Some commutativity theorems for Banach algebras

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A number of theorems in ring theory, mostly due to HERSTEIN, are devoted to showing that certain rings must be commutative as a consequence of conditions which are seemingly too weak to imply commutativity. See [4, Chapter 3]. In [7] we showed that, in the special case of a Banach algebra, some of these results can be sharpened. We continue this program here.

Let $R$ be a ring with center $Z$. We let $[a, b]$ denote the Lie product $ab - ba$ and $a \cdot b$ the Jordan product $ab + ba$. In a recent paper [2] R. D. GIRI and A. R. DHOBLE showed the following.

**Theorem 1.** Suppose that $R$ is a semiprime ring and that $n, m$ are fixed positive integers each larger than one. Suppose that either (a) $[x^n, y^m] \in Z$ for every $x, y \in R$ or (b) $x^n \cdot y^m \in Z$ for every $x, y \in R$. Then $R$ is commutative.

Henceforth $A$ will denote a Banach algebra over the complex field with center $Z$. For this special case we prove a sharper version of Theorem 1.

**Theorem 2.** Suppose that there are non-empty open subsets $G_1, G_2$ of $A$ such that for each $x \in G_1$ and $y \in G_2$ there are positive integers $n = n(x, y), m = m(x, y)$ depending on $x$ and $y$, $n > 1$, $m > 1$, such that either $[x^n, y^m] \in Z$ or $x^n \cdot y^m \in Z$. Then $A$ is commutative if $A$ is semiprime.

Let $p(t) = \sum_{r=0}^{n} b_r t^r$ be a polynomial in the real variable $t$ with coefficients in $A$ where $p(t) \in Z$ for an infinite set of real values $t$. Then every $b_r \in Z$. For let $f(x)$ be any bounded linear functional on $A$ which vanishes on $Z$. Then $\sum_{r=0}^{n} f(b_r) t^r = 0$ for an infinite set of reals so that each $f(b_r) = 0$. As $Z$ is a closed linear subspace of $A$ this implies that each $b_r \in Z$. 

We begin the proof of Theorem 2. Fix $x \in G_1$. For positive integers $n \geq 2$, $m \geq 2$ let $V(n, m)$ be the set of $y \in A$ for which $[x^n, y^m] \notin Z$ and $x^n \cdot y^m \notin Z$. Each $V(n, m)$ is open in $A$. If every $V(n, m)$ is dense then, by the Baire category theorem, so is the intersection $W$ of all the sets $V(n, m)$. But $W$ being dense would violate the nature of $G_1$ and $G_2$. Hence there are integers $r \geq 2$ and $s \geq 2$ so that $V(r, s)$ is not dense. Therefore there is a non-empty open subset $G_3$ in the complement of $V(r, s)$. For each $y \in G_3$ either $[x^r, y^s] \in Z$ or $x^r \cdot y^s \in Z$. Let $y_0 \in G_3$ and $w \in A$. There is positive real number $a > 0$ such that $y_0 + tw \in G_3$ for all $t$, $0 \leq t \leq a$. For each such $t$ either

\[
[x^r, (y_0 + tw)^s] \in Z
\]

or

\[
x^r \cdot (y_0 + tw)^s \in Z.
\]

Therefore at least one of (1) and (2) must be valid for infinitely many real $t$. Suppose (1) is valid for these $t$. Now $[x^r, (y_0 + tw)^s]$ can be written as a polynomial in $t$ with coefficients in $A$. The coefficient of $t^s$ in that polynomial is $[x^r, w^s]$. Therefore $[x^r, w^s] \in Z$. Likewise if (2) is valid for infinitely many values of $t$ then $x^r \cdot w^s \in Z$.

Thus, given $x \in G_1$, there are positive integers $r > 1$, $s > 1$ so that, for each $w \in A$, either $[x^r, w^s] \in Z$ or $x^r \cdot w^s \in Z$. Let $F_1 = \{w \in A : [x^r, w^s] \in Z\}$ and $F_2 = \{w \in A : x^r \cdot w^s \in Z\}$. Now $A = F_1 \cup F_2$ and each $F_k$ is closed. Then, by the Baire category theorem, at least one of $F_1$ and $F_2$ must contain a non-empty open subset of $A$.

Suppose $F_1$ contains a ball with center $v_0$ and radius $r > 0$. Let $z \in A$. For infinitely many $t$ we must have $[x^r, (v_0 + tz)^s] \in Z$. Therefore $[x^r, z^s] \in Z$ for every $z \in A$. Likewise if $F_2$ has non-void interior then $x^r \cdot z^s \in Z$ for every $z \in A$.

Consequently, given $x \in G_1$, there are positive integers $r > 1$, $s > 1$ so that either $[x^r, z^s] \in Z$ for all $z \in A$ or $x^r \cdot z^s \in Z$ for all $z \in A$.

Now we note that in our set-up with $G_1$ and $G_2$ we could replace $G_2$ by $A$. Next we reverse the roles of $G_1$ and $G_2$ (now replaced by $A$) in the above arguments. Thus, for each $y \in A$, there are positive integers $r > 1$, $s > 1$ depending on $y$ so that either $[x^r, y^s] \in Z$ for all $x \in A$ or $x^r \cdot y^s \in Z$ for all $x \in A$.

For positive integers $m > 1$ and $n > 1$ let $W(n, m)$ be the set of all $y \in A$ so that either $[x^n, y^m] \in Z$ for all $x \in A$ or $x^n \cdot y^m \in Z$ for all $x \in A$. We check that $W(n, m)$ is closed. For let $\{y_k\}$ be a sequence in $W(n, m)$ and $y_k \to w$. Then either there is an infinite subsequence $\{y_{k_j}\}$ so that $[x^n, y_{k_j}] \in Z$ for all $x \in A$ and each $k_j$ or such a subsequence $\{y_{k_j}\}$ where $x^n \cdot y_{k_j} \in Z$ for all $x \in A$ and each $k_j$. Thus $w \in W(n, m)$. Inasmuch
as \( A \) is the union of all the sets \( W(n, m) \) we see by the Baire category theorem that some \( W(p, q) \) must contain a non-void open subset \( G_4 \) of \( A \). Let \( y_0 \in G_4 \). For each \( v \in A \) there is some real number \( b > 0 \) so that when \( 0 \leq t \leq b \) either \( [x^p, (y_0 + tv)^q] \in Z \) for all \( x \in A \) or \( x^p \cdot (y_0 + tv)^q \in Z \) for all \( x \in A \). Now at least one of these alternatives is valid for infinitely many real \( t \). Reasoning already used shows that either \( [x^p, v^q] \in Z \) for all \( x \in Z, v \in A \) or \( x^p \cdot v^q \in Z \) for all \( x \in A, v \in A \). If \( A \) is semiprime then \( A \) is now seen to be commutative by Theorem 1.

In the proof of Theorem 2 we needed \( m > 1 \) and \( n > 1 \) in order to use Theorem 1. If \( A \) has an identity we can do with \( m \geq 1, n \geq 1 \), as we do not then cite Theorem 1.

**Theorem 3.** Suppose that \( A \) has an identity \( e \) and that there are non-empty open subsets \( G_1, G_2 \) of \( A \) where, for each \( x \in G_1, y \in G_2 \), there are integers \( m = m(x, y), n = n(x, y) \), \( m \geq 1, n \geq 1 \), such that either \( [x^n, y^m] \in Z \) or \( x^n \cdot y^m \in Z \). If \( Z \) is semisimple then \( A \) is commutative.

By the proof of Theorem 2 there exist positive integers \( p, q \), \( p \geq 1, q \geq 1 \), so that either \( [x^p, v^q] \in Z \) for all \( x, v \in A \) or \( x^p \cdot v^q \in Z \) for all \( x, v \in A \). In case \( [x^p, v^q] \in Z \) for all \( x, v \in A \) we may replace \( v \) by \( e + tv \). Then \( [x^p, (e + tv)^q] \in Z \) for all \( t \). The coefficient of \( t \) in the polynomial \( [x^p, (e + tv)^q] = [x^p, v] \). Then \( [x^p, v] \in Z \) for all \( x, v \in A \). Now replace \( x \) by \( e + tx \) and \( [(e + tx)^p, v] \in Z \) for all \( t \). Then \( [x, v] \in Z \) for all \( x, v \in Z \). Likewise if \( x^p \cdot v^q \in Z \) for all \( x, v \in A \), we see that \( x \cdot v \in Z \) for all \( x, v \in A \).

In the case that \( x \cdot v \in Z \) for all \( x, v \), set \( v = e \) to see that \( 2x \in Z \) for all \( x \in A \). Then \( A \) is commutative. It remains to consider the case where \( [x, v] \in Z \) for all \( x, v \in A \). By the Kleinecke–Shirokov theorem [1, Prop. 13, p.91] each \( w = [x, v] \) is a generalized nilpotent element of \( A \), that is, \( \lim \|w^n\|^{1/n} = 0 \). Then \( [x, v] \) is a generalized nilpotent element in the commutative Banach algebra \( Z \) and so is in the radical of \( Z \). As \( Z \) is semisimple \( [x, v] = 0 \) so that \( A \) is commutative.

We point out that is easy to show that \( Z \) is semisimple if \( A \) is semisimple. See, for example, [6, Lemma 2.1].

In the situation of Theorem 3 we next drop the requirement that \( Z \) be semisimple. Then \( A \) need not be commutative as the following example shows.

First let \( B \) be the three-dimensional complex algebra with basis \( \{a, b, c\} \) and multiplication given by

\[
(\lambda_1 a + \mu_1 b + \nu_1 c)(\lambda_2 a + \mu_2 b + \nu_2 c) = (\lambda_1 \mu_2 - \lambda_2 \mu_1)c
\]
where the $\lambda_k, \mu_k$ and $\nu_k$ are complex scalars. With the norm, say,

$$\|\lambda a + \mu b + \nu c\| = (|\lambda|^2 + |\mu|^2 + |\nu|^2)^{1/2}.$$ 

$B$ is a Banach algebra (as the product of any three elements of $B$ is zero, $B$ is associative). Now let $A$ be the Banach algebra obtained by adjoining an identity $e$ to $B$ where $\|\gamma e + x\| = |\gamma| + \|x\|$ for $x \in B$ and $\gamma$ complex. For $x, y$ in $B$ we have

$$[\gamma_1 e + x, \gamma_2 e + y] = [x, y]$$

which is a multiple of $c$. Therefore, as $c$ is in the center of $A$, we have $[v, w] \in Z$ for all $v, w \in A$. Hence the requirements of Theorem 3 for $G_1$ and $G_2$ hold if $G_1 = G_2 = A$. However $A$ is not commutative.

For the purposes of the next theorem we discuss a point in the theory of non-associative algebras. Let $K$ be a non-associative algebra. By the center of $K$ is meant [5, p.14] the set of all $z \in K$ where $zx = xz$ for all $x \in K$ and where

$$(x, y, z) = (z, x, y) = (x, z, y) = 0$$

for all $x, y \in K$. Here $(a, b, c)$ is the associator of the elements $a, b \text{ and } c$,

$$(a, b, c) = (ab)c - a(bc).$$

Now we consider $A$ as a non-associative algebra $A^J$ with its multiplication the Jordan multiplication $x \cdot y = xy + yx$. Let $Z^J$ be the center of $A^J$ according to the above definition of center.

For a Lie ideal $U$ of $A$ as in [3, p.5] we set

$$T(U) = \{x \in A : [x, A] \subset U\}.$$ 

As noted there $T(U)$ is both a subalgebra and a Lie ideal of $A$ and $T(U) \supset U$.

**Lemma.** For $A$ we have $Z^J = T(Z)$.

**Proof.** A straight-forward calculation shows that

$$(a \cdot b) \cdot c - a \cdot (b \cdot c) = [b, [a, c]]$$

for all $a, b \text{ and } c \in A$. Then $Z^J$ is the set of all $z \in A$ such that

$$[x, [y, z]] = [z, [x, y]] = [y, [z, x]] = 0$$

for all $x, y \in A$. Thus we see that $Z^J \subset T(Z)$. Conversely suppose that $z \in T(Z)$ so that $[[z, x], y] = 0$ for all $x, y \in A$. Inasmuch as the Jacobi identity gives us

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$$

for all $x, y, z \in A$, we also get $T(Z) \subset Z^J$. Also, as $Z \subset T(Z)$, we have $Z \subset Z^J$. 
Theorem 4. Let $A$ be a Banach algebra with identity $e$ which satisfies the requirements on $G_1$ and $G_2$ of Theorem 3. Then $A = Z^J$.

Proof. As shown in the proof of Theorem 3 either $A$ is commutative (so that also $A = Z^J$) or $[x, y] \in Z$ for all $x, y \in A$. Then $A = T(Z) = Z^J$ by the above lemma.

References


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