cdot \nabla_{\epsilon_\beta} \Phi = -\epsilon_\beta \cdot \epsilon_\beta \cdot \left( \frac{1}{n} \partial \Phi \right) = -\frac{1}{n} \epsilon_\beta \cdot \partial \Phi.
\]
It follows that
\[ \nabla_X \Phi + \frac{1}{n} X \cdot \psi = \nabla_X \epsilon_\alpha \Phi = X^\alpha \nabla_{\epsilon_\alpha} \Phi = X^\alpha \epsilon_\alpha \cdot \partial \Phi \]
for arbitrary \( X = X^\alpha \epsilon_\alpha \in \Gamma(TM) \). Thus we see that \( \Phi \) is a twistor spinor.

Conversely, if \( \Phi \) is a twistor spinor, then the spinor field
\[ X \cdot \nabla_X \Phi \]
does not depend on the unit vector field \( X \) [1, page 23, Theorem 2]. □

3. Dirac-harmonic maps along an isometric immersion

In this section, we are going to give some sufficient conditions for a Dirac-harmonic map along an isometric immersion to be trivial.

Let \( \psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN) \) be a spinor field along \( \phi : M \to N \). We call \( \psi \) to be harmonic if \( \nabla \psi = 0 \) [4].

Let \( \phi : M \hookrightarrow N \) be an isometric immersion. This means that the Riemannian metric on \( M \) induced from the ambient space \( N \) coincides with the original one on \( M \). We identify \( M \) with its immersed image in \( N \). For each \( x \in M \) the tangent space \( T_xN \) can be decomposed into a direct sum of \( T_xM \) and its orthogonal complement \( T_x^\perp M \). Such a decomposition is differentiable. Thus, we have an orthogonal decomposition of the tangent bundle \( TN \) along \( M \)
\[ TN|_M = \phi^{-1}TN = TM \oplus T_x^\perp M. \]

Let \((\ldots)^T\) denote the orthogonal projection into the subbundle \( \Sigma M \otimes TM \) from the twisted bundle \( \Sigma M \otimes \phi^{-1}TN \).

For a global section \( R(\phi, \psi) \) on \( \phi^{-1}TN \) (see Lemma 2.1), we have
\[ R(\phi, \psi) = R^T(\phi, \psi) + R^N(\phi, \psi) \]
where
\[ R^T(\phi, \psi) \in \Gamma(TM), \quad R^N(\phi, \psi) \in \Gamma(T_x^\perp M). \]

Similarly, for \( D\psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN) \), we have
\[ D\psi = D^T\psi + D^N\psi \]
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where

\[ D^T \psi \in \Gamma(\Sigma M \otimes TM), \quad D^N \psi \in \Gamma(\Sigma M \otimes T^\perp M). \]

The mean curvature vector of \( M \) in \( N \) is

\[ H = \frac{1}{n} \tau(\phi) \in \Gamma(T^\perp M) \]

where \( \tau(\phi) \) is the tension field of the map \( \phi \). Hence we have the following:

**Lemma 3.1.** Let \( \phi : M \hookrightarrow N \) be an isometric immersion with the mean curvature vector \( \xi \) and \( \psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN) \). Then \( (\phi, \psi) \) is a Dirac-harmonic map from \( M \) into \( N \) if and only if

1. \( R^T(\phi, \psi) = 0 \);
2. \( R^N(\phi, \psi) = n\xi \) where \( n = \dim M \);
3. \( D^T \psi = 0 \);
4. \( D^N \psi = 0 \).

We shall be using the following ranges of indices:

\[ 1 \leq \alpha, \beta, \cdots \leq n, \quad n + 1 \leq s, t, \cdots \leq n', \quad 1 \leq i, j, \cdots \leq n'. \]

Choose a local orthonormal frame field \( \{ \epsilon_i \} \) of \( \phi^{-1}TN \) such that \( \{ \epsilon_\alpha \} \) lies in the tangent bundle \( TM \) and \( \{ \epsilon_s \} \) in the normal bundle \( T^\perp M \) of \( M \). We put

\[ (d\phi)^\sharp = \nabla \phi^i \otimes \epsilon_i \]  (3.1)

where \( \sharp : T^\ast M \otimes \phi^{-1}TN \to TM \otimes \phi^{-1}TN \) is the musical isomorphism. By using (3.1) we have

\[ \nabla \phi^i = \sum \delta^i_\alpha \epsilon_\alpha. \]  (3.2)

Now we assume that \( N = N(c) \) is a Riemannian manifold of constant curvature \( c \). Then the components of the Riemannian curvature tensor of \( N \) satisfy

\[ R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \]  (3.3)

**Proof of Theorem 1.1.** Let \( \psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN) \) be a spinor field along the isometric immersion \( \phi \) and \( \psi = \psi^i\epsilon_i \). From (3.2) and (3.3) we obtain

\[ R_{ijkl}(\nabla \phi^k \cdot \psi^i, \nabla \phi^j \cdot \psi^l) = c \left[ \langle \nabla \phi^i \cdot \psi^i, \nabla \phi^j \cdot \psi^j \rangle - \langle \nabla \phi^j \cdot \psi^i, \nabla \phi^i \cdot \psi^j \rangle \right] \]

\[ = c \left[ \langle \epsilon_\alpha \cdot \psi^\alpha, \epsilon_\beta \cdot \psi^\beta \rangle - \langle \epsilon_\beta \cdot \psi^\alpha, \epsilon_\alpha \cdot \psi^\beta \rangle \right] \]

\[ = c \sum_{\alpha \neq \beta} \left[ \langle \epsilon_\alpha \cdot \psi^\alpha, \epsilon_\beta \cdot \psi^\beta \rangle - \langle \epsilon_\beta \cdot \psi^\alpha, \epsilon_\alpha \cdot \psi^\beta \rangle \right]. \]  (3.4)
By using the skew-symmetry relation of the Clifford product and the Clifford relation we have
\[
\sum_{\alpha \neq \beta} (\epsilon_{\beta} \cdot \psi^\alpha, \epsilon_{\alpha} \cdot \psi^\beta) = -\sum_{\alpha \neq \beta} (\psi^\alpha, \epsilon_{\beta} \cdot \epsilon_{\alpha} \cdot \psi^\beta) = \sum_{\alpha \neq \beta} (\epsilon_{\alpha} \cdot \epsilon_{\beta} \cdot \psi^\beta).
\]

Plugging this into (3.4) yields
\[
R_{ijkl}(\nabla\phi^k \cdot \psi^i, \nabla\phi^l \cdot \psi^j) = 2c \sum_{\alpha \neq \beta} (\epsilon_{\alpha} \cdot \psi^\alpha, \epsilon_{\beta} \cdot \psi^\beta).
\]

Now we assume
\[
\psi^T = \sum_{\alpha} \epsilon_{\alpha} \cdot \Psi \otimes \phi_{e}(\epsilon_{\alpha}).
\]
where \(\Psi \in \Gamma(\Sigma M)\) is a spinor (field). It follows that \(\psi^\alpha = \epsilon_{\alpha} \cdot \Psi\), and therefore
\[
R_{ijkl}(\nabla\phi^k \cdot \psi^i, \nabla\phi^l \cdot \psi^j) = 2c \sum_{\alpha \neq \beta} (\epsilon_{\alpha} \cdot \epsilon_{\alpha} \cdot \Psi, \epsilon_{\beta} \cdot \epsilon_{\beta} \cdot \Psi).
\]

Assume that \(\psi\) is a harmonic spinor field along the isometric immersion \(\phi\). From Proposition 3.4 in [4], we have the following Bochner-type formula
\[
\frac{1}{2}\Delta|\psi|^2 = |\tilde{\nabla}\psi|^2 + \frac{1}{4} R|\psi|^2 - \frac{1}{2} R_{ijkl}(\nabla\phi^k \cdot \psi^i, \nabla\phi^l \cdot \psi^j)
\]
where \(R\) is the scalar curvature of \(M\). Substituting (3.5) into (3.6) yields
\[
\frac{1}{2}\Delta|\psi|^2 = |\tilde{\nabla}\psi|^2 + \frac{1}{4} R|\psi|^2 - 2(n - 1)nc|\Psi|^2.
\]

Therefore, under the assumption \(R > 0\) and \(c \leq 0\), (3.7) shows that \(|\psi|^2\) is subharmonic on \(M\). By the Hopf maximum principle, we see that this function must be a constant and the right hand side of (3.7) must be zero. In particular \(|\psi| = 0\).

Proof of Proposition 1.2. Plugging (3.2) into (2.7) yields
\[
\calR(\phi, \psi) = \frac{1}{2} R_{ijkl}(x) (\psi^k, \epsilon_{\alpha} \cdot \psi^\beta, \epsilon_{\alpha} \cdot \psi^\beta, x).
\]

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From which together with (3.3) we obtain

\[
R(\phi, \psi) = c(\delta^i_k \delta_{\alpha l} - \delta^i_l \delta_{\alpha k}) \text{Re}(\psi^k, \epsilon_\alpha \cdot \psi^l) \epsilon_i
\]

\[
= c \left[ \text{Re}(\psi^i, \epsilon_\alpha \cdot \psi^\alpha) - \text{Re}(\psi^\alpha, \epsilon_\alpha \cdot \psi^i) \right] \epsilon_i = 2c \text{Re}(\psi^i, \epsilon_\alpha \cdot \psi^\alpha) \epsilon_i.
\]

It follows that

\[
R^N(\phi, \psi) = 2c \text{Re}(\psi^s, \epsilon_\alpha \cdot \psi^\alpha) \epsilon_s = 2c \text{Re}(\psi^s, \epsilon_\alpha \cdot \Psi) \epsilon_s
\]

\[
= -2nc \text{Re}(\psi^s, \Psi) \epsilon_s.
\]

Together with (ii) of Lemma 3.1, we obtain

\[
-2c \text{Re}(\psi^{n+1}, \Psi) = \xi
\]

(3.9)

where \(\xi\) is the mean curvature of \(\phi\). Choose a local orthonormal frame field \(\{\epsilon_\alpha\}\) near \(x \in M\) with \(\nabla_{\epsilon_\alpha} \epsilon_\beta|_x = 0\). By (2.5) we have

\[
D\psi = D(\psi^i \otimes \epsilon_i) = \epsilon_\alpha \cdot \nabla_{\epsilon_\alpha}(\psi^i \otimes \epsilon_i)
\]

\[
= \epsilon_\alpha \cdot \left[ (\nabla_{\epsilon_\alpha} \psi^j) \otimes \epsilon_i + \psi^j \otimes \nabla_{\epsilon_\alpha} \epsilon_i \right]
\]

\[
= (\epsilon_\alpha \cdot \nabla_{\epsilon_\alpha} \psi^i) \otimes \epsilon_i + \epsilon_\alpha \cdot \left[ \psi^\beta \otimes \nabla_{\epsilon_\alpha} \epsilon_\beta + \psi^s \otimes \nabla_{\epsilon_\alpha} \epsilon_s \right]
\]

\[
= \partial \psi^i \otimes \epsilon_i + \epsilon_\alpha \cdot \psi^s \otimes \nabla_{\epsilon_\alpha} \epsilon_s
\]

(3.10)

at \(x\).

Let \(A_\nu\) be the shape operator and \(\nabla_X^\perp\) the normal connection of \(M\) in \(N\) where \(X\) denotes a tangent vector of \(M\) and \(\nu\) a normal vector to \(M\). Then

\[
\nabla_{\epsilon_\alpha} \epsilon_s = -A_\nu \epsilon_\alpha + \nabla_{\epsilon_\alpha}^\perp \epsilon_s.
\]

(3.11)

Let \(B\) be the second fundamental form of \(M\) in \(N\). Then \(B\) satisfies the Weingarten equation

\[
\langle B(X, Y), \nu \rangle = \langle A_\nu(X), Y \rangle
\]

(3.12)

where \(X, Y \in \Gamma(TM)\). By using (3.11) and (3.12) we have

\[
\nabla_{\epsilon_\alpha} \epsilon_s = -(B(\epsilon_\alpha, \epsilon_\beta), \epsilon_s) \epsilon_\beta + \nabla_{\epsilon_\alpha}^\perp \epsilon_s.
\]

(3.13)

By plugging (3.13) into (3.10) we obtain

\[
\partial \psi = \partial \psi^i \otimes \epsilon_i - (B(\epsilon_\alpha, \epsilon_\beta), \epsilon_s) \epsilon_\alpha \cdot \psi^s \otimes \epsilon_\beta + \epsilon_\alpha \cdot \psi^s \otimes \nabla_{\epsilon_\alpha}^\perp \epsilon_s.
\]

(3.14)
Let \( \psi^T \) be defined by
\[
\psi^T = \sum_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_s(\epsilon_\alpha).
\]
Choose a local orthonormal frame field \( \{\epsilon_\alpha\} \) near \( x \in M \) with \( \nabla_{\epsilon_\alpha} \epsilon_\delta |_x = 0. \)
\[
\tilde{\phi} \psi^T = \tilde{\phi}(\epsilon_\alpha \cdot \Psi) = \epsilon_\beta \cdot \nabla_{\epsilon_\beta} (\epsilon_\alpha \cdot \Psi) = \epsilon_\beta \left[ (\nabla_{\epsilon_\beta} \epsilon_\alpha) \cdot \Psi + \epsilon_\alpha \cdot \nabla_{\epsilon_\beta} \Psi \right] = \epsilon_\beta \cdot \epsilon_\alpha \cdot \nabla_{\epsilon_\beta} \Psi = - \nabla_{\epsilon_\alpha} \Psi = \epsilon_\beta \cdot \epsilon_\alpha \cdot \nabla_{\epsilon_\beta} \Psi.
\]
Substituting (3.15) into (3.14) and taking the tangent projection yield
\[
\tilde{\mathcal{D}} \psi^T = \epsilon_\beta \cdot \epsilon_\alpha \cdot \nabla_{\epsilon_\beta} \Psi = \epsilon_\beta \cdot \epsilon_\alpha \cdot \nabla_{\epsilon_\beta} \Psi.
\]
It is easy to see that
\[
\langle B(\epsilon_\alpha, \epsilon_\beta), \epsilon_s \rangle \epsilon_\alpha \cdot \psi^s \otimes \epsilon_\beta.
\]
does not depend on the choice of \( \{\epsilon_\alpha\} \). Since the normal bundle of \( M \) is flat, we choose \( \{\epsilon_\alpha\} \) such that
\[
\langle B(\epsilon_\alpha, \epsilon_\beta), \epsilon_s \rangle = \lambda_s^{\alpha} \delta_{\alpha\beta}.
\]
Therefore we have
\[
\sum_\beta \lambda^{n+1}_\beta = n \xi, \quad \text{and} \quad \sum_\beta \lambda^s_\beta = 0 \quad \text{for} \ s \neq n + 1.
\]
Plugging (3.17) into (3.16) yields
\[
\tilde{\mathcal{D}} \psi^T = - \sum_\beta \left[ 2 \nabla_{\epsilon_\beta} \Psi + \epsilon_\beta \cdot \tilde{\phi} \Psi + \sum_\alpha, A \lambda^A_\beta \epsilon_\beta \cdot \psi^A \right] \otimes \epsilon_\beta.
\]
Thus \( \tilde{\mathcal{D}} \psi^T = 0 \) if and only if
\[
2 \nabla_{\epsilon_\beta} \Psi + \epsilon_\beta \cdot \tilde{\phi} \Psi + \sum_\alpha, A \lambda^A_\beta \epsilon_\beta \cdot \psi^A = 0
\]
for all \( \beta \). By (2.4), (3.20) holds if and only if
\[
2 \epsilon_\beta \cdot \nabla_{\epsilon_\beta} \Psi - \tilde{\phi} \Psi = \sum_s \lambda^s_\beta \psi^s.
\]
Summing on \( \beta \) and using (3.18) we have
\[
(2 - n) \tilde{\phi} \Psi = n \xi \psi^{n+1}.
\]
Note that \( n = \text{dim} \ M = 2 \). It follows that
\[
\xi \psi^{n+1} = 0.
\]
Suppose that \( \xi(x) \neq 0 \) for some \( x \in M \), then (3.22) implies that \( \psi^{n+1}(x) = 0 \). Plugging this into (3.9) yields \( \xi(x) = 0 \) which is a contradiction and therefore \( \xi \equiv 0. \) □
Corollary 3.2. Let $\phi : M \hookrightarrow N$ be an $n(\geq 3)$-dimensional submanifold in a Riemannian manifold of constant curvature $c$ with flat normal bundle and let $(\phi, \psi)$ be a Dirac-harmonic map where

$$\psi^T = \sum_{\alpha} \epsilon_{\alpha} \cdot \Psi \otimes \phi_\ast(\epsilon_{\alpha})$$

for some $\Psi \in \Gamma(\Sigma M)$. Then $\phi$ is minimal if and only if $\Psi$ is harmonic.

4. Dirac-harmonic maps from a Riemann surface

In this section, we extend Chen–Jost–Li–Wang’ result and give a structure theorem of Dirac-harmonic maps from a Riemann surface.

Proof of Proposition 1.3. We claim that

$$R(\phi, \psi_\phi, \psi) \equiv 0, \quad \mathcal{D}\psi_\phi, \psi = -\Psi \otimes \tau(\phi) - 2 \left( \nabla_{\epsilon_\alpha} \Psi + \frac{1}{2} \epsilon_{\alpha} \cdot \partial \phi_\ast(\epsilon_{\alpha}) \right) \otimes \phi_\ast(\epsilon_{\alpha})$$

(4.1)

where $\epsilon_{\alpha}$ $(\alpha = 1, 2)$, as always, is a local orthonormal basis of $M$.

In fact, we define local vector fields $\nabla \phi^i$ on $M$ by

$$\nabla \phi^i := (d\phi)^\sharp(dy^i)$$

where $\{dy^i\}$ is the natural local dual basis on $N$. By using (1.1), we have

$$\psi^i := \psi_{\phi, \psi}(dy^i) = \nabla \phi^i \cdot \Psi.$$ 

Set $d\phi = \phi^\ast \theta^\alpha \otimes \frac{\partial}{\partial y^i}$ where $\theta^\alpha$ is the dual basis for $\epsilon_{\alpha}$. Then $\nabla \phi^i = \sum \phi^i_{\alpha} \epsilon_{\alpha}$ and

$$\langle \psi^k, \nabla \phi^j \cdot \psi^l \rangle = \phi^i_{\alpha} \phi^j_{\beta} \phi^l_{\gamma} \langle \epsilon_{\alpha}, \Psi, \epsilon_{\beta}, \epsilon_{\gamma} \cdot \Psi \rangle.$$ 

Note that $\text{Re}(\epsilon_{\alpha} \cdot \Psi, \epsilon_{\beta} \cdot \epsilon_{\gamma} \cdot \Psi) = 0$ [10, Lemma 3.1]. We conclude that $R_{jkl}^i(\psi^k, \nabla \phi^j \cdot \psi^l)$ is purely imaginary. On the other hand, from the proof of Lemma 2.1, $R_{jkl}^i(\psi^k, \nabla \phi^j \cdot \psi^l)$ must be real, and hence

$$R(\phi, \psi_{\phi, \psi}) \equiv \frac{1}{2} R_{jkl}^i(\psi^k, \nabla \phi^j \cdot \psi^l) \frac{\partial}{\partial y^i} \equiv 0.$$ 

By using (2.4) we have

$$\nabla_{\epsilon_{\alpha}} \Psi + \frac{1}{2} \epsilon_{\alpha} \cdot \partial \Psi = \nabla_{\epsilon_{\alpha}} \Psi + \frac{1}{2} \epsilon_{\alpha} \cdot \left[ \Sigma \epsilon_{\beta} \cdot \nabla \epsilon_{\gamma} \Psi \right]$$
Then
\[
\psi_{\phi_0,\psi} = \psi_{\phi_0,\phi_0} = \epsilon_0 \cdot \nabla_{\epsilon_0} (\epsilon_0 \cdot \Psi \otimes \phi_0 (\epsilon_0)),
\]
\[
\psi_{\phi_1,\psi} = \epsilon_1 \cdot \nabla_{\epsilon_1} (\epsilon_1 \cdot \Psi \otimes \phi_1 (\epsilon_1)),
\]
\[
\psi_{\phi_2,\psi} = \epsilon_2 \cdot \nabla_{\epsilon_2} (\epsilon_2 \cdot \Psi \otimes \phi_2 (\epsilon_2)),
\]
\[
\psi_{\phi_0,\phi_1} = \epsilon_0 \cdot \nabla_{\epsilon_0} (\epsilon_0 \cdot \phi_1 (\epsilon_0)),
\]
\[
\psi_{\phi_0,\phi_2} = \epsilon_0 \cdot \nabla_{\epsilon_0} (\epsilon_0 \cdot \phi_2 (\epsilon_0)),
\]
\[
\psi_{\phi_1,\phi_2} = \epsilon_1 \cdot \nabla_{\epsilon_1} (\epsilon_1 \cdot \phi_2 (\epsilon_1)).
\]
We choose a local orthonormal frame field \(\epsilon_\alpha\) such that \(\nabla_{\epsilon_\alpha} \epsilon_\beta = 0\) at \(x \in M\).

Then
\[
\hat{D} \psi_{\phi_0,\psi} = \epsilon_\beta \cdot \nabla_{\epsilon_\beta} \psi_{\phi_0,\psi} = \epsilon_\beta \cdot \nabla_{\epsilon_\beta} (\epsilon_\alpha \cdot \Psi \otimes \phi_\epsilon (\epsilon_\alpha))
\]
\[
= \epsilon_\beta \cdot \left[ \nabla_{\epsilon_\beta} (\epsilon_\alpha \cdot \Psi) \otimes \phi_\epsilon (\epsilon_\alpha) + \epsilon_\alpha \cdot \Psi \otimes \nabla_{\epsilon_\beta} (\phi_\epsilon (\epsilon_\alpha)) \right]
\]
\[
= \epsilon_\beta \cdot \left[ \left[ (\nabla_{\epsilon_\beta} (\epsilon_\alpha) \cdot \Psi + \epsilon_\alpha \cdot \nabla_{\epsilon_\beta} \Psi) \otimes \phi_\epsilon (\epsilon_\alpha) + \epsilon_\alpha \cdot \Psi \otimes \nabla_{\epsilon_\beta} (\phi_\epsilon (\epsilon_\alpha)) \right] \right]
\]
\[
= \epsilon_\beta \cdot \epsilon_\alpha \cdot \left[ \nabla_{\epsilon_\beta} \Psi \otimes \phi_\epsilon (\epsilon_\alpha) + \Psi \otimes \nabla_{\epsilon_\beta} (\phi_\epsilon (\epsilon_\alpha)) \right]
\]
\[
= (\Sigma_{\alpha=\beta} + \Sigma_{\alpha \neq \beta}) \epsilon_\beta \cdot \epsilon_\alpha \cdot \left\{ \nabla_{\epsilon_\beta} \Psi \otimes \phi_\epsilon (\epsilon_\alpha) + \Psi \otimes \nabla_{\epsilon_\beta} (\phi_\epsilon (\epsilon_\alpha)) \right\}
\]
\[
= (I) + (II)
\]
where

\[
(I) = \epsilon_\alpha \cdot \epsilon_\alpha \cdot \left\{ \nabla_{\epsilon_\alpha} \Psi \otimes \phi_\epsilon (\epsilon_\alpha) + \Psi \otimes \nabla_{\epsilon_\alpha} (\phi_\epsilon (\epsilon_\alpha)) \right\}
\]
\[
= - \{ \nabla_{\epsilon_\alpha} \Psi \otimes \phi_\epsilon (\epsilon_\alpha) + \Psi \otimes \nabla_{\epsilon_\alpha} (\phi_\epsilon (\epsilon_\alpha)) \} - \phi_\epsilon (\nabla_{\epsilon_\alpha} (\phi_\epsilon (\epsilon_\alpha)))
\]
\[
= - \{ \nabla_{\epsilon_\alpha} \Psi \otimes \phi_\epsilon (\epsilon_\alpha) + \Psi \otimes \tau (\phi) \}
\]
\[
(II) = \epsilon_1 \cdot \epsilon_2 \cdot \left\{ \nabla_{\epsilon_1} \Psi \otimes \phi_\epsilon (\epsilon_2) + \Psi \otimes \nabla_{\epsilon_1} (\phi_\epsilon (\epsilon_2)) \right\}
\]
\[
+ \epsilon_2 \cdot \epsilon_1 \cdot \left\{ \nabla_{\epsilon_2} \Psi \otimes \phi_\epsilon (\epsilon_1) + \Psi \otimes \nabla_{\epsilon_2} (\phi_\epsilon (\epsilon_1)) \right\}
\]
\[
= \epsilon_1 \cdot \epsilon_2 \cdot \left\{ \nabla_{\epsilon_1} \Psi \otimes \phi_\epsilon (\epsilon_2) - \nabla_{\epsilon_2} \Psi \otimes \phi_\epsilon (\epsilon_1) + \Psi \otimes \nabla_{\epsilon_1} (\phi_\epsilon (\epsilon_2))
\]
\[
- \Psi \otimes \nabla_{\epsilon_2} (\phi_\epsilon (\epsilon_1)) \right\}
\]
\[
= \epsilon_1 \cdot \epsilon_2 \cdot \left\{ \nabla_{\epsilon_1} \Psi \otimes \phi_\epsilon (\epsilon_2) - \nabla_{\epsilon_2} \Psi \otimes \phi_\epsilon (\epsilon_1) \right\}
\]
\[
(4.3)
\]
here we have used the following
\[
\nabla_{\epsilon_1} (\phi_\epsilon (\epsilon_2)) = (\nabla_{\epsilon_1} \phi_\epsilon) (\epsilon_2) = (\nabla_{\epsilon_2} \phi_\epsilon) (\epsilon_1) = \nabla_{\epsilon_2} (\phi_\epsilon (\epsilon_1)).
\]

Substituting (4.4) and (4.5) into (4.3) yields
\[
\hat{D} \psi_{\phi_0,\psi} = - \{ \nabla_{\epsilon_\alpha} \Psi \otimes \phi_\epsilon (\epsilon_\alpha) + \Psi \otimes \tau (\phi) \}
\]
\[
+ \epsilon_1 \cdot \epsilon_2 \cdot \{ \nabla_{\epsilon_1} \Psi \otimes \phi_\epsilon (\epsilon_2) - \nabla_{\epsilon_2} \Psi \otimes \phi_\epsilon (\epsilon_1) \}
\]
\[
= - \Psi \otimes \tau (\phi) - (\nabla_{\epsilon_1} \Psi + \epsilon_1 \cdot \epsilon_2 \cdot \nabla_{\epsilon_2} \Psi) \otimes \phi_\epsilon (\epsilon_1)
\]
\[
+ (\epsilon_1 \cdot \epsilon_2 \cdot \nabla_{\epsilon_1} \Psi - \nabla_{\epsilon_2} \Psi) \otimes \phi_\epsilon (\epsilon_2).
\]
\[
(4.6)
\]
Plugging (4.2) into (4.6) yields the second equation of (4.1).

By using the Clifford relation, one obtains

\[ \nabla_{\epsilon_1} \Psi + 1/2 \epsilon_1 \cdot \phi \psi = 1/2 (\nabla_{\epsilon_1} \Psi + \epsilon_1 \cdot \epsilon_2 \cdot \nabla_{\epsilon_2} \Psi) = 1/2 \epsilon_1 \cdot \Phi \]  

(4.7)

where

\[ \Phi := -\epsilon_1 \cdot \nabla_{\epsilon_1} \Psi + \epsilon_2 \cdot \nabla_{\epsilon_2} \Psi. \]

We recall that \( \Phi = 0 \) if and only if \( \Psi \) is a twistor spinor, equivalently, \( \Psi \) belongs to the kernel of the twistor operator (cf. Lemma 2.2). Similarly, we have

\[ \nabla_{\epsilon_2} \Psi + 1/2 \epsilon_2 \cdot \phi \psi = 1/2 (\nabla_{\epsilon_2} \Psi + \epsilon_2 \cdot \epsilon_1 \cdot \nabla_{\epsilon_1} \Psi) = -1/2 \epsilon_2 \cdot \Phi. \]  

(4.8)

Plugging (4.7) and (4.8) into (4.1) yields

\[ \mathcal{D}_\psi \phi, \psi = -\psi \otimes \tau(\phi) + 1/2 \epsilon_1 \cdot \Phi \otimes \phi_*(\epsilon_1) - 1/2 \epsilon_2 \cdot \Phi \otimes \phi_*(\epsilon_2). \]  

(4.9)

Note that \( (\phi, \psi, \psi, \tau) \) is a Dirac-harmonic map, i.e.

\[ \tau(\phi) = \mathcal{R}(\phi, \psi, \psi), \]

(4.10)

\[ \mathcal{D}_\psi \phi, \psi = 0. \]  

(4.11)

(4.1) and (4.10) imply that

\[ \tau(\phi) = 0. \]  

(4.12)

Hence \( \phi \) is a harmonic map, equivalently, it is a branched minimal immersion. Substituting (4.12) into (4.9) and using (4.11) yield

\[ \epsilon_1 \cdot \Phi \otimes \phi_*(\epsilon_1) - \epsilon_2 \cdot \Phi \otimes \phi_*(\epsilon_2) = 0. \]  

(4.13)

Since \( \phi : (M, g) \to (N, h) \) is conformal, we can assume that \( \phi^* h = e^\lambda g \). It follows that

\[ h(\phi_*(\epsilon_\alpha), \phi_*(\epsilon_\beta)) = \delta_{\alpha\beta} e^\lambda. \]  

(4.14)

Note that \( \phi \) is non-constant, there exists an \( \alpha \) such that \( \phi_*(\epsilon_\alpha) \neq 0 \). Without loss of generality, we assume \( \phi_*(\epsilon_1) \neq 0 \). From (4.13) and (4.14) we have \( \epsilon_1 \cdot \Phi = 0 \). It follows that

\[ \Phi = -\epsilon_1 \cdot (\epsilon_1 \cdot \Phi) = 0. \]

Thus \( \Psi \) is a twistor spinor.
Remark. Note that the Dirac-harmonicity of \((\phi, \psi, \varphi, \Psi)\) implies the harmonicity of \(\phi\) and any harmonic map from a sphere is conformal. Hence Proposition 1.3 is a natural generalization of Proposition 2.2 of [4].

**Proof of Theorem 1.4.** We only consider the case that \(g > 0\) and \(\deg \phi > g - 1\) where \(g\) is the genus of compact Riemann surface \(M\). Let \((\phi, \psi)\) is a Dirac-harmonic map from a compact Riemann surface \(M_g\) of genus \(g\) to the sphere \(M_0\) and \(\phi\) is non-constant. By using Theorem 1.1 in [15], \(\phi\) is a harmonic map. Note that \(M_0\) is homeomorphic to \(S^2 = \mathbb{C}P^1\) and \(\phi\) is a non-constant map. Hence \(\phi\) is linearly full into \(\mathbb{C}P^1\). By using Liao’s result \(\phi\) is isotropic [13, Corollary 1]. Recall that isotropic harmonic maps are generated from holomorphic maps by a process of taking derivatives. Therefore \(\phi\) is \pm\text{-holomorphic for } n = 1. Consider the Fubini–Study metric on \(\mathbb{C}P^1\) with the constant holomorphic sectional curvature 4. The degree of \(\phi\) can be computed as follows [7], [14]

\[
\deg(\phi) = \frac{1}{\pi} [E'(\phi) - E''(\phi)]
\]

where \(E'(\phi)\) (resp. \(E''(\phi)\)) is the holomorphic (resp. anti-holomorphic) energy of \(\phi\). If \(\phi\) is anti-holomorphic, then \(E'(\phi) = 0\). It follows that

\[
0 \leq g - 1 < \deg(\phi) = -\frac{E''(\phi)}{\pi}
\]

Thus \(E''(\phi) \leq 0\). Hence \(\phi\) is also holomorphic. We conclude that \(\phi\) is constant which is a contradiction.

The twisted bundle \(\Sigma M_g \otimes \phi^{-1}T M_0\) can be divided into the following

\[
\Sigma M_g \otimes \phi^{-1}T M_0 = \Sigma M_g \otimes (\phi^{-1}T M_0)^C
\]

\[= (\Sigma^+ M_g \otimes \phi^{-1}T' M_0) \oplus (\Sigma^+ M_g \otimes \phi^{-1}T'' M_0)
\]

\[\oplus (\Sigma^- M_g \otimes \phi^{-1}T' M_0) \oplus (\Sigma^- M_g \otimes \phi^{-1}T'' M_0) \quad (4.15)
\]

where

\[
\Sigma^\pm M_g := \{ \Psi \in \Sigma M_g | \sqrt{-1} \epsilon_1 \cdot \epsilon_2 \cdot \Psi = \pm \Psi \}
\]

(4.16)

for some orthonormal frame field \(\epsilon_\alpha\) of \(M_g\) and \(T' M_0\) (resp. \(T'' M_0\)) denote the tangent bundle of \(M_0\) of type (1, 0) (resp. (0, 1)). Denote by \(\pi^+\) (resp. \(\pi^-\)) the projection of the twisted bundle \(\Sigma M_g \otimes \phi^{-1}T M_0\) onto the subbundle \(\Sigma^+ M_g \otimes \phi^{-1}T' M_0\) (resp. \(\Sigma^- M_g \otimes \phi^{-1}T' M_0\)). Let \(m\) denote the sum of the multiplicities of the zeros of the function \(|\pi^+(\psi)|\). If \(|\pi^+(\psi)|\) is not identically zero, then (cf. [15, Theorem 4.2])

\[
m = g - 1 - 2\deg(\phi).
\]
Some rigidity results for Dirac-harmonic maps

Note that
\[ g - 1 - 2 \deg(\phi) < g - 1 - 2(g - 1) = 1 - g \leq 0. \]
It follows that \( m \leq -1 \) which is a contradiction, therefore
\[ |\pi^+(\psi)| \equiv 0. \] (4.17)
Similarly we have
\[ |\pi^-(\psi)| \equiv 0. \] (4.18)

For non-constant holomorphic map \( \phi \)
\[ \phi^{-1}T' M_0 = \text{Span}\{\phi_*(\epsilon_1) - \sqrt{-1}\phi_*(\epsilon_2)\}, \]
\[ \phi^{-1}T'' M_0 = \text{Span}\{\phi_*(\epsilon_1) + \sqrt{-1}\phi_*(\epsilon_2)\}. \]
We write
\[ \Sigma^\pm M_g = \text{Span}\{\psi^\pm\} \]
where
\[ \psi^- = \epsilon_1 \cdot \psi^+. \] (4.19)
By using (4.15), (4.17) and (4.18), we have
\[ \psi = f\psi^+ \odot [\phi_*(\epsilon_1) - \sqrt{-1}\phi_*(\epsilon_2)] + g\psi^- \odot [\phi_*(\epsilon_1) + \sqrt{-1}\phi_*(\epsilon_2)]. \] (4.20)
From (4.16), (4.19) and the Clifford relation ones obtain
\[ \psi^+ = -\epsilon_1 \cdot \psi^- = -\sqrt{-1}\epsilon_2 \cdot \psi^-; \] (4.21)
\[ \psi^- = \epsilon_1 \cdot \psi^+ = -\sqrt{-1}\epsilon_2 \cdot \psi^+. \] (4.22)
Plugging (4.21) and (4.22) into (4.20) yields
\[ \psi = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha) \]
where \( \Psi = g\psi^+ - f\psi^- \). Note that arbitrary isotropic harmonic map is conformal.
From Proposition 1.3, \( \Psi \) is a twistor spinor. \( \square \)

In particular, we have the following

**Corollary 4.1.** Let \((\phi, \psi)\) is a non-constant Dirac-harmonic map from a torus \(T^2\) to a sphere with non-zero degree. Then \( \phi \) is \( \pm \) holomorphic, and \( \psi \) could be written in the form
\[ \psi = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha) \]
where \( \epsilon_\alpha (\alpha = 1, 2) \) is a local orthonormal basis of \( T^2 \), and \( \Psi \) is a twistor spinor.

We have several special case of Theorem 1.4.

(1) When \( g = 0 \), our corollary have been given by Yang Ling [15];
(2) When \( \psi = 0 \), our result is reduced to Liao’s isotropy work [13, Corollary 1].
References


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