Weighted composition operators on Dirichlet-type spaces and related $Q_p$ spaces

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Abstract. We generalize Gallardo-Gutiérrez and Partington’s results on BMOA to the Dirichlet-type spaces with similar strategy. That is, we use the generalized Nevanlinna counting function associated to the weight function to characterize the boundedness and compactness of weighted composition operators on the Dirichlet-type spaces.

1. Introduction

Let $D$ denote the open unit disk of the complex plane $\mathbb{C}$. For $p \in (0, \infty)$, the Dirichlet-type space $D_p$ is the Hilbert space of holomorphic functions on $D$ for which the norm

$$
\|f\|_{D_p} = |f(0)| + \left( \int_{D} |f'(z)|^2 (1 - |z|^2)^p dA(z) \right)^{1/2} < \infty,
$$

where $dA(z) = \frac{1}{\pi} dx dy$ is the normalized Lebesgue measure on the unit disk. The inner product of $f$ and $g$ in $D_p$ is given by

$$
\langle f, g \rangle = |f(0)g(0)| + \left( \int_{D} f'(z)\overline{g'(z)}(1 - |z|^2)^p dA(z) \right)^{1/2} < \infty.
$$

Mathematics Subject Classification: 47B38, 47B33, 46E15, 32A37.

Key words and phrases: Dirichlet space, weighted composition operators, $Q_p$ space.

The second author is supported by NBHM (DAE) Post-Doctoral Fellowship grant no. 2/40(32)/2009-R&D-II/1337.

The third author is the corresponding author. Supported in part by the National Natural Science Foundation of China (Grant Nos.10971153, 10671141).
\[ D_p \] is a reproducing kernel Hilbert space with the kernel function \( K_w(z) = K(z, w) = (1 - \overline{w}z)^{-p} \). The space \( D_p \) has been extensively studied in number of papers, e.g., [13], [14], [17], [18]. It is well known that when \( p = 1 \), \( D_p \) is the classic Hardy space \( H^2 \) and when \( p > 1 \), \( D_p \) is the weighted Bergman space with weight \((1 - |z|^2)^{p-2}\). In particular, \( D_2 \) is the classic Bergman space \( L^2 \). See [17], for example.

The space of Möbius bounded functions in \( D_p \) is written as \( Q_p \). That is, \( Q_p \) is the Banach space of function \( f \in D_p \) with the norm
\[
\|f\|_{Q_p} = |f(0)| + \sup_{w \in D} \|f \circ \varphi_w - f\|_{D_p} < \infty,
\]
where \( \varphi_w(z) = (w - z)/(1 - \overline{w}z) \). For different \( p \in (0, \infty) \), \( Q_{p_1} \subset Q_{p_2} \) when \( 0 < p_1 < p_2 \leq 1 \). In particular, \( Q_1 = BMOA \), the bounded mean oscillation space of analytic functions; and when \( p > 1 \), \( Q_p = B \), the Bloch space on \( D \). See [21], [22], and the reference therein.

For \( \varphi \), a non-constant holomorphic map of the unit disk into itself, the composition operator \( C_\varphi \) with the symbol \( \varphi \) is defined by
\[
C_\varphi(f) = f \circ \varphi,
\]
where \( f \) is a holomorphic function in \( D \). It is well known that \( C_\varphi \) on \( D_p \) is always bounded when \( p \geq 1 \) and may be unbounded when \( 0 < p < 1 \). For \( u \) holomorphic on \( D \), the weighted composition operator \( uC_\varphi \) is defined by \( uC_\varphi(f) = u \cdot f \circ \varphi \). Apparently, if \( u = 1 \), the weighted composition operator \( uC_\varphi \) becomes \( C_\varphi \); and if \( \varphi(z) = z \) for \( z \in D \), \( uC_\varphi \) becomes the multiplier \( M_u \).

It is a long story to characterize the properties of (weighted) composition operator on several function spaces. The boundedness, compactness, Schatten class property are usually related to the pullback Carleson measure, Nevanlinna counting function and generalized Berezin transform, see [3], [4], [12], [15], [16].

Recently, Gallardo-Gutiérrez and Partington characterized the boundedness and compactness of weighted composition operators on Hardy and weighted Bergman spaces in [6] and [7]. Their characterization involved a condition related to a Nevanlinna counting function associated to the symbols \( \varphi \) and the weight \( u \). Their method is based on the analyzing the weighted composition operator acting the normalized kernel function. It is shown that the spaces BMOA and VMOA (resp. the Bloch space and little Bloch space) play key roles in the boundedness and compactness of weighted composition operators on the Hardy spaces (reps. the weighted Bergman spaces).

This manuscript is a generalization of Gallardo-Gutiérrez and Partington’s work [6]. Consider the Hardy spaces and weighted Bergman spaces as
the Dirichlet-type spaces $D_p$, [6] is about the case $p \geq 1$. We characterize the case for all $p \in (0, \infty)$. The idea is the same as that of [6]. We show that the $Q_p$ plays the role on the boundedness of weighted composition operators on $D_p$ as $BMOA$ related to the Hardy space.

Notation: Throughout the paper, we will denote $a \approx b$ whenever there exist two positive universal constants $c$ and $C$, such that $cb \leq a \leq Cb$. Further, for the sake of simplicity, $C$ will always denote an independent constant, which can be different from one display to another.

2. The main result

Let $T$ denote the boundary of $D$. Recall that a Carleson disk in $D$ centered at $\zeta \in T$ of radius $r \in (0,1)$ is given by $D(\zeta, r) = \{ z \in D : |z - \zeta| < r \}$. It is easy to check that $A(D(\zeta, r)) \approx r^2$. For $p \in (0, \infty)$, a positive Borel measure $\mu$ on $D$ is called a $p$-Carleson measure, if

$$\|\mu\|^p_p = \sup_{\zeta \in T} \frac{\mu(D(\zeta, r))}{r^p} < \infty.$$ 

When $p = 1$, we get the standard definition of the original Carleson measure.

The main result of this manuscript is the following theorem.

**Theorem 1.** Let $\varphi$ be an analytic self-map of $D$. For $p \in (0, \infty)$, suppose $u \in Q_p$. Then $uC_{\varphi}$ is bounded on $D_p$ if and only if

$$\sup_{\zeta \in T} \int_{D(\zeta, r)} \left( \sum_{\varphi(v) = v} |u(z)|^2 (1 - |z|)^p \right) dA(v) = O(r^{2+p}). \quad (1)$$

The case for $p \geq 1$, has been proved in [6], so we focus on the case for $p \in (0,1)$. To give the proof we need some preliminary results. The following lemmas are quoted from [21] or [2].

**Lemma 1.** Let $p \in (0, \infty)$ and let $\mu$ be a positive Borel measure on $D$. Then $\mu$ is a $p$-Carleson measure if and only if

$$\sup_{w \in D} \int_D \left( \frac{1 - |w|}{|1 - z\bar{w}|^2} \right)^p d\mu(z) < \infty.$$

**Lemma 2.** Let $p \in (0, \infty)$ and $f$ holomorphic on $D$ with $d_{\mu f, p}(z) = |f'(z)|^2 \times (1 - |z|^p) dA(z)$. Then $f \in Q_p$ if and only if $\mu f_p$ is a $p$-Carleson measure.
Proof of Theorem 1. We only consider \( p \in (0,1) \). Firstly fix a Carleson disk \( D(\zeta_0, r_0) \) in \( D \), centered at \( \zeta_0 \in T \) of radius \( r_0 \). Consider \( u_0 = (1 - r_0)\zeta_0 \). Then, if \( k_{u_0} \) denotes the normalized reproducing kernel in \( D_p \) at \( u_0 \in D \), that is,

\[
k_{u_0}(z) = \frac{K_{u_0}(z)}{\|K_{u_0}\|_{D_p}} = \frac{(1 - |u_0|^2)p/2}{(1 - u_0z)^p}.
\]

For all \( w \in D \), it is well known that \( \{k_w\} \) spans a dense subset of \( D_p \). Consider \( uC_p \) acting on \( k_{u_0}(z) \), we have

\[
\|uC_p(k_{u_0})\|_{D_p}^2 = |u(0)|^2|k_{u_0}(\varphi(0))|^2 + \int_D |(u(k_{u_0} \circ \varphi))(z)|^2(1 - |z|^2)^pdA(z).
\]

It follows that

\[
|(u(k_{u_0} \circ \varphi))'(z)|^2 = |u'(z)k_{u_0}(\varphi(z)) + u(z)k'_{u_0}(\varphi(z))\varphi'(z)|^2
\]

\[
= |u'(z)k_{u_0}(\varphi(z))|^2 + |u(z)k'_{u_0}(\varphi(z))\varphi'(z)|^2 + 2\Re u'(z)k_{u_0}(\varphi(z))u(z)k'_{u_0}(\varphi(z))\varphi'(z) = T_1 + T_2 + T_3.
\]

If we write \( I_j = \int_D T_j(1 - |z|^2)^pdA(z) \) for \( j = 1, 2, 3 \), the Cauchy–Schwarz inequality implies that \( I_2^2 \leq 4I_1I_2 \). We have

\[
I_1 = \int_D |u'(z)k_{u_0}(\varphi(z))|^2(1 - |z|^2)^pdA(z)
\]

\[
= \int_D \left| \frac{(1 - |u_0|^2)p/2}{(1 - u_0\varphi(z))^p} \right|^2 |u'(z)|^2(1 - |z|^2)^pdA(z)
\]

and the Area Formula of change variables gives that

\[
I_2 = \int_D |u(z)k'_{u_0}(\varphi(z))\varphi'(z)|^2(1 - |z|^2)^pdA(z)
\]

\[
= \int_D |k'_{u_0}(v)|^2 \left( \sum_{\varphi(z) = v} |u(z)|^2(1 - |z|^2)^p \right) dA(z).
\]

Now, since \( u \in Q_p \), it follows from Lemma 2 that \( |u'(z)|^2(1 - |z|^2)^pdA(z) \) is a \( p \)-Carleson measure. So

\[
I_1 \leq C \int_D \left| \frac{(1 - |u_0|^2)p/2}{(1 - u_0\varphi(z))^p} \right|^2 dA(z) = C\|C_p(k_{u_0})\|_{L^2_p}^2.
\]
Because $k_{w_0} \in \mathcal{D}_p \subset L^2_a$ is bounded and $C_{\varphi}$ is bounded on $L^2_a$, we conclude that $I_1$ is bounded.

If further (1) holds, denote

$$\Delta_n(w_0) = \{ v \in D : 2^n r_0 \leq |\zeta_0 - v| < 2^{n+1} r_0 \}$$

for $n \geq 1$ and denote $\Delta_0(w_0) = D(\zeta_0, r_0)$. Notice that for any $v \in \Delta_n(w_0)$ one has $|1 - \overline{w} v| \approx 2^n r_0$. We can get

$$I_2 = \int_{\Delta_n(w_0)} |k'_{w_0}(v)|^2 \left( \sum_{\varphi(z)=v} |u(z)|^2 (1 - |z|^2)^p \right) dA(v)$$

$$= \int_{\Delta_0(w_0)} |k'_{w_0}(v)|^2 \left( \sum_{\varphi(z)=v} |u(z)|^2 (1 - |z|^2)^p \right) dA(v)$$

$$= \sum_{n=0}^{\infty} \int_{\Delta_n(w_0)} |k'_{w_0}(v)|^2 \left( \sum_{\varphi(z)=v} |u(z)|^2 (1 - |z|^2)^p \right) dA(v)$$

$$\approx \sum_{n=0}^{\infty} \frac{(1 - |w|^2)^p}{(2^n r_0)^{2+2p}} \int_{\Delta_0(w_0)} \left( \sum_{\varphi(z)=v} |u(z)|^2 (1 - |z|^2)^p \right) dA(v).$$

Since $1 - |w|^2 \approx r_0$, $\Delta_n(w_0) \subset D(\zeta_0, 2^n r_0)$ and $A(D(\zeta_0, 2^n r_0)) \approx (2^n r_0)^2$. Combine these to (1) we have that

$$\int_{\Delta_n(w_0)} \left( \sum_{\varphi(z)=v} |u(z)|^2 (1 - |z|^2)^p \right) dA(v) \leq C (2^n r_0)^{2+p},$$

and thus

$$I_2 \leq C \sum_{n=0}^{\infty} \frac{1}{2^{np}}.$$

The above series converges when $p > 0$. Since $k_{w_0}$ spans a dense set of $\mathcal{D}_p$, we conclude that if (1) holds, then $\overline{u} C_{\varphi}$ is bounded.

Now suppose $u C_{\varphi}$ is bounded, then $I_1$ and $I_1 + I_2$ are bounded. Therefore,

$$I_2 = \int_{D(\zeta_0, r_0)} |k'_{w_0}(v)|^2 \left( \sum_{\varphi(z)=v} |u(z)|^2 (1 - |z|^2)^p \right) dA(v),$$

is bounded. So

$$\int_{D(\zeta_0, r_0)} |k'_{w_0}(v)|^2 \left( \sum_{\varphi(z)=v} |u(z)|^2 (1 - |z|^2)^p \right) dA(v)$$

$$= \int_{D(\zeta_0, r_0)} \frac{(1 - |w|^2)^p}{(1 - \overline{w} v)^{2+2p}} \left( \sum_{\varphi(z)=v} |u(z)|^2 (1 - |z|^2)^p \right) dA(v) \leq C.$$
For \( v \in D(\zeta_0, r_0) \), using the fact \(|1 - \overline{w_0}v| \approx 1 - |w|^2 \approx r_0\) again, we have

\[
\sup_{\zeta \in T} \int_{D(\zeta, r)} \left( \sum_{\varphi(z)=v} |u(z)|^2 (1 - |z|)^p \right) dA(v) = O(r^{2+p}).
\]

This means that we can get (1) from the boundedness of \( uC_\varphi \). \( \square \)

Let \( u = 1 \), we have the following corollary on the boundedness of composition operators on \( D_p \)

**Corollary 1.** Let \( \varphi \) be an analytic self-map of \( D \) and \( p \in (0, \infty) \). Then \( C_\varphi \) is bounded on \( D_p \) if and only if

\[
\sup_{\zeta \in T} \int_{D(\zeta, r)} \left( \sum_{\varphi(z)=v} (1 - |z|)^p \right) dA(v) = O(r^{2+p}).
\]

If we let \( \varphi(z) = z \) for all \( z \in D \), then the following corollary follows immediately.

**Corollary 2.** Let \( u \in D_p \). For \( p \in (0, \infty) \), consider the following three properties that \( u \) may possess:

(a) \( u \in Q_p \);
(b) \( M_u \) is bounded on \( D_p \);
(c) \[
\sup_{\zeta \in T} \int_{D(\zeta, r)} |u(z)|^2 (1 - |z|)^p dA(z) = O(r^{2+p}).
\]

Then any two of these properties implies the third.

**Proof.** Theorem 1 has shown that (b) and (c) are equivalent under the assumption (a). We need to prove that (b) and (c) imply (a). Indeed, according to [17], \( u \) is a multiplier on \( D_p \) if and only if \( u \) is bounded and \( |u'(z)|^2 (1 - |z|)^p dA(z) \) is a \( p \)-Carleson measure. It follows from Lemma 2 that \( u \in Q_p \). This completes the proof. \( \square \)

If \( Q_{p,0} \) stands for the space of all functions \( f \in Q_p \) with

\[
\lim_{|u| \to 1} \int_D |f'(z)|^2 (1 - |\varphi_u(z)|^2)^p dA(z) = 0,
\]

then \( Q_{p,0} \) is a closed subspace of \( Q_p \). Since \( k_w \) converges to 0 weakly in \( D_p \), we have the little-\( o \) version of Theorem 1.
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**Theorem 2.** Let $\varphi$ be an analytic self-map of $D$. For $p \in (0, \infty)$, suppose $u \in Q_{p,0}$. Then $uC_\varphi$ is compact on $D_p$ if and only if

$$r^{-2} \sup_{\zeta \in T} \int_{D(\zeta, r)} \left( \sum_{\varphi(z) = v} |u(z)|^2 (1 - |z|)^p \right) dA(v) = o(r^p) \quad \text{as } r \to 0.$$ 

**Corollary 3.** Let $\varphi$ be an analytic self-map of $D$ and $p \in (0, \infty)$. Then $C_\varphi$ is compact on $D_p$ if and only if

$$r^{-2} \sup_{\zeta \in T} \int_{D(\zeta, r)} \left( \sum_{\varphi(z) = v} (1 - |z|)^p \right) dA(v) = o(r^p) \quad \text{as } r \to 0.$$ 

**Corollary 4.** Let $u \in D_p$. For $p \in (0, \infty)$, consider the following three properties that $u$ may possess:

(a) $u \in Q_{p,0}$;
(b) $M_u$ is compact on $D_p$;
(c) $r^{-2} \sup_{\zeta \in T} \int_{D(\zeta, r)} |u(z)|^2 (1 - |z|)^p dA(z) = o(r^p) \quad \text{as } r \to 0.$

Then any two of these properties implies the third.

### 3. Weighted composition operators on $D_q^p$ spaces

For $0 < q < \infty$ and $-1 < p < \infty$, the spaces of Dirichlet type $D_q^p$ consist of those functions $f$ holomorphic on $D$ such that

$$\|f\|_{D_q^p} = \left( |f(0)|^q + \int_D |f'(z)|^q (1 - |z|^2)^p dA(z) \right)^{1/q} < \infty.$$ 

Obviously, $D_q^p$ is a generalization of $D_p$ and in particular, when $q = 2$, $D_q^p$ is $D_p$. For more details about $D_q^p$, see [1], [19], [20]. Further, $D_{q_1}^p \subseteq D_{q_2}^p$, if $1 \leq q_2 < q_1$. For detail about Dirichlet type spaces one can refer to [8], [9], [10] and the references therein.

In this section we characterize boundedness and compactness of $D_q^p$ by taking different values of $p$ and $q$.

**Theorem 3.** Let $\varphi$ be an analytic self-map of $D$. For $p \in (0, \infty)$, suppose $u \in D_{p-1}^p$ with $|u'(z)|^p (1 - |z|)^{p-1} dA(z)$ is a Carleson measure. Then $uC_\varphi$ is bounded on $D_p$ if and only if

$$\sup_{\zeta \in T} \int_{D(\zeta, r)} \left( \sum_{\varphi(z) = v} |u(z)|^p (1 - |z|)^{p-1} \right) dA(v) = O(r^{2+p}). \quad (2)$$
**Theorem 4.** Let \( \varphi \) be an analytic self-map of \( D \). For \( 0 < q < \infty \), \( q < p + 2 \) and \( p > -1 \), let \( u \in \mathcal{D}_p^q \) with \( |u'(z)|^q(1-|z|)^p \, dA(z) \) is a Carleson measure. Then \( uC_{\varphi} \) is bounded operator on \( \mathcal{D}_p^q \) if and only if

\[
\sup_{\zeta \in T} \int_{D(\zeta,r)} \left( \sum_{\varphi(z)=v} |u(z)|^q(1-|z|)^p \right) dA(v) = O(r^{2+p}).
\]

**Theorem 5.** Let \( \varphi \) be an analytic self-map of \( D \). For \( 0 < q < \infty \), \( q = p + 2 \) and \( p > -1 \), let \( u \in \mathcal{D}_p^q \) with \( |u'(z)|^q(1-|z|^2)^p \, dA(z) \) is a Carleson measure. Then \( uC_{\varphi} \) is bounded operator on \( \mathcal{D}_p^q \) if and only if

\[
\sup_{\zeta \in T} \int_{D(\zeta,r)} \left( \sum_{\varphi(z)=v} |u(z)|^q(1-|z|)^p \right) dA(v) = O(r^{2+p} \log \frac{1}{r^2}).
\]

The proof of Theorem 3, Theorem 4 and Theorem 5 follows on similar lines as proof of Theorem 1, with the following minor modifications:

For different values of \( p \) and \( q \), we have the different test functions. If \( 0 < q < \infty, p > -1 \):

- For \( q < p + 2 \) with \( q \neq p + 1 \), the test function is
  \[
f_z(w) = \frac{(1-|z|^2)^{\frac{p+2-q}{4}}}{(1\!-\!\overline{z}w)^{\frac{p+2-q}{4}}}, \quad |z| < 1.
  \]

- For \( q = p + 1 \), the test function is
  \[
f_z(w) = \frac{(1-|z|^2)^{\frac{1}{2}}}{(1\!-\!\overline{z}w)^{\frac{1}{2}}}, \quad |z| < 1.
  \]

- If \( q = p + 2 \), then
  \[
f_z(w) = \frac{\log \frac{1}{1-\overline{z}w}}{(\log \frac{1}{1-|z|^2})^{\frac{1}{2}}}, \quad |z| < 1.
  \]

**Acknowledgements.** The authors would like to thank the editor and the referees for their helpful comments and valuable suggestions for improving this manuscript.
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(Received April 13, 2010; revised April 26, 2011)