A note on $n$-clean group rings

By ANGELINA Y. M. CHIN (Kuala Lumpur) and KIAT TAT QUA (Kuala Lumpur)

Abstract. Let $R$ be an associative ring with identity. An element $x \in R$ is clean if $x$ can be written as the sum of a unit and an idempotent in $R$. $R$ is said to be clean if all of its elements are clean. Let $n$ be a positive integer. An element $x \in R$ is $n$-clean if it can be written as the sum of an idempotent and $n$ units in $R$. $R$ is said to be $n$-clean if all of its elements are $n$-clean. In this paper we obtain conditions which are necessary or sufficient for a group ring to be $n$-clean.

1. Introduction

Throughout this paper all rings are associative with identity. The notion of clean rings was first introduced by Nicholson in [4]. An element $x$ in a ring $R$ is said to be clean if $x$ can be written as the sum of a unit and an idempotent in $R$. The ring $R$ is clean if every element in $R$ is clean. In [6], Xiao and Tong generalised clean rings to $n$-clean rings. For a positive integer $n$, an element $x$ in a ring $R$ is $n$-clean if $x$ can be written as the sum of an idempotent and $n$ units in $R$. A ring $R$ is $n$-clean if all of its elements are $n$-clean. Clearly, a 1-clean ring is a clean ring and vice versa.

It is known by work of Chen and Zhou [1], as well as McGovern [3], that for a commutative ring $R$ and an abelian group $G$, if $RG$ is clean, then $G$ is locally finite. We extend this result to $n$-clean rings in this paper. We also show that a partial converse of this result is true. That is, we show that if $R$ is a commutative clean ring and $G$ is a locally finite $p$-group where $p$ is some prime with $p \in J(R)$,
then $RG$ is $n$-clean. The notation $J(R)$ as usual denotes the Jacobson radical of the ring $R$.

2. Some preliminary results

In this section we obtain some results which will be used in the proofs of the main results.

Proposition 2.1. Let $n$ be a positive integer. Then every homomorphic image of an $n$-clean ring is $n$-clean.

Proof. Let $R$ be an $n$-clean ring and let $\phi : R \to S$ be a ring epimorphism. Let $x \in S$. Then $x = \phi(y)$ for some $y \in R$. Since $R$ is $n$-clean, then $y = e + u_1 + \cdots + u_n$ for some idempotent $e$ and units $u_1, \ldots, u_n$ in $R$. Since $\phi$ is an epimorphism, we then have that $\phi(e)$ is an idempotent, $\phi(u_i)$ is a unit in $S$ $(i = 1, \ldots, n)$ and $x = \phi(y) = \phi(e) + \phi(u_1) + \cdots + \phi(u_n)$, that is, $x$ is $n$-clean in $S$. It follows that $\phi(R) = S$ is $n$-clean. □

Proposition 2.2. Let $R$ be a ring and let $n$ be a positive integer. If $R$ is local, then $R$ is $n$-clean.

Proof. Since $R$ is local, the only idempotents in $R$ are 0 and 1. Let $x \in R$.

Case 1: $x$ is a unit

Note that

$$x = \begin{cases} 0 + (x + (-x)) + \cdots + (x + (-x)) + x, & \text{if } n \text{ is odd} \\ 1 + (x + (-1)) + (x + (-x)) + \cdots + (x + (-x)), & \text{if } n \text{ is even} \end{cases}.$$

Case 2: $x$ is not a unit

In this case $1 - x$ is a unit. Hence $x - 1$ is also a unit and we have

$$x = \begin{cases} 1 + ((1 - x) + (x - 1)) + \cdots + ((1 - x) + (x - 1)) + (x - 1), & \text{if } n \text{ is odd} \\ 0 + ((x - 1) + 1) + ((1 - x) + (x - 1)) + \cdots + ((1 - x) + (x - 1)), & \text{if } n \text{ is even} \end{cases}.$$

In both cases, we have that $x$ can be written as the sum of an idempotent and $n$ units; hence $x$ is $n$-clean. □

Proposition 2.3. A clean ring without any nontrivial idempotents is local.
Let $R$ be a clean ring with no nontrivial idempotents. Then for any $x \in R$, either $x$ or $1 - x$ is a unit. Suppose that $R$ has two distinct maximal right ideals $M_1$ and $M_2$. Then there exists an element $a \in M_1$, $a \notin M_2$. Thus for every $r \in R$, $ar \in M_1$ and since $M_1$ is a maximal right ideal (hence, a proper ideal), it follows that $ar$ is not a unit. Therefore $1 - ar$ is a unit for every $r \in R$. Now from $M_2 + aR = R$, we have that $1 - as \in M_2$ for some $s \in R$. But since $1 - as$ is a unit, we have that $M_2 = R$ which contradicts the fact that $M_2$ is proper. Hence, $R$ only has one maximal right ideal, that is, $R$ must be local. \(\square\)

**Proposition 2.4.** Let $R$ be a ring, let $G$ be a group and let $n$ be a positive integer. If $x \in RG$ is not $n$-clean, then there exists a proper ideal $J$ of $R$ such that $R/J$ does not have any nontrivial central idempotent and $x + JG \in RG/JG$ is not $n$-clean.

**Proof.** Suppose that $x \in RG$ is not $n$-clean. Let $\mathcal{C} = \{I \triangleleft R \mid I \neq R, x + IG \in RG/IG$ is not $n$-clean$\}$. Then $\mathcal{C} \neq \emptyset$ because $\{0\} \in \mathcal{C}$. Let $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_j \subseteq I_{j+1} \subseteq \ldots$ be a chain of ideals in $\mathcal{C}$ and let $I = \bigcup I_i$. Then $I$ is a proper ideal of $R$ and $\bar{x} = x + IG \in RG/IG$ is not $n$-clean. Indeed, if $\bar{x}$ is $n$-clean, then $x + I_t G \in RG/I_t G$ is $n$-clean for some $t \in \mathbb{N}$; a contradiction. Therefore $I \in \mathcal{C}$. By Zorn’s Lemma, $\mathcal{C}$ has a maximal element, say $J$.

If $R/J$ has a nontrivial central idempotent $e + J$, then \[R/J = (e + J)R/J \oplus ((1 - e) + J)R/J \cong I_1/J \times I_2/J\] for some ideals $I_1, I_2$ of $R$ properly containing $J$. Then \[RG/JG \cong (R/J)G \cong (I_1/J)G \times (I_2/J)G \cong I_1G/JG \times I_2G/JG.\] (1) Note that \[RG/I_1G \cong (RG/JG)/(I_1G/JG) \cong I_2G/JG\] (2) and \[RG/I_2G \cong (RG/JG)/(I_2G/JG) \cong I_1G/JG.\] (3) Let $(x_1 + JG, x_2 + JG) \in I_1G/JG \times I_2G/JG$ be the image of $x + JG$ under the isomorphism in (1). By the maximality of $J$ in $\mathcal{C}$, $x_1 + I_t G$ is $n$-clean in $RG/I_t G$ ($t = 1, 2$). Hence, by (2) and (3), $x_1 + JG$ is $n$-clean in $I_t G/JG$ ($t = 1, 2$). It follows by (1) that $x + JG \in RG/JG$ is $n$-clean. This is a contradiction since $J \in \mathcal{C}$. Hence, $R/J$ does not have any nontrivial central idempotent. \(\square\)

**Corollary 2.5.** Let $R$ be a commutative clean ring, let $G$ be a group and let $n$ be a positive integer. If $x \in RG$ is not $n$-clean, then there exists a proper ideal $J$ of $R$ such that $R/J$ is local and $x + JG \in RG/JG$ is not $n$-clean.
PROOF. By Proposition 2.4, there exists a proper ideal $J$ of $R$ such that $R/J$ does not have any nontrivial idempotent and $x + JG \in RG/JG$ is not $n$-clean. Since $R/J$ is clean (by Proposition 2.1), it follows by Proposition 2.3 that $R/J$ is local.

In [5], Nicholson obtained sufficient conditions for a group ring to be local as follows:

**Proposition 2.6.** Let $R$ be a local ring and let $G$ be a locally finite $p$-group where $p$ is some prime with $p \in J(R)$. Then $RG$ is local.

3. Main results

For a ring $R$, let $Id(R)$ and $U(R)$ denote the set of all idempotents and the set of all units of $R$, respectively. We first extend an idea in [1] to obtain necessary conditions for a commutative group ring to be $n$-clean.

**Theorem 3.1.** Let $R$ be a commutative ring, let $G$ be an abelian group and let $n$ be a positive integer. If $RG$ is $n$-clean, then $R$ is $n$-clean and $G$ is locally finite.

**Proof.** Since $RG/\Delta \cong R$ where $\Delta$ is the augmentation ideal of $RG$, it follows readily by Proposition 2.1 that $R$ is $n$-clean. Suppose that $G$ is not locally finite. Then $G$ is not torsion; hence $G/t(G)$ is nontrivial and torsion-free, where $t(G)$ is the torsion subgroup of $G$. Since $R(G/t(G)) \cong RG/R(t(G))$ is a homomorphic image of $RG$ and $RG$ is $n$-clean, it follows by Proposition 2.1 that $R(G/t(G))$ is $n$-clean. We may therefore assume that $G$ is torsion-free. If $G$ has rank greater than 1, then $G$ has a torsion-free quotient $G'$ of rank 1. But since $RG'$ is also $n$-clean, we can assume that $G$ is of rank 1. Thus, $G$ is isomorphic to a subgroup of $(\mathbb{Q}, +)$. Since $R$ is commutative, then $R/M$ is a field where $M$ is a maximal ideal of $R$. Furthermore, $(R/M)G$ is $n$-clean because $(R/M)G \cong RG/MG$ is a homomorphic image of $RG$ (by Proposition 2.1). Hence, we can assume that $R$ is a field. Since $G$ is torsion-free, there exists a $g \in G$ such that $g^{-1} \neq g$. Now since $g + \cdots + g^n + g^{-1} + \cdots + g^{-n}$ is $n$-clean in $RG$, there exists a finitely generated subgroup $G_1$ of $G$ such that $g \in G_1$ and $g + \cdots + g^n + g^{-1} + \cdots + g^{-n}$ is $n$-clean in $RG_1$. From above, $G_1$ is isomorphic to a finitely generated subgroup of $(\mathbb{Q}, +)$. Since every finitely generated subgroup of $(\mathbb{Q}, +)$ is cyclic, so is $G_1$, and we can write $G_1 = \langle h \rangle$. Thus, $g = h^k, g^{-1} = h^{-k}$ for some $k \in \mathbb{N}$. Note that there is a natural isomorphism $R\langle h \rangle \cong R[x, x^{-1}]$ with
A note on n-clean group rings

$h^k + \cdots + h^{nk} + h^{-k} + \cdots + h^{-nk} \leftrightarrow x^k + \cdots + x^{nk} + x^{-k} + \cdots + x^{-nk}$. This implies that $x^k + \cdots + x^{nk} + x^{-k} + \cdots + x^{-nk}$ is n-clean in $R[x, x^{-1}]$ which is impossible because $\text{Id}(R[x, x^{-1}]) \subseteq R$ and $U(R[x, x^{-1}]) \subseteq \{ax^i | 0 \neq a \in R, i \in \mathbb{Z}\}$. Hence, $G$ must be locally finite.

In [2], it was shown that if $R$ is a clean ring and $G$ is a finite group such that $|G|$ is a unit in $R$, then $RG$ is not necessarily clean. Here we prove the following:

**Theorem 3.2.** Let $R$ be a commutative clean ring and let $G$ be a locally finite $p$-group with $p \in J(R)$. Then $RG$ is an n-clean ring for any positive integer $n$.

**Proof.** Let $n$ be a positive integer and suppose that $x \in RG$ is not n-clean. By Corollary 2.5, there exists a proper ideal $I$ of $R$ such that $R/I$ is local and $x + IG \in RG/IG$ is not n-clean. If $p + I$ is a unit in $R/I$, then $pr - 1 \in I$ for some $r \in R$. But $p \in J(R)$ implies that $pr - 1$ is a unit in $R$. Hence $I = R$; a contradiction. Thus $p + I$ is not a unit in $R/I$ and therefore, $p + I \in J(R/I)$. By Proposition 2.6, $RG/IG \cong (R/I)G$ is local. It follows by Proposition 2.2 that $RG/IG$ is n-clean; a contradiction. Hence, $x \in RG$ must be n-clean. $\square$

**Remark.** Let $Z_{(7)} = \{ \frac{a}{b} \in \mathbb{Q} | 7 \text{ does not divide } n \}$ and let $C_3$ be the cyclic group of order 3. The example in [2] that the group ring $Z_{(7)}C_3$ is not clean shows that the condition $p \in J(R)$ in Theorem 3.2 is not superfluous.

**References**


ANGELINA Y. M. CHIN
INSTITUTE OF MATHEMATICAL SCIENCES
UNIVERSITY OF MALAYA
50603 KUALA LUMPUR
MALAYSIA
E-mail: aymc@pc.jaring.my

KIAT TAT QUA
INSTITUTE OF MATHEMATICAL SCIENCES
UNIVERSITY OF MALAYA
50603 KUALA LUMPUR
MALAYSIA
E-mail: qkt84@um.edu.my

(Received December 15, 2009; revised July 31, 2010)