Weakly-peripherally multiplicative conditions and isomorphisms between uniform algebras

By RUMI SHINDO (Niigata)

Abstract. Suppose that $A$ and $B$ are uniform algebras on compact Hausdorff spaces $X$ and $Y$, respectively. Let $\rho, \tau : \Lambda \to A$ and $S, T : \Lambda \to B$ be mappings on a non-empty set $\Lambda$. Suppose that $\rho(\Lambda), \tau(\Lambda)$ and $S(\Lambda), T(\Lambda)$ are closed under multiplications and contain $\exp A$ and $\exp B$ respectively and that $S(e_1) \in S(\Lambda)^{-1}, T(e_2) \in T(\Lambda)^{-1}$ with $|S(e_1)T(e_2)| = 1$ on $\text{Ch}(B)$ for some fixed $e_1, e_2 \in A_1$ with $\rho(e_1) = \tau(e_2) = 1$. If $\sigma_\pi (S(f)T(g)) \cap \sigma_\pi (\rho(f)\tau(g)) \neq \emptyset$ for all $f, g \in \Lambda$ and there exists a first-countable dense subset $D_B$ in $\text{Ch}(B)$, or a first-countable dense subset $D_A$ in $\text{Ch}(A)$, then there exists an algebra isomorphism $\tilde{S} : A \to B$ such that $\tilde{S}(\rho(f)) = S(e_1)^{-1}S(f)$ and $\tilde{S}(\tau(f)) = T(e_2)^{-1}T(f)$ for every $f \in \Lambda$.

1. Introduction

The search for sufficient conditions for mappings between Banach algebras to be algebra isomorphisms has a long and interesting history. Such results demonstrate that linear maps between Banach algebras that preserve the norm, the spectrum, or a subset of the spectrum must be multiplicative. For example, one of the corollaries of the classical theorem of Gleason–Kahane–Żelazko [Ze] states that a surjection $T : A \to B$ between uniform algebras is an algebra isomorphism if it is linear and preserves the spectra, i.e. $\sigma(T(f)) = \sigma(f)$ for all $f, g \in A$. A theorem by Kowalski and Słodkowski [K-S] considers alternative spectral conditions for not necessarily linear surjections.

Mathematics Subject Classification: Primary: 46J10, 46J20; Secondary: 46H40.
Key words and phrases: uniform algebra, spectrum-preserving, peripheral spectrum.
MOLNÁR [Mo] have introduced an interesting spectral multiplicativity condition that contributes to the matter. In particular, he proved that if \( T \) is a surjection from the Banach algebra \( C(X) \) of all complex-valued continuous functions on a first countable compact Hausdorff space \( X \) onto itself such that \( \sigma(T(f)T(g)) = \sigma(fg) \) for all \( f, g \in C(X) \), then \( T \) is an algebra isomorphism. In the case where \( T \) is a surjection from a uniform algebra \( A \) onto itself, this result was proven by RAO and ROY [RR1]. HATORI, MIURA and TAKAGI [HMT06] showed that if \( T : A \to B \) is a surjection between uniform algebras such that the range of \( T(f)T(g) \) equals that of \( fg \) for all \( f, g \in A \), then \( T(1)^{-1}T \) is an algebra isomorphism. Maps between uniform algebras and more general semi-simple commutative Banach algebras that satisfy \( \sigma(T(f)T(g)) = \sigma(fg) \) [HMT07], [HMT], [RR2] or \( \sigma \left( T(f)T(g) \right) = \sigma \left( fg \right) \) [G-T], [LT] were analyzed further (see also [Hon]).

Maps \( T \) such that for some positive integers \( m \) and \( n \), \( \sigma(T(f)^mT(g)^n) \subset \sigma(f^m g^n) \), or, such that \( \sigma(T(f)T(g)) \) and \( \sigma(fg) \) meet only, without being necessarily equal, were analyzed recently (see [HHMO], [LLT], [T.talk], [JLV]).

Most recently, TONEV [T.talk], [T10] characterized a surjection \( T : A \to B \) between function algebras, without assuming the existences of the units, such that \( \sigma(T(f)T(g)) \cap \sigma(fg) \neq \emptyset \) and \( \sigma(T(f)) = \sigma(f) \) for all \( f, g \in A \). HATORI, MIURA, SHINDO and TAKAGI [HMST] have characterized maps \( \rho, \tau : I \to A \) and \( S, T : I \to B \) from a non-empty set into uniform algebras that satisfy \( \sigma(S(f)T(g)) \subset \sigma(\rho(f)\tau(g)) \) for all \( f, g \in I \). In this paper, we analyze maps \( \rho, \tau : \Lambda \to A \) and \( S, T : \Lambda \to B \) from a non-empty set into uniform algebras such that \( \sigma(S(f)T(g)) \cap \sigma(\rho(f)\tau(g)) \neq \emptyset \) for all \( f, g \in \Lambda \) and give conditions for isomorphisms between uniform algebras.

2. Main result

We begin by providing definitions and notations. Let \( C(X) \) be the space of all complex-valued continuous functions on a compact Hausdorff space \( X \). \( C(X) \) is Banach algebra with pointwise multiplication and the supremum norm \( \| \cdot \|_\infty \).

Let \( A \) be a uniform algebra on a compact Hausdorff space \( X \). Denote by \( M_A \) the maximal ideal space of \( A \), by \( \sigma(f) \) the spectrum of \( f \in A \), and by \( \hat{f} \) the Gelfand transform of \( f \in A \). Note that \( \sigma(f) = \hat{f}(M_A) \) and \( \text{sup} \{ |\lambda| : \lambda \in \sigma(f) \} = \| f \|_\infty \).

The peripheral spectrum of an element \( f \in A \) is the maximum modulus set of the spectrum of \( f \), that is \( \sigma_p(f) = \{ \lambda \in \sigma(f) : |\lambda| = \| f \|_\infty \} \). If \( \sigma_u(1) = \{ 1 \} \) for \( u \in A \), then \( u \) is called a peak function of \( A \). In this case \( u^{-1}(\{ 1 \}) \) is a peak set of \( A \). For a fixed \( x \in X \) denote by \( P_A(x) \) the set of all peak functions \( u \) of \( A \).
Weakly-peripherally multiplicative conditions

with \( u(x) = 1 \). A point \( x \in X \) that equals the intersection of peak sets is called a weak peak point of \( A \). The set of all weak peak points of \( A \) is the Choquet boundary of \( A \), denoted by \( \text{Ch}(A) \). It is known that \( \text{Ch}(A) \) is a boundary for \( A \), that is \( \| f \|_\infty = \max \{ |f(x)| : x \in \text{Ch}(A) \} \) for every \( f \in A \). An \( x \in X \) is said to be a peak point of \( A \) if \( \{ x \} \) is a peak set of \( A \). Note that a weak peak point \( x \) which has a countable neighborhood basis is a peak point of \( A \) [Br, Lemma 2.3.1 and Theorem 2.3.4]. Denote by \( \exp A \) the range of the exponential map on \( A \). In the sequel we will need the following corollary of [HHMO, Proposition 2.2] (see also [HMST, Proposition 2.3] (see also [S.Ce, Proposition 2.1]).

**Lemma 2.1.** If \( x \in X \) is a peak point and \( f(x) \neq 0 \) for some \( f \in A \), then there exists a \( u \in P_A(x) \cap \exp A \) such that \( \sigma_\pi (fu) = \{ f(x) \} \) and \( (fu)^{-1}(\{ f(x) \}) = u^{-1}(\{ 1 \}) = \{ x \} \).

**Proof.** By Proposition 2.2 in [HHMO], there exists a \( u_1 \in P_A(x) \cap \exp A \) such that \( \sigma_\pi (fu_1) = \{ f(x) \} \). Since \( x \) is a peak set, there exists a \( u' \in P_A(x) \) such that \( u'^{-1}(\{ 1 \}) = \{ x \} \). Let \( u_2 = (u' + 1)/2 \). Then \( u = u_1u_2 \in P_A(x) \cap \exp A \) satisfies \( \sigma_\pi (fu) = \{ f(x) \} \) and \( (fu)^{-1}(\{ f(x) \}) = u^{-1}(\{ 1 \}) = \{ x \} \).

Throughout this paper we assume that \( A \) and \( B \) are uniform algebras on compact Hausdorff spaces \( X \) and \( Y \) respectively and that \( \Lambda \) is a non-empty set. Denote by \( f^{-1} \) an inverse element of \( f \in A \) and by \( E^{-1} \) the set of invertible elements of \( E \). We will also use the following proposition, which is a corollary of [HMST, Proposition 2.3] (see also [S.Ce, Proposition 2.1]).

**Proposition 2.2.** Let \( h_1, h_2 : \Lambda \to A \) and \( H_1, H_2 : \Lambda \to B \) be mappings on \( \Lambda \). Suppose that \( h_1(\Lambda), h_2(\Lambda) \) and \( H_1(\Lambda), H_2(\Lambda) \) are closed under multiplications and contain \( \exp A \) and \( \exp B \), respectively. If

\[
\| H_1(f)H_2(g) \|_\infty = \| h_1(f)h_2(g) \|_\infty ,
\| H_1(f) \|_\infty = \| h_1(f) \|_\infty \text{ and } \| H_2(f) \|_\infty = \| h_2(f) \|_\infty
\]

for all \( f, g \in \Lambda \), then there exists a homeomorphism \( \psi : \text{Ch}(B) \to \text{Ch}(A) \) such that

\[
|H_1(f)(y)| = |h_1(f)(\psi(y))| \text{ and } |H_2(f)(y)| = |h_2(f)(\psi(y))|
\]

for every \( f \in \Lambda \) and \( y \in \text{Ch}(B) \).

**Proposition 2.3.** Let \( \rho, \tau : \Lambda \to A \) and \( S, T : \Lambda \to B \) be mappings on \( \Lambda \). Suppose that \( \rho(\Lambda), \tau(\Lambda) \) and \( S(\Lambda), T(\Lambda) \) are closed under multiplications and contain \( \exp A \) and \( \exp B \), respectively. Suppose that \( S(e_1) \in S(\Lambda)^{-1}, T(e_2) \in
$T(\Lambda)^{-1}$ with $|S(e_1)T(e_2)| = 1$ on $\text{Ch}(B)$ for some $e_1, e_2 \in \Lambda$ with $\rho(e_1) = \tau(e_2) = 1$. If
\[ \sigma_\pi(S(f)T(g)) \cap \sigma_\pi(\rho(f)\tau(g)) \neq \emptyset \]
for all $f, g \in \Lambda$, then there exists a homeomorphism $\psi : \text{Ch}(B) \rightarrow \text{Ch}(A)$ such that
\begin{align*}
|S(e_1)^{-1}S(f)(y)| &= |\rho(f)(\phi(y))|, \\
|T(e_2)^{-1}T(f)(y)| &= |\tau(f)(\phi(y))| 
\end{align*}
for every $y \in \text{Ch}(B)$. If, in addition, $S(e_1)^{-1}S(f)(y_0) = 1$ and $y_0 \in \text{Ch}(B)$ is a peak point of $B$, or $\phi(y_0)$ is a peak point of $A$, then
\begin{align*}
(S(e_1)^{-1}S(f))(y_0) &= \rho(f)(\phi(y_0)), \\
(T(e_2)^{-1}T(f))(y_0) &= \tau(f)(\phi(y_0)) 
\end{align*}
for every $f \in \Lambda$.

**Proof.** Since $|S(e_1)T(e_2)| = 1$ on $\text{Ch}(B)$, we obtain
\[ \|S(e_1)^{-1}S(f)T(e_2)^{-1}T(g)\|_\infty = \|S(f)T(g)\|_\infty, \]
which implies that
\begin{align*}
\|S(e_1)^{-1}S(f)T(e_2)^{-1}T(g)\|_\infty &= \|\rho(f)\tau(g)\|_\infty, \\
\|S(e_1)^{-1}S(f)\|_\infty &= \|\rho(f)\|_\infty \quad \text{and} \quad \|T(e_2)^{-1}T(g)\|_\infty = \|\tau(g)\|_\infty
\end{align*}
for all $f, g \in \Lambda$. Then the mappings $\rho, \tau, S(e_1)^{-1}S$ and $T(e_2)^{-1}T$ satisfy the hypotheses of Proposition 2.2. Hence there exists a homeomorphism $\phi : \text{Ch}(B) \rightarrow \text{Ch}(A)$ satisfying (2.1).

Suppose that $S(e_1)^{-1}S(f)(y_0) = 1$, that is $S(e_1)^{-1} = T(e_2)$. Let $f \in \Lambda$ and $y_0 \in \text{Ch}(B)$. Note that $(S(e_1)^{-1}S(f))(y_0) = 0$ if and only if $\rho(f)(\phi(y_0)) = 0$ and that $(T(e_2)^{-1}T(f))(y_0) = 0$ if and only if $\tau(f)(\phi(y_0)) = 0$. If $y_0$ is a peak point of $B$ and $(S(e_1)^{-1}S(f))(y_0) \neq 0$, then, by Lemma 2.1, there exists a $p \in P_B(y_0) \cap \exp B$ such that
\begin{align*}
\sigma_\pi(S(e_1)^{-1}S(f)p) &= \{(S(e_1)^{-1}S(f))(y_0)\}, \\
(S(e_1)^{-1}S(f)p)^{-1}(\{(S(e_1)^{-1}S(f))(y_0)\}) &= p^{-1}(\{1\}) = \{y_0\}. 
\end{align*}
(2.2)

Note that, by the hypotheses,
\[ \sigma_\pi(S(e_1)^{-1}S(f)T(e_2)^{-1}T(g)) \cap \sigma_\pi(\rho(f)\tau(g)) \neq \emptyset \]
for every \( g \in \Lambda \). Let \( g_0 \in \Lambda \) with \( T(e_2)^{-1}T(g_0) = p \). Since
\[
\sigma_{\pi}(S(e_1)^{-1}S(f)p) \cap \sigma_{\pi}(\rho(f)\tau(g_0)) \neq \emptyset,
\]
there exists a \( y' \in \text{Ch}(B) \) such that
\[
(\rho(f)\tau(g_0))(\phi(y')) = (S(e_1)^{-1}S(f))(y_0).
\]
We also have \( \sigma_{\pi}(p) \cap \sigma_{\pi}(\tau(g_0)) \neq \emptyset \). Thus there exists a \( y'' \in \text{Ch}(B) \) such that \( \tau(g_0)(\phi(y'')) = 1 \). Equation (2.1) shows that
\[
|(S(e_1)^{-1}S(f)p)(y')| = |(\rho(f)\tau(g_0))(\phi(y'))| = |(S(e_1)^{-1}S(f))(y_0)|, \quad \text{and}
\]
\[
|p(y'')| = |\tau(g_0)(\phi(y''))| = 1.
\]
Together with (2.2), we obtain \( y' = y'' = y_0 \). We conclude that
\[
(S(e_1)^{-1}S(f))(y_0) = (\rho(f)\tau(g_0))(\phi(g_0)) = \rho(f)(\phi(g_0)).
\]
If we consider the maps \( T(e_2)^{-1}T \) and \( \tau \), the same arguments imply that
\[
(T(e_2)^{-1}T(f))(y_0) = \tau(f)(\phi(g_0)).
\]
Similar arguments complete the proof for the case where \( \phi(g_0) \) is a peak point of \( A \).

We will now prove the main theorem by using Proposition 2.3.

**Theorem 2.4.** Let \( \rho, \tau : \Lambda \to A \) and \( S, T : \Lambda \to B \) be mappings on \( \Lambda \). Suppose that \( \rho(\Lambda), \tau(\Lambda) \) and \( S(\Lambda), T(\Lambda) \) are closed under multiplications and contain \( \exp A \) and \( \exp B \) respectively. Suppose that \( S(e_1) \in S(\Lambda)^{-1}, T(e_2) \in T(\Lambda)^{-1} \) with \( |S(e_1)T(e_2)| = 1 \) on \( \text{Ch}(B) \) for some \( e_1, e_2 \in \Lambda \) with \( \rho(e_1) = \tau(e_2) = 1 \). If there exists a first-countable dense subset \( D_B \) in \( \text{Ch}(B) \), or a first-countable dense subset \( D_A \) in \( \text{Ch}(A) \), and
\[
\sigma_{\pi}(S(f)T(g)) \cap \sigma_{\pi}(\rho(f)\tau(g)) \neq \emptyset
\]
for all \( f, g \in \Lambda \), then \( S(e_1)T(e_2) = 1 \) and there exists a homeomorphism \( \phi : \text{Ch}(B) \to \text{Ch}(A) \) such that
\[
(S(e_1)^{-1}S(f))(y) = \rho(f)(\phi(y)), \quad (T(e_2)^{-1}T(f))(y) = \tau(f)(\phi(y)) \quad (2.3)
\]
for every \( f \in \Lambda \) and \( y \in \text{Ch}(B) \). Moreover, there exist an algebra isomorphism \( \hat{S} : A \to B \) and a homeomorphism \( \Phi : M_B \to M_A \) satisfying
\[
\hat{S}(f) = \hat{f} \circ \Phi
\]
for every \( f \in A \),
\[
\hat{S}(\rho(f)) = S(e_1)^{-1}S(f) \quad \text{and} \quad \hat{S}(\tau(f)) = T(e_2)^{-1}T(f)
\]
for every \( f \in \Lambda \).
Applying Proposition 2.3 to \( \rho, \tau, S(e_1)^{-1}S \) and \( T(e_2)^{-1}T \), there exists a homeomorphism \( \phi : \text{Ch}(B) \to \text{Ch}(A) \) such that
\[
|(S(e_1)^{-1}S(f))(y)| = |\rho(f)(\phi(y))|,
\]
\[
|(T(e_2)^{-1}T(f))(y)| = |\tau(f)(\phi(y))|
\]
for every \( f \in \Lambda \) and \( y \in \text{Ch}(B) \).

We will prove that \( S(e_1)T(e_2) = 1 \) if there exists a first-countable dense subset \( D_B \) in \( \text{Ch}(B) \). Let \( y \in \text{Ch}(B) \). If \( y \in D_B \), then \( y \) is a peak point of \( B \). By Lemma 2.1, there exist \( T(u_1) \in P_B(y) \cap \exp B \) such that
\[
\sigma_\pi (S(e_1)T(u_1)) = \{S(e_1)(y)\},
\]
\[
(S(e_1)T(u_1))^{-1}(\{S(e_1)(y)\}) = T(u_1)^{-1}(\{1\}) = \{y\}
\]
and \( S(u_2) \in P_B(y) \cap \exp B \) such that
\[
\sigma_\pi (S(u_2)T(e_2)) = \{T(e_2)(y)\},
\]
\[
(S(u_2)T(e_2))^{-1}(\{T(e_2)(y)\}) = S(u_2)^{-1}(\{1\}) = \{y\}.
\]
(2.4)

Since
\[
\sigma_\pi (S(e_1)T(u_1)) \cap \sigma_\pi (\rho(e_1)\tau(u_1)) \neq \emptyset
\]
and
\[
\sigma_\pi (S(u_2)T(e_2)) \cap \sigma_\pi (\rho(u_2)\tau(e_2)) \neq \emptyset,
\]
there exist \( y_1, y_2 \in \text{Ch}(B) \) such that
\[
\tau(u_1)(\phi(y_1)) = S(e_1)(y) \quad \text{and} \quad \rho(u_2)(\phi(y_2)) = T(e_2)(y).
\]

Note that
\[
|(S(e_1)T(u_1))(y_1)| = |\tau(u_1)(\phi(y_1))| = |S(e_1)(y)|
\]
and
\[
|(S(u_2)T(e_2))(y_2)| = |\rho(u_2)(\phi(y_2))| = |T(e_2)(y)|.
\]

By (2.4) and (2.5), we obtain \( y_1 = y_2 = y \). Since
\[
\sigma_\pi (S(u_2)T(u_1)) \cap \sigma_\pi (\rho(u_2)\tau(u_1)) \neq \emptyset,
\]
there exists a \( y_3 \in \text{Ch}(B) \) such that \( (\rho(u_2)\tau(u_1))(\phi(y_3)) = 1 \). We also have
\[
|(S(u_2)T(u_1))(y_3)| = |(\rho(u_2)\tau(u_1))(\phi(y_3))| = 1.
\]
Equations (2.4) and (2.5) imply that $y_3 = y$. Hence we obtain

$$S(e_1)(y)T(e_2)(y) = \tau(u_1)(\phi(y))p(u_2)(\phi(y)) = 1.$$  

Consequently, $S(e_1)T(e_2) = 1$ on $D_B$. Since $D_B$ is dense in $\text{Ch}(B)$ and $\text{Ch}(B)$ is a boundary for $B$, we obtain $S(e_1)T(e_2) = 1$. Similar arguments show that the equation $S(e_1)T(e_2) = 1$ holds for the case where there exists a first-countable dense subset $D_A$ in $\text{Ch}(A)$.

If there exists a first-countable dense subset $D_B$ in $\text{Ch}(B)$, then by Proposition 2.3 we obtain (2.3) for every $f \in \Lambda$ and $y \in D_B$, that is for every $f \in \Lambda$ and $y \in \text{Ch}(B)$. Similar arguments imply (2.3) for the case where there exists a first-countable dense subset $D_A$ in $\text{Ch}(A)$. The existence of $\tilde{S} : A \rightarrow B$ and $\Phi : M_B \rightarrow M_A$ is follows by the arguments from the proof of Theorem 3.6 in [HMT06] (see also [MHS]).

Below we give examples of topological spaces that have first-countable dense subnets.

**Example 1** (cf. [S-S]).

(1) Let $[0, 1]$ be the unit interval and $p$ a fixed point of $[0, 1]$. Consider the topology on $[0, 1]$ consisting of open set $G$ such that:

(a) $G$ excludes $p$; or

(b) $G$ contains all but a finite number of the points of $[0, 1]$.

We call this space Fort space [S-S, Part II, 24]. The subset $[0, 1] \setminus \{p\}$ is first-countable and dense in the Fort space $[0, 1]$.

(2) For the square $[0, 1] \times [0, 1]$, we define the topology by taking as a neighborhood basis of all points $(a, b)$ off the diagonal $\Delta = \{(x, x) : x \in [0, 1]\}$ the intersection of $X - \Delta$ with an open vertical line segment centered at $(a, b) : N_{\varepsilon}(a, b) = \{(a, y) \in X - \Delta : |b - y| < \varepsilon\}$. Neighborhoods of points $(a, a) \in \Delta$ are defined by the intersection with $X$ of open horizontal strips less a finite number of vertical lines: $N_{\varepsilon}(a, a) = \{(x, y) \in X : |y - a| < \varepsilon, x \neq x_0, x_1, \ldots, x_n\}$. We call this space the Alexandroff square [S-S, Part II, 101]. The subset $[0, 1] \times [0, 1] \setminus \Delta$ is first-countable and dense in $[0, 1] \times [0, 1]$.

(3) Let $\beta \mathbb{N}$ be the Stone–Cech compactification of the set $\mathbb{N}$ of natural numbers. Then $\mathbb{N}$ is first-countable and dense in $\beta \mathbb{N}$. 
3. Applications

In this section we give some corollaries of Theorem 2.4.

**Corollary 3.1.** Suppose that the sets $A_0 \subset A$ and $B_0 \subset B$ are closed under multiplications and contain $\exp A$ and $\exp B$, respectively. Let $T$ be a surjection from $A_0$ onto $B_0$ and $m, n$ positive integers such that

$$\sigma_\pi(T(f)^m T(g)^n) \cap \sigma_\pi(f^m g^n) \neq \emptyset, \quad (3.1)$$

for all $f, g \in A_0$.

If there exists a first-countable dense subset $D_B$ in $\text{Ch}(B)$, or a first-countable dense subset $D_A$ in $\text{Ch}(A)$, then $T(1)^{m+n} = 1$ and there exist an algebra isomorphism $\tilde{\varphi} : A \to B$ and a homeomorphism $\Phi : M_B \to M_A$ satisfying $\tilde{\varphi} = \hat{f} \circ \Phi$ for every $f \in A_0$, where $d$ is the greatest common divisor of $m$ and $n$. If, in addition, $d = 1$ and $T(1) = 1$, then $T$ can be extended to an algebra isomorphism.

For the case where $m = n = 1$, $A_0 = A$ and $B_0 = B$, this result is proven in [S.Ni, Theorem 4.6]. If, moreover, $A = B = C(X)$, then it generalizes the result proven by Molnár [Mo].

**Proof.** By (3.1), we have

$$\|T(f)^m T(g)^n\|_\infty = \|f^m g^n\|_\infty$$

for all $f, g \in A_0$. In particular, $\|T(f)^{m+n}\|_\infty = \|f^{m+n}\|_\infty$, that is

$$\|T(f)\|_\infty = \|f\|_\infty, \|T(f)^m\|_\infty = \|f^m\|_\infty \quad \text{and} \quad \|T(f)^n\|_\infty = \|f^n\|_\infty$$

for every $f, g \in A_0$. Proposition 2.2 shows that there exists a homeomorphism $\phi : \text{Ch}(B) \to \text{Ch}(A)$ such that $|T(f)(g)| = |f(\phi(g))|$ for every $f \in A_0$ and $g \in \text{Ch}(B)$. In particular, $|T(1)| = 1$ on $\text{Ch}(B)$. By similar arguments to the second paragraph of the proof of Theorem 2.4, we obtain $T(1)^{m+n} = 1$, that is $T(1) \in B_0^{-1}$. According to Theorem 2.4 for $S(f) = T(f)^m$, $T(f) = T(f)^n$, $\rho(f) = f^m$, and $\tau(f) = f^n$, we obtain the conclusion. \hfill \Box

**Corollary 3.2.** Suppose that the sets $A_1 \subset A^{-1}$ and $B_1 \subset B^{-1}$ are closed under multiplications and contain $\exp A$ and $\exp B$, respectively. Let $T$ be a surjection from $A_1$ onto $B_1$ and $k, l$ are non-zero integers such that

$$\sigma_\pi(T(f)^k T(g)^l) \cap \sigma_\pi(f^k g^l) \neq \emptyset, \quad (3.2)$$
for all $f, g \in A_1$. If $T(1) \in B_0^{-1}$, $|T(1)| = 1$ on $Ch(B)$ and there exists a first-countable dense subset $D_B$ in $Ch(B)$, or a first-countable dense subset $D_A$ in $Ch(A)$, then $T(1)^{k+l} = 1$ and there exist an algebra isomorphism $\phi : A \to B$ and a homeomorphism $\Phi : M_B \to M_A$ satisfying $\phi(f) = f \circ \Phi$ for every $f \in A$ and $\Phi(f) = T(1)^{-d}T(f)^d$ for every $f \in A_1$, where $d$ is the greatest common divisor of $k$ and $l$. If, in addition, $d = 1$ and $T(1) = 1$, then $T$ can be extended to an algebra isomorphism.

This follows from Theorem 2.4 with $S(f) = T(f)^k, T(f) = T(f)^l, \rho(f) = f^k$, and $\tau(f) = f^l$.

Remark 3.1. Let $A_1, B_1, k$ and $l$ be as in Corollary 3.2. Suppose $T$ is a surjection from $A_1$ onto $B_1$ satisfying (3.2) for all $f, g \in A_1$. By (3.2), we have

$$\|T(f)^kT(g)^l\|_\infty = \inf\|f^kg^l\|_\infty$$

for all $f, g \in A_1$. If $kl > 0$, then $T(f)^{k+l} = f^{k+l}$, that is

$$\|T(f)^{k+l}\|_\infty = \|f^{k+l}\|_\infty$$

and

$$\|T(f)^k\|_\infty = \|f^k\|_\infty \quad \text{and} \quad \|T(f)^l\|_\infty = \|f^l\|_\infty$$

for every $f \in A_1$. Proposition 2.2 implies that there exists a homeomorphism $\phi : Ch(B) \to Ch(A)$ with $|T(f)(g)| = |f(\phi(g))|$ for every $f \in A_1$ and $g \in Ch(A)$. In particular, $T(1)|_B = 1$ on $Ch(B)$. If, in addition, there exists a first-countable dense subset $D_B \subset Ch(B)$ or $D_A \subset Ch(A)$, then, by similar arguments to the second paragraph of the proof of Theorem 2.4, we obtain $T(1)^{k+l} = 1$.

This shows that Corollary 3.2 holds in the case when $kl > 0$ or $k + l = 0$ without assuming that $|T(1)| = 1$ on $Ch(B)$.

Corollary 3.3. Let $T$ be a surjection from $A$ onto $B$ such that

$$\sigma_\pi(T(f) \exp T(g)) \cap \sigma_\pi(f \exp g) \neq \emptyset$$

(3.3)

for all $f, g \in A$. If $T(1) \in B^{-1}$, $|T(1) \exp T(0)| = 1$ on $Ch(B)$ and there exists a first-countable dense subset $D_B$ in $Ch(B)$, or a first-countable dense subset $D_A$ in $Ch(A)$, then $T(1) \exp T(0) = 1$ and there exists a homeomorphism $\Phi : M_B \to M_A$ such that

$$\Phi^{-1}(T(f) \exp T(0)) = \Phi$$

and

$$\Phi^{-1}(T(f) \exp T(0)) = \Phi$$

(3.4)

for every $f \in A$. Moreover, $T(0) = 0$ and $T(1) = 1$, hence $T$ is an algebra isomorphism.
Proof. According to Theorem 2.4 for $S(f) = T(f)$, $T(f) = \exp T(f)$, $\rho(f) = f$, and $\tau(f) = \exp f$, we obtain

$$T(1) \exp T(0) = 1$$

and equation (3.4). In particular, $\widehat{T}(1)^{-1} \widehat{T}(0) = 0$, that is $T(0) = 0$. Consequently, $T(1) = T(1) \exp T(0) = 1$. \hfill \Box

Acknowledgements. The author would like to thank the referee for their suggestions to improve the manuscript.

References


Weakly-peripherally multiplicative conditions


RUMI SHINDO
NSG ACADemy Co. LTD., 11-32 HIGASHIODORI 1-CHO-ME CHUO-KU, NIIGATA, 950-0087 JAPAN
E-mail: rumi_shindo@email.plala.or.jp

(Received June 14, 2010; revised January 2, 2011)