Diffeomorphic theorems for open Riemannian manifolds with curvature decay

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Abstract. In this paper, we study the topology of complete non-compact Riemannian manifolds with curvature decay to a non-positive constant. We show that such a complete open manifold $M$ is diffeomorphic to a Euclidean $n$-space $\mathbb{R}^n$ if it contains enough rays starting from the base point. As applications, we also show that this kind of manifolds with Ricci curvature bounded from below by a non-positive constant are diffeomorphic to $\mathbb{R}^n$ if the volumes of geodesic balls in $M$ grow properly. Our results generalize the main theorems of Wang–Xia for manifolds with quadratic curvature decay to zero.

1. Introduction

Let $(M, g)$ be a complete non-compact $n(\geq 2)$-dimensional Riemannian manifold. For a fixed point $p \in M$ and any $r > 0$, denote by $B(p, r)$ the open geodesic ball around $p$ with radius $r$ in $M$, and $S(p, r)$ the corresponding geodesic sphere.

Denote by $K_M$ the sectional curvature of $M$, and let

$$k_p(r) = \inf_{M \setminus B(p, r)} K_M,$$

where the infimum is taken over all the sections at all points on $M \setminus B(p, r)$. It is easy to see that we can choose $k_p(r)$ to be a non-positive monotone increasing function of $r$.

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In this paper, we consider complete open Riemannian manifolds with curvature decay to a non-positive constant, that is to say, there exists a positive monotone decreasing function $K(r)$ with $\lim_{r \to \infty} K(r) \geq 0$ satisfying

$$k_p(r) \geq - (K(r))^2, \quad \text{for all } r > 0.$$  

Let $R_p$ denote the (point set) union of rays issuing from $p$. One can show that $R_p$ is a closed subset of $M$. Define a function $h_p$ on $M$ by

$$h_p(x) = d(x, R_p),$$

where $d$ is the distance function on $M$. We set for $r > 0$ (cf. [8], [17], [21])

$$H(p, r) = \max_{x \in S(p, r)} d(x, R_p). \quad (1.1)$$

By definition, we always have

$$H(p, r) \leq \max_{x \in S(p, r)} d(x, p) = r, \quad \text{for all } r > 0.$$  

If $M$ is a complete simply connected Riemannian manifold with non-negative sectional curvature, then $H(p, x) \equiv 0$. This follows from the fact that any point in $M$ lies in some ray starting from $p$.

For a constant $c \geq 0$, we denote by $M^n(-c)$ an $n$-dimensional complete simply connected Riemannian manifold of constant curvature $-c$. If the Ricci curvature of $M$ satisfies $\text{Ric}_M \geq -(n-1)c$, the relative volume comparison theorem [4] tells us that the function $r \to \frac{\text{vol}[B(p,r)]}{\alpha_n(r, -c)}$ is monotone decreasing, where $\text{vol}[B(p,r)]$ is the volume of $B(p,r)$ and $\alpha_n(r, -c)$ the volume of a geodesic ball of radius $r$ in $M^n(-c)$. It is well known that

$$\alpha_n(r, -c) = \omega_{n-1} \int_0^r f_{-c}^{n-1}(t) dt,$$

where

$$f_{-c}(t) = \begin{cases} t, & c = 0, \\ \frac{\sinh(\sqrt{c}t)}{\sqrt{c}}, & c > 0, \end{cases} \quad (1.2)$$

and $\omega_m$ is the volume of $S^m(1)$.

For any $p \in M$, we set

$$v_{-c}(p) = \lim_{r \to \infty} \frac{\text{vol}[B(p, r)]}{\alpha_n(r, -c)}.$$
and define

\[ v_{-c} = \inf_{p \in M} v_{-c}(p). \]  

One always has

\[ \frac{\text{vol}[B(p,r)]}{\alpha_n(r,-c)} \geq v_{-c}(p) \geq v_{-c}, \quad \forall r > 0, \quad \forall p \in M. \]

Abresch [1] proved that if \( \int_0^{\infty} r k_p(r) dr > -\infty \), then \( M \) is of finite topological type. Xia [21] proved that if \( M \) is an \( n \)-dimensional complete open Riemannian manifold with nonnegative sectional curvature in which there exists a \( p \in M \) such that \( H(p,r) < r \), for all \( r > 0 \), then \( M \) is diffeomorphic to \( \mathbb{R}^n \). Wang–Xia [19] proved that there exists a constant \( \epsilon = \epsilon(n) > 0 \) such that an \( n \)-dimensional open manifold with quadratic curvature decay to zero and \( H(p,r) < \epsilon r \) for all \( r > 0 \) is diffeomorphic to \( \mathbb{R}^n \). For recent progress on manifolds with quadratic curvature decay, we refer to the paper of Yeganefar [24] for more details.

In this paper, we first obtain the following pinching theorem, which generalizes the result of Wang–Xia in [19]

**Theorem 1.1.** Given a positive monotone decreasing function \( K(r) \), there are positive constants \( \epsilon, \delta \in (0,1) \) with \( \epsilon + \delta < 1 \), such that any \( n \)-dimensional complete open manifold \( M \) satisfying

\[ k_p(r) \geq -(K(r))^2, \]

and

\[ H(p,r) < \epsilon r - \frac{1}{2K(\delta r)} \ln \left\{ \cosh^2(K(\delta r) \epsilon r) + \sqrt{\cosh^4(K(\delta r) \epsilon r) - 1} \right\}, \]

for some \( p \in M \) and all \( r > 0 \) is diffeomorphic to \( \mathbb{R}^n \).

Our second theorem is on a Riemannian manifold with large volume growth, i.e. \( v_{-c} > 0 \). There have been many articles studying complete noncompact Riemannian manifold with large volume growth (cf. [3], [7], [8], [11], [12], [14]–[23]). If \( M \) has nonnegative Ricci curvature, it has been proven by Li [12] and Anderson [3] that \( \pi_1(M) \) is finite. Perelman [15] has shown that there is a small constant \( \epsilon(n) > 0 \) depending only on \( n \) such that if \( \epsilon_0 > 1 - \epsilon(n) \), then \( M \) is contractible, and Cheeger–Colding [7] showed that the manifold in Perelman’s theorem is actually diffeomorphic to \( \mathbb{R}^n \). Shen [17] has shown that \( M \) has finite topological type, provided that \( \frac{\text{vol}[B(p,r)]}{\alpha_n(r,-c)} = v_0 + o\left(\frac{1}{r^n}\right) \) and, either the conjugate radius
conj\textsubscript{M} \geq c > 0 or the sectional curvature $K_{M} \geq K_{0} > -\infty$. Sha–Shen [16] proved that these manifolds have finite topological type if in addition the manifolds have quadratic curvature decay to zero. One can find some other topological uniqueness theorems about $M$, e.g. in [8], [14], [22].

Given a point $p \in M$, we denote by crit\textsubscript{p} the criticality radius of $M$ at $p$, i.e., crit\textsubscript{p} is the smallest critical value for the distance function $d(p, .) : M \to \mathbb{R}$. Recall that $q$ is not a critical point of this distance function iff there exists a vector $v \in S_{q}M$ such that for all minimizing geodesics $\sigma$ from $\sigma(0) = q$ to $p$, we have $\angle(v, \sigma'(0)) > \frac{\pi}{2}$ (cf. [6, 9]). In view of Theorem 1.1, we have the following topological rigidity theorem for manifolds with Ricci curvature bounded from below by a non-positive constant and large volume growth.

**Theorem 1.2.** Given a positive monotone decreasing function $K(r)$, there are positive constants $\epsilon, \delta \in (0, 1)$ with $\epsilon + \delta < 1$, such that any $n$-dimensional complete open manifold $M$ with $\text{Ric}_{M} \geq - (n-1)c$ ($c \geq 0$), $v_{-c} > 0$,

$$k_{p}(r) \geq -(K(r))^{2},$$

and

$$\frac{\text{vol}[B(p, r)]}{\omega_{n}r^{n}} < \left\{ 1 + \frac{\int_{0}^{A} f_{n-1}(t)dt}{\int_{0}^{\epsilon r} f_{n-1}(t)dt} \right\} v_{-c},$$

for some $p \in M$ and all $r > 0$ is diffeomorphic to $\mathbb{R}^{n}$, where

$$A = \epsilon r - \frac{1}{2K(\delta r)} \ln \left\{ \cosh^{2}(K(\delta r)\epsilon r) + \sqrt{\cosh^{4}(K(\delta r)\epsilon r) - 1} \right\}.$$

Especially, if $M$ is of nonnegative Ricci curvature, we have

**Theorem 1.3.** Given a positive monotone decreasing function $K(r)$, there are positive constants $\epsilon, \delta \in (0, 1)$ with $\epsilon + \delta < 1$, such that any $n$-dimensional complete open manifold $M$ with $\text{Ric}_{M} \geq 0$, $v_{0} > 0$,

$$k_{p}(r) \geq -(K(r))^{2},$$

and

$$\frac{\text{vol}[B(p, r)]}{\omega_{n}r^{n}} < \left\{ 1 + 2^{2-n}r^{1-n} \left[ \epsilon r - \frac{1}{2K(\delta r)} \ln \left( \cosh^{2}(K(\delta r)\epsilon r) \right. \right. \\

$$+ \sqrt{\cosh^{4}(K(\delta r)\epsilon r) - 1} \left. \right]^{n-1} \right\} v_{0},$$

for some $p \in M$ and all $r > 0$ is diffeomorphic to $\mathbb{R}^{n}$.
Remark 1.4. (i) If the curvature of $M$ is quadratic decay to zero, i.e., there exists a constant $C > 0$ such that $K(r)r \leq C$ for all $r > 0$, then we can choose $\epsilon$ in Theorem 1.1 small enough so that (1.4) becomes

$$H(p, r) < \tilde{\epsilon}r,$$

for some $\tilde{\epsilon} \in (0, 1)$ depending on $\epsilon$, $\delta$ and $C$. Then by Theorem 1.1, $M$ is diffeomorphic to $\mathbb{R}^n$, and this recovers Wang–Xia’s result in [19].

(ii) If the function $K(r) = Cr^{-\beta}$ for $C > 0$, $\beta \in [0, 1]$ and the Ricci curvature of $M$ is nonnegative, (1.6) can be written in an explicit form, therefore we again obtain a Wang–Xia’s type theorem in [19].

2. Preliminaries

Let $(M, g)$ be a complete non-compact $n$-dimensional Riemannian manifold. For a fixed point $p \in M$. We say that $K^\min_p \geq c$ if for any minimal geodesic $\gamma$ issuing from $p$ all sectional curvatures of planes which are tangent to $\gamma$ are greater than or equal to $c$. For $p, q \in M$, the excess function $e_{pq}(x)$ is defined by

$$e_{pq}(x) = d(p, x) + d(q, x) - d(p, q).$$

We denote by $M^2(c)$ the complete simply connected surface of constant curvature $c$. Throughout this paper, all geodesics are assumed to have unit speed. In [13], Machigashira proved the following Toponogov-type comparison theorem for complete manifolds with $K^\min_p \geq c$.

Lemma 2.1. Let $M$ be a complete $n$-manifold and $p$ be a point of $M$ with $K^\min_p \geq c$. Let $\gamma_i : [0, l_i] \to M$, $i = 0, 1, 2$ be minimal geodesics with $\gamma_1(0) = \gamma_2(l_2) = p$, $\gamma_0(0) = \gamma_1(l_1)$ and $\gamma_0(l_0) = \gamma_2(0)$. Then there exist minimal geodesics $\tilde{\gamma}_i : [0, l_i] \to M^2(c)$, $i = 0, 1, 2$ with $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(l_2)$, $\tilde{\gamma}_0(0) = \tilde{\gamma}_1(l_1)$ and $\tilde{\gamma}_0(l_0) = \tilde{\gamma}_2(0)$ which are such that

$$L(\gamma_i) = L(\tilde{\gamma}_i) \quad \text{for } i = 0, 1, 2,$$

and

$$\angle(-\gamma'_1(l_1), \gamma'_0(0)) \geq \angle(-\tilde{\gamma}'_1(l_1), \tilde{\gamma}'_0(0)),$$

$$\angle(-\gamma'_0(l_0), \gamma'_2(0)) \geq \angle(-\tilde{\gamma}'_0(l_1), \tilde{\gamma}'_2(0)).$$
Lemma 2.2 ([22]). Let $M$ be a complete $n$-manifold with $\text{Ric}_M \geq 0$ and $v_0 > 0$. Then for any $r > 0$ and any $x \in S(p, r)$, we have
\[
d(x, R_p) \leq 2v_0^{-\frac{1}{n}} \left\{ \frac{\text{vol}[B(p, r)]}{\omega_n r^n} - v_0 \right\}^\frac{1}{2} r.
\]

Lemma 2.3 ([2]). Let $(M, g)$ be a complete $n$-manifold with $\text{Ric}_M \geq 0$. Let $\gamma : [0, a] \to M$ be a minimal geodesic from $p$ to $q$. Then for any $x \in M$
\[
\epsilon_{pq}(x) \leq 8 \left( \frac{s^n}{r} \right)^{\frac{1}{n+1}},
\]
where $s = d(x, \gamma)$ and $r = \min(d(p, x), d(q, x))$.

Let $\Sigma$ be a closed subset of the unit tangent sphere $S_p M$ of $M$ at $p$. Denote by $B_{\Sigma}(p, r)$ the set of points $x \in B(p, r)$ such that there is a minimizing geodesic $\gamma$ from $p$ to $x$ with $\gamma'(0) \in \Sigma$. For $0 < r \leq \infty$, let $\Sigma_p(r)$ denote the set of unit vectors $v \in \Sigma$ such that the geodesic $\gamma(t) = \exp_p(tv)$ is minimizing on $[0, r)$.

Notice that $\Sigma_p(r_2) \subset \Sigma_p(r_1)$, for $0 < r_1 < r_2$; $\Sigma_p(\infty) = \bigcap_{r>0} \Sigma_p(r)$.

The standard argument [4, 5] gives the following generalized Bishop’s comparison theorem.

Lemma 2.4 ([22]). Let $(M, g)$ be a complete $n$-manifold with $\text{Ric}_M \geq 0$ and $v_0 > 0$. Then
\[
\frac{\text{vol}[B_{\Sigma_p(\infty)}(p, r)]}{\alpha_n(r, 0)} \geq v_0.
\]

Lemma 2.5 ([23]). Let $(M, g)$ be a complete $n$-manifold with $\text{Ric}_M \geq -(n-1)$ and $v_{-1} > 0$. Then
\[
\frac{\text{vol}[B_{\Sigma_p(\infty)}(p, r)]}{\alpha_n(r, -1)} \geq v_{-1}.
\]

It is not difficult to check that Lemma 2.5 also holds for $\text{Ric}_M \geq -(n-1)c$ ($c > 0$). We then have the following corollary.

Corollary 2.6. Let $(M, g)$ be a complete $n$-manifold with $\text{Ric}_M \geq -(n-1)c$, $c \geq 0$ and $v_{-c} > 0$. Then for any $p \in M$ and any $r > 0$, we have
\[
\int_0^{\mathcal{H}(p, r)} f_{-c}^{n-1}(t) dt \leq v_{-c}^{-1} \int_0^{2r} f_{-c}^{n-1}(t) dt \left\{ \frac{\text{vol}[B(p, r)]}{\alpha_n(r, -c)} - v_{-c} \right\},
\]
where $f_{-c}(t)$ is defined in (1.2), and $v_{-c}$ is defined in (1.3).
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Proof. Fix a point \( x \in S(p, r) \), and set \( s = d(x, R_p) \), then \( s \leq r \) and
\[
B(p, r) \bigcup B_{S_p(\infty)}(p, 2r) \subset B(p, 2r).
\]
The left hand side is a disjoint union. We have
\[
\text{vol}[B(p, r)] \geq v_{-c}\{\alpha_{n}(s, -c)\}.
\]
From Lemma 2.4 and 2.5, one obtains
\[
\text{vol}[B(p, 2r)] \geq \text{vol}[B(x, s)] + \text{vol}[B_{S_p(\infty)}(p, 2r)] \\
\geq v_{-c}\alpha_{n}(s, -c) + v_{-c}\alpha_{n}(2r, -c).
\]
By Bishop’s comparison theorem, we have
\[
\text{vol}[B(p, 2r)] \alpha_{n}(2r, -c) \leq \text{vol}[B(p, r)] \alpha_{n}(r, -c).
\]
We then obtain
\[
\int_{0}^{H(p, r)} f_{-c}^{n-1}(t)dt \leq v_{-c}^{-1} \int_{0}^{2r} f_{-c}^{n-1}(t)dt \left\{ \frac{\text{vol}[B(p, r)]}{\alpha_{n}(r, -c)} - v_{-c}\right\}.
\]
□

3. Proofs of Theorems

Proof of Theorem 1.1. We shall prove that \( M \) contains no critical points of the distance function \( d(p, \cdot) \) other than \( p \), and therefore it is diffeomorphic to \( \mathbb{R}^n \) (cf. [9], Disk Theorem). We refer to [6], [9], [10] for the notion of critical points of the distance functions and their applications.

For any \( \epsilon, \delta \in (0, 1) \) with \( \epsilon + \delta < 1 \) and any \( r > 0 \), we see
\[
cosh(2K(\delta r)er) - \cosh^2(K(\delta r)er) = \cosh^2(K(\delta r)er) - 1 > 0. \tag{3.1}
\]
By definition, \( \cosh(2K(\delta r)er) = \frac{1}{2}(e^{2K(\delta r)er} + e^{-2K(\delta r)er}) \), we have from (3.1)
\[
(e^{2K(\delta r)er})^2 - 2\cosh^2(K(\delta r)er)e^{2K(\delta r)er} + 1 > 0,
\]
which implies
\[
er - \frac{1}{2K(\delta r)} \ln \left\{ \cosh^2(K(\delta r)er) + \sqrt{\cosh^4(K(\delta r)er) - 1} \right\} > 0. \tag{3.2}
\]
Take an arbitrary point \( x(\neq p) \in M \) and set \( r = d(p, x) \). By (1.1), \( \mathcal{H}(p, r) \) must be nonnegative. By (3.2), we see that our condition (1.4) is reasonable, and this enables us to find a ray \( \gamma : [0, +\infty) \to M \) such that

\[
s = d(x, \gamma) < \epsilon r - \frac{1}{2K(\delta r)} \ln \left\{ \cosh^{2}(K(\delta r)\epsilon r) + \sqrt{\cosh^{4}(K(\delta r)\epsilon r) - 1} \right\}. \tag{3.3}
\]

Fix a minimizing geodesic \( \sigma \) from \( x \) to \( q = \gamma(2r) \). For any minimal geodesic \( \sigma_1 \) from \( x \) to \( p \), let \( \tilde{p} = \sigma_1(\epsilon r) \) and \( \tilde{q} = \sigma(\epsilon r) \). The choice of \( \epsilon \) and \( \delta \) indicates that \( \sigma|_{[0, \epsilon r]} \) and \( \sigma_1|_{[0, \epsilon r]} \) are disjoint with \( B(p, \delta r) \). Moreover, the sectional curvature of \( M \) satisfies \( K_M \geq -(K(\delta r))^2 \) on \( M \setminus B(p, \delta r) \). Applying the Toponogov comparison theorem Lemma 2.1 to the hinge \( (\sigma|_{[0, \epsilon r]}, \sigma_1|_{[0, \epsilon r]}) \) in \( M \setminus B(p, \delta r) \), we have

\[
\cos \theta \sinh^{2}(K(\delta r)\epsilon r) \leq \cosh^{2}(K(\delta r)\epsilon r) - \cosh(K(\delta r)d(\tilde{p}, \tilde{q})), \tag{3.4}
\]

where \( \theta = \angle(\sigma'(0), \sigma_1'(0)) \) is the angle of \( \sigma \) and \( \sigma_1 \) at \( x \).

Let \( m \in \gamma \) be such that \( d(x, m) = d(x, \gamma) \), then \( m \in \gamma(0, 2r) \). It follows from the triangle inequality that

\[
d(\tilde{p}, \tilde{q}) \geq d(p, q) - d(p, \tilde{p}) - d(q, \tilde{q})
\]

\[
= d(p, m) + d(q, m) - [d(p, x) - d(\tilde{p}, x)] - [d(x, q) - d(x, \tilde{q})]
\]

\[
= 2\epsilon r + [d(p, m) - d(p, x)] + [d(q, m) - d(q, x)] \geq 2\epsilon r - 2d(x, m).
\]

Introducing (3.3) into the above inequality we see that

\[
d(\tilde{p}, \tilde{q}) > \frac{1}{K(\delta r)} \ln \left\{ \cosh^{2}(K\epsilon r) + \sqrt{\cosh^{4}(K\epsilon r) - 1} \right\}. \tag{3.5}
\]

This implies that

\[
\cosh^{2}(K(\delta r)\epsilon r) - \cosh(K(\delta r)d(\tilde{p}, \tilde{q})) < 0. \tag{3.6}
\]

From (3.4) and (3.6), we obtain

\[
\cos \theta \sinh^{2}(K(\delta r)\epsilon r) \leq \cosh^{2}(K(\delta r)\epsilon r) - \cosh(K(\delta r)d(\tilde{p}, \tilde{q})) < 0.
\]

Thus

\[
\theta > \frac{\pi}{2}.
\]

Therefore any minimizing geodesic \( \sigma_1 \), from \( x \) to \( p \) has \( \angle(\sigma_1'(0), \sigma'(0)) > \frac{\pi}{2} \), which implies that \( x \) is not a critical point of \( d(p, \cdot) \). Theorem 1.1 follows. \( \square \)
Proof of theorem 1.2. We take the constants $\epsilon$ and $\delta$ in Theorem 1.2 to be the same as in Theorem 1.1. Therefore in order to prove Theorem 1.2, it suffices to show that

$$H(p, r) < \epsilon r - \frac{1}{2K(\delta r)} \ln \left\{ \cosh^2(K(\delta r)er) + \sqrt{\cosh^4(K(\delta r)er) - 1} \right\}. \tag{1.5}$$

Since $\text{Ric}_M \geq -(n-1)c$, where $c \geq 0$, we have by Corollary 2.6

$$\int_0^{H(p, r)} f_{-c}^{n-1}(t) dt \leq v_{-c}^{-1} \int_0^{2r} f_{-c}^{n-1}(t) dt \left\{ \frac{\text{vol}[B(p, r)]}{\alpha_n(r, -c)} - v_{-c} \right\}. \tag{2.6}$$

Substituting the assumption (1.5) into the above inequality, we have

$$\int_0^{H(p, r)} f_{-c}^{n-1}(t) dt < \int_0^{A} f_{-c}^{n-1}(t) dt.$$

Thus

$$H(p, r) < \epsilon r - \frac{1}{2K(\delta r)} \ln \left\{ \cosh^2(K(\delta r)er) + \sqrt{\cosh^4(K(\delta r)er) - 1} \right\}. \tag{3.7}$$

This completes the proof of Theorem 1.2. \hfill \Box

Before proving Theorem 1.3, we need the following lemma.

Lemma 3.1. Given a positive monotone decreasing function $K(r)$, there are positive constants $\epsilon, \delta \in (0, 1)$ with $\epsilon + \delta < 1$, such that any $n$-dimensional complete open manifold $M$ with $\text{Ric}_M \geq 0$,

$$k_p(r) \geq -(K(r))^2,$$

and

$$H(p, r) < r^\frac{1}{2} \left[ \frac{1}{4} \epsilon r - \frac{1}{8K(\delta r)} \ln \left( \cosh^2(K(\delta r)er) + \sqrt{\cosh^4(K(\delta r)er) - 1} \right) \right]^{n-1},$$

for some $p \in M$ and all $r > 0$ is diffeomorphic to $\mathbb{R}^n$.

Proof. We take the constants $\epsilon$ and $\delta$ as same in Theorem 1.1. Fix any point $q \in M$ and set $r = d(p, q)$. We only need to show that $q$ is not a critical point of $d(p, \cdot)$.

Take a point $m \in R_p$ so that $d(q, m) = d(q, R_p)$. Set $s = d(q, m)$. It then follows from the assumption of Lemma 3.1 that

$$s < r^\frac{1}{2} \left[ \frac{1}{4} \epsilon r - \frac{1}{8K(\delta r)} \ln \left( \cosh^2(K(\delta r)er) + \sqrt{\cosh^4(K(\delta r)er) - 1} \right) \right]^{n-1}. \tag{3.7}$$
Take a ray $\gamma : [0, +\infty) \to M$ starting from $p$ and passing through $m$. It follows from the triangle inequality that $\min(d(p, q), d(\gamma(t), q)) = r$ for all $t \geq 2r$. Thus $m \in \gamma(0, 2r)$ and so $d(\gamma(t), q) = r$ for all $t \geq 2r$.

Thus $m \in \gamma(0, 2r)$ and so $d(q, \gamma|_{[0, 2r]}) = s$. By Lemma 2.3 and (3.7) we have

$$e_{p, \gamma(2r)}(q) < 2er - \frac{1}{K(\delta r)} \ln \left( \cosh^2(K(\delta r)er) + \sqrt{\cosh^4(K(\delta r)er) - 1} \right).$$  (3.8)

Set $z = \gamma(2r)$ and take a minimal geodesic $\tilde{\sigma}$ from $q$ to $z$. For any minimal geodesic $\tilde{\sigma}_1$ from $q$ to $p$, let $p' = \tilde{\sigma}_1(\epsilon r)$ and $z' = \tilde{\sigma}(\epsilon r)$, and set $\tilde{\theta} = \angle(\tilde{\sigma}'(0), \tilde{\sigma}_1'(0))$. Since $K_M \geq -(K(\delta r))^2$ on $M \setminus B(p, \delta r)$ we can apply the Toponogov comparison theorem to the hinge $\{\tilde{\sigma}|_{[0, \epsilon r]}, \tilde{\sigma}_1|_{[0, \epsilon r]}\}$ in $M \setminus B(p, \delta r)$ to get

$$\cos \tilde{\theta} \sinh^2(K(\delta r)er) \leq \cosh^2(K(\delta r)er) - \cosh(K(\delta r)d(p', z')).$$  (3.9)

It follows from the triangle inequality that

$$d(p', z') \geq -d(p, p') - d(z, z') + d(p, z)$$

$$= -d(p, q) + d(p', q) - d(q, z) + d(z', q) + d(p, z) = 2er - e_{p, z}(q).$$

Inserting (3.8) into the above inequality and noticing (3.9), one obtains

$$\cos \tilde{\theta} \sinh^2(K(\delta r)er) \leq \cosh^2(K(\delta r)er) - \cosh(K(\delta r)(2er - e_{p, z}(q))) < 0.$$  (3.10)

Thus $\tilde{\theta} > \frac{\pi}{2}$. Consequently, $q$ is not a critical point of $d(p, \cdot)$. Lemma 3.1 follows.

**Proof of Theorem 1.3.** We take the constants $\epsilon$ and $\delta$ to be the same as in Lemma 3.1. In order to prove Theorem 1.3, it suffices to show that

$$\mathcal{H}(p, r) < r^\frac{1}{2} \left[ \frac{\sqrt{\frac{3}{4}r - \frac{1}{8K(\delta r)}} \ln \left( \cosh^2(K(\delta r)er) + \sqrt{\cosh^4(K(\delta r)er) - 1} \right)}{\omega_n r^{n-1}} \right].$$  (3.10)

By Lemma 2.2

$$d(x, R_p) \leq 2\omega_0^{-\frac{1}{n}} \left( \frac{\text{vol}[B(p, r)]}{\omega_n r^n} - \omega_0 \right)^{\frac{n}{2}} r.$$  (3.11)

By the definition (1.1), it then follows from the assumption (1.6) of Theorem 1.3 that (3.10) holds, and therefore we complete the proof of the theorem.
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