Automorphisms on algebras of operator-valued Lipschitz maps

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Abstract. Let Lip($X, B(H)$) and lip$_{\alpha}(X, B(H))$ ($0 < \alpha < 1$) be the big and little Banach $^*$-algebras of $B(H)$-valued Lipschitz maps on $X$, respectively, where $X$ is a compact metric space and $B(H)$ is the $C^*$-algebra of all bounded linear operators on a complex infinite-dimensional Hilbert space $H$. We prove that every linear bijective map that preserves zero products in both directions from Lip($X, B(H)$) or lip$_{\alpha}(X, B(H))$ onto itself is biseparating. We give a Banach–Stone type description for the $^*$-automorphisms on such Lipschitz $^*$-algebras, and we show that the algebraic reflexivity of the $^*$-automorphism groups of Lip($X, B(H)$) and lip$_{\alpha}(X, B(H))$ holds for $H$ separable.

1. Introduction

Let $A$ be a Banach $^*$-algebra. A continuous linear map $\Phi : A \to A$ is a local automorphism if for every $a \in A$, there exists an automorphism $\Phi_a$ of $A$, possibly depending on $a$, such that $\Phi(a) = \Phi_a(a)$. Similarly, a continuous linear map $\Phi : A \to A$ is an approximate local automorphism if for every $a \in A$, there exists a sequence of automorphisms of $A$, $\{\Phi_n\}$, that may depend on $a$, such that $\Phi(a) = \lim_{n \to \infty} \Phi_n(a)$.

Obviously, every automorphism of $A$ is an (approximate) local automorphism of $A$, but the converse is not true in general. Precisely, the automorphism

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group of $\mathcal{A}$ is said to be \textit{algebraically reflexive} (\textit{topologically reflexive}) if every local automorphism (respectively, approximate local automorphism) of $\mathcal{A}$ is an automorphism of $\mathcal{A}$. Analogous definitions for $^*$-automorphisms of $\mathcal{A}$ can be given.

The study of local automorphisms of Banach algebras was started by Larson in [19, Some concluding remarks (5), p. 298], and since then they have been a matter of interest, mainly in the theory of operator algebras. Concretely, if $\mathcal{B}(X)$ is the algebra of all bounded linear operators on a complex infinite-dimensional Banach space $X$, Larson and SourOUR [20] proved that every surjective local automorphism of $\mathcal{B}(X)$ is an automorphism. The real case was solved by BrŠšar and ŠemrJ in [7]. Moreover, they showed in [8] that the automorphism group of $\mathcal{B}(\mathcal{H})$ (no surjectivity is assumed now) is algebraically reflexive provided that $\mathcal{H}$ is an infinite-dimensional separable Hilbert space. In fact, this group is topologically reflexive as MolnÁE showed in [21]. Concerning local automorphisms on other operator algebras, we refer to [3], [6], [22].

These results motivated further research on reflexivity in the setting of group algebras and function algebras. Along this line, MolnÁE and Zalar, [24], studied the algebraic reflexivity of the isometric automorphism group of the convolution algebra $L_p(G)$ of a compact metric group $G$. Concerning function algebras, Cabello SÁNchez and MolnÁE investigated in [10] the reflexivity of the automorphism group of Banach algebras of holomorphic functions, Fréchet algebras of holomorphic functions, and algebras of continuous functions (see also [9]). In [11], Cabello SÁNchez proved that the automorphism group of $L_\infty$ is algebraically reflexive. Recently, Botelho and Jamison [4] have studied the algebraic and topological reflexivity properties of $\ell_p(X)$ spaces.

In this manuscript, we deal with the reflexivity of $^*$-automorphisms on $^*$-algebras of big and little Lipschitz maps taking values in $\mathcal{B}(\mathcal{H})$, the $C^*$-algebra of all bounded linear operators on a separable complex infinite-dimensional Hilbert space $\mathcal{H}$. Recently, Botelho and Jamison have established in [5], under a different approach, the algebraic reflexivity of the class of $^*$-automorphisms preserving the constant functions on algebras of $\mathcal{B}(\mathcal{H})$-valued big Lipschitz maps.

The study of Banach algebras of complex-valued Lipschitz functions begins with the works by Sherbert [26], [27]. We refer to Weaver’s book [28], mainly Chapter 4, for a very comprehensive description of these algebras. The research into spaces of vector-valued Lipschitz functions was initiated by Johnson in [16]. He examined the Banach space properties of scalar-valued and Banach-valued Lipschitz functions. Cao, Zhang and Xu [12] characterized Banach-valued Lipschitz functions (known as \textit{Lipschitz $\alpha$-operators} in [12]), and studied the Lipschitz extension of such functions.
Automorphisms on algebras of operator-valued Lipschitz maps

This paper is organized as follows. In Section 2 we introduce the big and little Banach algebras (*-algebras), Lip_α(X, A) and lip_α(X, A), where 0 < α ≤ 1, of Lipschitz functions on a compact metric space X with values in a Banach algebra (respectively, *-algebra) A. For α = 1, we write Lip(X, A) and lip(X, A).

Assuming that A is a prime unital Banach algebra, we prove in Section 3 that every linear bijective map that preserves zero products in both directions from Lip(X, A) or lip_α(X, A) onto itself is biseparating. The proof uses a technique introduced by Araujo and Jarosz in [2] to state an analogous result in the setting of spaces of operator-valued continuous functions.

Section 4 deals with a Banach–Stone type description of the automorphisms and *-automorphisms on Lip(X, B(H)) and lip_α(X, B(H)) with α ∈ (0, 1) (no separability is assumed now). Here we apply the results obtained in the preceding section, and some results on biseparating linear maps between spaces of Banach-valued Lipschitz functions established by Araujo and Dubarbie in [1], and by the second and third author of the present manuscript in a joint work with Wang, [15].

In Section 5, taking into account the characterization of the automorphisms of Lip(X, C) and lip_α(X, C) with α ∈ (0, 1) by Sherbert [26], we prove that they are algebraically reflexive.

In the last section, we state the algebraic reflexivity of the *-automorphism groups of the Banach *-algebras Lip(X, B(H)) and lip_α(X, B(H)), with α ∈ (0, 1) and H being now separable.

We must point out that our study is motivated by a very nice work by Molnár and Győry, [23], concerning the algebraic reflexivity of the automorphism group of the C*-algebra C_0(X, B(H)) of all continuous functions from X to B(H) that vanish at infinity, where X is a locally compact Hausdorff space and H is a separable complex infinite-dimensional Hilbert space.

2. Preliminaries and notation

Let X and Y be metric spaces. We will use the letter d to denote the distance in any metric space. A map f : X → Y is Lipschitz if there exists a constant k ≥ 0 such that
\[ d(f(x), f(y)) \leq k \, d(x, y), \quad \forall x, y \in X. \]

The smallest k fulfilling this condition is the Lipschitz constant for f, and we denote it by L(f). A map f : X → Y is a Lipschitz homeomorphism if f is bijective, and both f and f^{-1} are Lipschitz.
Let \((X, d)\) be a compact metric space, \(\alpha\) a real parameter in \((0, 1]\), and \(E\) a complex Banach space. Clearly, the set \(X\) with the distance \(d^\alpha\), defined by 
\[d^\alpha(x, y) = d(x, y)^\alpha,\]
for all \(x, y \in X\), is also a compact metric space.

The big Lipschitz space \(\text{Lip}_\alpha(X, E)\) is the Banach space of all functions \(f : X \rightarrow E\) such that
\[L_\alpha(f) := \sup \left\{ \frac{\|f(x) - f(y)\|}{d(x, y)^\alpha} : x, y \in X, x \neq y \right\}\]
is finite, endowed with the norm 
\[\|f\|_\alpha := L_\alpha(f) + \|f\|_\infty,
\]
where 
\[\|f\|_\infty := \sup \{\|f(x)\| : x \in X\}.
\]

The little Lipschitz space \(\text{lip}_\alpha(X, E)\) is the norm-closed linear subspace of \(\text{Lip}_\alpha(X, E)\) formed by all functions \(f\) satisfying the condition \(\forall \varepsilon > 0 \exists \delta > 0 : x, y \in X, 0 < d(x, y) < \delta \Rightarrow \frac{\|f(x) - f(y)\|}{d(x, y)^\alpha} < \varepsilon\)
(see [12, Theorem 2.1]). When \(\alpha = 1\), we will drop the subscript and write simply \(\text{Lip}(X, E)\) and \(\text{lip}(X, E)\). For \(E = \mathbb{C}\), it is usual to denote \(\text{Lip}_\alpha(X)\) and \(\text{lip}_\alpha(X)\).

The space \(\text{Lip}(X)\) is contained in \(\text{lip}_\alpha(X)\) for any \(\alpha \in (0, 1]\), contains the constant functions, and separates the points of \(X\) [27, Proposition 1.6]. However, there are spaces \(\text{lip}(X)\) whose elements are all constant functions (for instance, \(\text{lip}[0, 1]\) with the usual metric). Consequently, the spaces \(\text{Lip}(X, E)\) and \(\text{lip}_\alpha(X, E)\) with \(\alpha \in (0, 1]\) contain the constant functions and separate points. Notice that if \(g \in \text{Lip}(X)\) \((g \in \text{lip}_\alpha(X))\) and \(e \in E\), the map \(g \cdot e\) defined by setting \(g \cdot e(x) = g(x)e\) for all \(x \in X\), belongs to \(\text{Lip}(X, E)\) (respectively, \(\text{lip}_\alpha(X, E)\)) and \(\|g \cdot e\|_\alpha = \|g\|_\alpha \|e\|\).

Given a \(^\ast\)-algebra \(A\), it is straightforward to verify that \(\text{Lip}_\alpha(X, A)\) is a Banach \(^\ast\)-algebra with the multiplication and involution defined pointwise, and \(\text{lip}_\alpha(X, A)\) is a norm-closed \(^\ast\)-subalgebra of \(\text{Lip}_\alpha(X, A)\). We denote by \(\text{Aut}(A)\) and \(\text{Aut}^\ast(A)\) the group of all automorphisms and the group of all \(^\ast\)-automorphisms of \(A\), respectively.

For a metric space \(X\) and a Hilbert space \(\mathcal{H}\), the unity of \(\text{Lip}(X)\) and \(\text{lip}(X)\), that is, the function constantly equal to 1 on \(X\), is denoted by \(1_X\), and the unity of \(\mathcal{B}(\mathcal{H})\), that is, the identity operator on \(\mathcal{H}\), by \(1_\mathcal{H}\).

If \(E\) is a Banach space, \(\mathcal{L}(E)^{-1}\) stands for the set of all linear bijections of \(E\), \(\mathcal{B}(E)\) for the space of all bounded linear operators of \(E\) equipped with the operator canonical norm, and \(\text{Iso}(E)\) for the group of all surjective linear isometries of \(E\).
Automorphisms on algebras of operator-valued Lipschitz maps

For any \( f : X \to E \), let \( c(f) = \{ x \in X : f(x) \neq 0 \} \) denote its cozero set.

Throughout this paper, we will use the following family of Lipschitz functions on a compact metric space \( X \). For any \( x \in X \) and \( \delta > 0 \), let \( h_{x,\delta} : X \to [0,1] \) be defined by

\[
h_{x,\delta}(z) = \max\left\{ 0, 1 - \frac{d(z,x)}{\delta} \right\}, \quad (z \in X).
\]

Clearly, \( h_{x,\delta} \in \text{Lip}(X) \), \( h_{x,\delta}(x) = 1 \), and \( h_{x,\delta}(z) = 0 \) if and only if \( d(z,x) \geq \delta \).

In order to simplify the notation, from now on we denote by \( F^\alpha_X(E) \) either \( \text{Lip}(X,E) \) if \( \alpha = 1 \) or \( \text{lip}^\alpha(X,E) \) if \( \alpha \in (0,1) \). Similarly, \( F^\alpha_X(A) \) stands for \( \text{Lip}(X) \) if \( \alpha = 1 \) or \( \text{lip}^\alpha(X) \) if \( \alpha \in (0,1) \), and \( F^\alpha_X(A)^{-1} \) denotes the set of all nowhere vanishing functions in \( F^\alpha_X(A) \).

3. Zero product preserving maps and separating maps between Lipschitz algebras

Let \( X \) be a compact metric space. Given a Banach space \( E \), a linear map \( \Phi : F^\alpha_X(E) \to F^\alpha_X(E) \) is said to be separating if \( c(\Phi(f)) \cap c(\Phi(g)) = \emptyset \) whenever \( f, g \in F^\alpha_X(E) \) satisfy \( c(f) \cap c(g) = \emptyset \). Moreover, \( \Phi \) is called biseparating if it is bijective and both \( \Phi \) and \( \Phi^{-1} \) are separating maps.

Given a Banach algebra \( A \), a linear map \( \Phi : F^\alpha_X(A) \to F^\alpha_X(A) \) preserves zero products if \( fg = 0 \) implies \( \Phi(f)\Phi(g) = 0 \) for all \( f, g \in F^\alpha_X(A) \). It is said that \( \Phi \) preserves zero products in both directions if it is bijective and both \( \Phi \) and \( \Phi^{-1} \) preserve zero products.

Our main goal in this section is to show that every linear bijective map that preserves zero products in both directions from \( F^\alpha_X(B(H)) \) onto itself is biseparating. This fact will be a key tool to get a Banach–Stone type representation for automorphisms of \( F^\alpha_X(B(H)) \) in the next section.

Let \( X \) be a compact metric space and \( A \) be a unital Banach algebra with unity \( 1_A \), and \( f \in F^\alpha_X(A) \). We fix some additional notation according to [2]:

\[
L(f) = \{ g \in F^\alpha_X(A) : gf = 0 \},
\]

\[
R(f) = \{ g \in F^\alpha_X(A) : fg = 0 \},
\]

\[
\mathcal{A}L = \{ g \in F^\alpha_X(A) : L(g) \subset R(g) \},
\]

\[
C(f) = \{ g \in F^\alpha_X(A) : R(f) \cap \mathcal{A}L \subset R(g) \}.
\]

For every \( f, g \in F^\alpha_X(A) \), it is clear that \( fg = 0 \) whenever \( c(f) \cap c(g) = \emptyset \). If \( A \) is prime (that is, \( a \mathcal{A}b = \{0\} \) implies \( a = 0 \) or \( b = 0 \)) and \( g \) lies in \( \mathcal{A}L \), the
converse also holds. Indeed, assume \( fg = 0 \). If \( x \in c(f) \cap c(g) \), then \( f(x) \) and \( g(x) \) are nonzero elements in \( A \). Since \( A \) is prime, there is \( a \in A \setminus \{ 0 \} \) such that \( g(x)a \neq 0 \). Let \( h \in F_{\alpha}(X, A) \) be given by \( h = (1_X \cdot a)f \). It is clear that \( hg = 0 \) and \( gh(x) = g(x)a \neq 0 \). Hence \( g \notin A\mathcal{I} \).

**Lemma 3.1.** Let \( X \) be a compact metric space and \( A \) be a unital Banach algebra. For every \( f \in F_{\alpha}(X, A) \), we have

\[
C(f) \subset \left\{ g \in F_{\alpha}(X, A) : c(g) \subset \text{int}(c(f)) \right\},
\]

and the equality holds whenever \( A \) is prime.

**Proof.** Let \( f, g \in F_{\alpha}(X, A) \) be such that \( c(g) \) is not included in \( \text{int}(c(f)) \).

Let us show that \( g \notin C(f) \). We can choose \( x \in c(g) \), \( \varepsilon > 0 \) with \( B(x, \varepsilon) \subset c(g) \), \( y \in B(x, \varepsilon) \) and \( \delta > 0 \) such that \( B(y, \delta) \cap c(f) = \emptyset \). As usual, \( B(x, \varepsilon) = \{ z \in X : d(z, x) < \varepsilon \} \). Let \( h \) be the map \( h_{y, \delta} \cdot 1_A \) defined on \( X \). It is clear that \( h \in A\mathcal{I} \) and \( c(h) = B(y, \delta) \). Hence \( c(h) \cap c(f) = \emptyset \) and, in particular, \( fh = 0 \). Nevertheless as \( gh(y) = g(y) \neq 0, gh \neq 0 \), which proves that \( g \notin C(f) \).

Notice that by the comments above, if \( A \) is prime, we have

\[
C(f) = \{ g \in F_{\alpha}(X, A) : \text{for all } h \in A\mathcal{I} [c(f) \cap c(h) = \emptyset \Rightarrow c(g) \cap c(h) = \emptyset] \}.
\]

Let \( g \in F_{\alpha}(X, A) \) be for which \( c(g) \subset \text{int}(c(f)) \). Take \( h \in A\mathcal{I} \) with \( c(f) \cap c(h) = \emptyset \).

Then \( \text{int}(c(f)) \cap c(h) = \emptyset \) and thus \( \text{int}(c(f)) \cap c(h) = \emptyset \). It follows that \( c(g) \cap c(h) = \emptyset \), that is, \( g \in C(f) \). \( \square \)

**Lemma 3.2.** Let \( X \) be a compact metric space and \( A \) be a unital Banach algebra. If \( c(f_1) \cap c(f_2) = \emptyset \), then \( C(f_1) \cap C(f_2) = \{ 0 \} \) for every \( f_1, f_2 \in F_{\alpha}(X, A) \).

The converse holds if \( A \) is prime.

**Proof.** Let \( f_1, f_2 \in F_{\alpha}(X, A) \). By Lemma 3.1, if \( g \in C(f_1) \cap C(f_2) \) and \( g \neq 0 \), then \( 0 \neq c(g) \subset \text{int}(c(f_1)) \cap \text{int}(c(f_2)) \). It follows easily that \( c(f_1) \cap c(f_2) \neq \emptyset \).

Conversely, if \( A \) is prime and \( c(f_1) \cap c(f_2) \neq \emptyset \), let \( x \in X \) and \( \delta > 0 \) be so that \( B(x, \delta) \subset c(f_1) \cap c(f_2) \). For \( g = h_{x, \delta} \cdot 1_A \), it is clear that \( c(g) = B(x, \delta) \subset \text{int}(c(f_1)) \cap \text{int}(c(f_2)) \) and hence, by Lemma 3.1, \( g \in C(f_1) \cap C(f_2) \). \( \square \)

The following result is inspired in [2, Theorem 2].

**Theorem 3.3.** Let \( X \) be a compact metric space and let \( A \) be a prime unital Banach algebra. Let \( \Phi : F_{\alpha}(X, A) \to F_{\alpha}(X, A) \) be a bijective linear map preserving zero products in both directions. Then \( \Phi \) is biseparating.
Proof. As $\Phi$ is bijective and preserves zero products in both directions, an easy verification shows that $\mathcal{A}I = \Phi(\mathcal{A}I)$ and $C(\Phi(f)) = \Phi(C(f))$ for every $f \in F_\alpha(X, A)$.

Let $f_1, f_2 \in F_\alpha(X, A)$ be such that $c(f_1) \cap c(f_2) = \emptyset$. By Lemma 3.2, $C(f_1) \cap C(f_2) = \{0\}$, that is, $C(\Phi(f_1)) \cap C(\Phi(f_2)) = \{0\}$. As $A$ is prime, we have also $c(\Phi(f_1)) \cap c(\Phi(f_2)) = \emptyset$. Hence $\Phi$ is separating. The same argument applied to $\Phi^{-1}$ shows that $\Phi$ is biseparating. □

As a direct consequence we obtain the above announced result.

**Corollary 3.4.** Let $X$ be a compact metric space and let $\mathcal{H}$ be a complex infinite-dimensional Hilbert space. Then every bijective linear map from $F_\alpha(X, B(\mathcal{H}))$ onto itself that preserves zero products in both directions is biseparating.

4. Banach–Stone type representation of automorphisms of $F_\alpha(X, B(\mathcal{H}))$

We describe the general form of the automorphisms and $^*$-automorphisms on Lipschitz $^*$-algebras $F_\alpha(X, B(\mathcal{H}))$. For the little Lipschitz spaces we require the next result on Lipschitz functional calculus.

**Lemma 4.1.** Let $X$ be a compact metric space, $E$ be a Banach space, and $\alpha \in (0,1)$. If $h \in \text{lip}_\alpha(X, E)$ and $\varphi : X \to X$ is Lipschitz, then $h \circ \varphi \in \text{lip}_\alpha(X, E)$.

**Proof.** First observe that $h \circ \varphi \in \text{Lip}_\alpha(X, E)$ since

$$\|h(\varphi(x)) - h(\varphi(y))\| \leq L_\alpha(h) d(\varphi(x), \varphi(y))^\alpha \leq L_\alpha(h) L(\varphi)^\alpha d(x, y)^\alpha$$

for every $x, y \in X$. Now, let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that

$$x, y \in X, \ 0 < d(x, y) < \delta \ \Rightarrow \ \frac{\|h(x) - h(y)\|}{d(x, y)^\alpha} < \frac{\varepsilon}{1 + L(\varphi)^\alpha}.$$ 

Let $x, y \in X$ with $0 < d(x, y) < \delta/(1 + L(\varphi))$. If $\varphi(x) \neq \varphi(y)$ (otherwise, the result is trivial) then $0 < d(\varphi(x), \varphi(y)) < \delta$ and

$$\frac{\|h(\varphi(x)) - h(\varphi(y))\|}{d(x, y)^\alpha} = \frac{\|h(\varphi(x)) - h(\varphi(y))\|}{d(\varphi(x), \varphi(y))^\alpha} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} < \frac{\varepsilon}{1 + L(\varphi)^\alpha} L(\varphi)^\alpha < \varepsilon.$$ 

This shows that $h \circ \varphi \in \text{lip}_\alpha(X, E)$. □
Let us recall that every ∗-automorphism of a C∗-algebra A is an isometry, and every automorphism of A is continuous in the norm topology in A and its norm equals the norm of its inverse (see, for example, [25, Corollary 1.2.6, Lemma 4.1.12 and Proposition 4.1.13]). Therefore Aut∗(A) ⊂ Aut(A) ⊂ B(A). We next consider these sets as metric spaces with the metric induced by the operator canonical norm.

**Theorem 4.2.** Let X be a compact metric space, H be a complex infinite-dimensional Hilbert space, and α ∈ (0, 1). A map Φ of lipα(X, B(H)) into lipα(X, B(H)) is an automorphism if and only if there exist a unique Lipschitz map τ from (X, dα) into Aut(B(H)) and a unique Lipschitz homeomorphism φ : X → X such that

\[ Φ(f)(x) = τ(x)(f(φ(x))) \quad (f \in \text{lip}_α(X, B(H)), \; x \in X). \]  

Moreover, if Φ : lipα(X, B(H)) → lipα(X, B(H)) is an automorphism, and τ : X → Aut(B(H)) is the map given above, then Φ is a ∗-automorphism if and only if τ(x) is a ∗-automorphism for every x ∈ X.

**Proof.** Let Φ : lipα(X, B(H)) → lipα(X, B(H)) be a map of the form (1) with τ, φ being as in the statement above. It is straightforward to check that Φ is linear, injective and multiplicative. Observe that τ ∈ Lipα(X, B(B(H))) by hypothesis. We prove that τ ∈ lipα(X, B(B(H))). Indeed, by (1)

\[ τ(x)(a) = Φ(1_X \cdot a)(x) \quad (x \in X, \; a \in B(H)), \]

and since Φ(1_X ∙ a) ∈ lipα(X, B(B(H)), for every a ∈ B(H), the map τ(·)(a) belongs to lipα(X, B(B(H))). From this we show that τ ∈ lipα(X, B(B(H))). Suppose to the contrary that there exist ε > 0 and, for each n ∈ N, x_n, y_n ∈ X with x_n ≠ y_n such that d(x_n, y_n) → 0 as n → ∞, but

\[ \frac{∥τ(x_n) − τ(y_n)∥}{d(x_n, y_n)^α} ≥ ε \]

for all n. Then we can find some a ∈ B(H) with ∥a∥ = 1 such that

\[ \frac{∥(τ(x_n) − τ(y_n))(a)∥}{d(x_n, y_n)^α} ≥ \frac{ε}{2} \]

for all n, and this says us that τ(·)(a) is not in lipα(X, B(B(H))), which is impossible.

It remains to show that Φ is surjective. To this end, pick h ∈ lipα(X, B(H)) and let f : X → B(H) be defined by

\[ f(x) = τ(φ^{-1}(x))^{-1}(h(φ^{-1}(x))) \quad (x \in X). \]
We only need to prove that $f \in \text{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$, since $\Phi(f) = h$. Notice that for any $x, y \in X$ and $a \in \mathcal{B}(\mathcal{H})$

$$\|(\tau(x)^{-1} - \tau(y)^{-1})(a)\| = \|(\tau(x)^{-1} \tau(y) - \tau(x)^{-1} \tau(y)^{-1})(a)\|$$

$$\leq \|\tau(x)\| \|(\tau(x) - \tau(y))(\tau(y)^{-1}(a))\|.$$

From this inequality, it follows that for every $x, y \in X$

$$\|f(x) - f(y)\| = \|\tau(\varphi^{-1}(x))^{-1}(h(\varphi^{-1}(x)) - h(\varphi^{-1}(y))) + (\tau(\varphi^{-1}(x))^{-1} - \tau(\varphi^{-1}(y))^{-1})(h(\varphi^{-1}(y)))\|$$

$$\leq \|\tau(\varphi^{-1}(x))\||h(\varphi^{-1}(x)) - h(\varphi^{-1}(y))\|$$

$$+ \|\tau(\varphi^{-1}(x))\|\|\tau(\varphi^{-1}(x)) - \tau(\varphi^{-1}(y))\|(\tau(\varphi^{-1}(y))^{-1})(h(\varphi^{-1}(y)))\|$$

$$\leq \|\tau\|_\infty \|h(\varphi^{-1}(x)) - h(\varphi^{-1}(y))\|$$

$$+ \|\tau\|_\infty \|\tau(\varphi^{-1}(x)) - \tau(\varphi^{-1}(y))\|\|h \circ \varphi^{-1}\|_\infty$$

$$\leq \|\tau\|_\infty L_a(h \circ \varphi^{-1})d(x, y)^\alpha + \|\tau\|_\infty \|h \circ \varphi^{-1}\|_\infty L_a(\tau)d(\varphi^{-1}(x), \varphi^{-1}(y))^\alpha$$

$$\leq \|\tau\|_\infty L_a(h \circ \varphi^{-1})d(x, y)^\alpha + \|\tau\|_\infty \|h \circ \varphi^{-1}\|_\infty L_a(\tau)L(\varphi^{-1})^\alpha d(x, y)^\alpha.$$  

Hence $f \in \text{Lip}_\alpha(X, \mathcal{B}(\mathcal{H})).$ Moreover, $h \circ \varphi^{-1}$ belongs to $\text{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$ and $\tau \circ \varphi^{-1}$ lies in $\text{lip}_\alpha(X, \mathcal{B}(\mathcal{H}))$ by Lemma 4.1, so the second inequality yields $f \in \text{lip}_\alpha(X, \mathcal{B}(\mathcal{H})), \text{as desired.}$

To prove the converse implication, we need some results from [15] on biseparating linear maps between spaces $\text{lip}_\alpha(X, E)$, with $0 < \alpha < 1$. Let $\Phi$ be an automorphism of $\text{lip}_\alpha(X, \mathcal{B}(\mathcal{H})).$ It is clear that $\Phi$ preserves zero products in both directions, and according to Corollary 3.4, $\Phi$ is a biseparating linear map. Then, by [15, Theorem 4.1], there exist a map $\tau : X \to \mathcal{L}(\mathcal{B}(\mathcal{H}))^{-1}$ and a homeomorphism $\varphi : X \to X$ such that

$$\Phi(f)(x) = \tau(x)(f(\varphi(x))), \forall f \in \text{lip}_\alpha(X, \mathcal{B}(\mathcal{H})), \forall x \in X.$$  

Since $\Phi$ is a homomorphism, it follows easily that $\tau(x)$ is multiplicative for every $x \in X$. Hence $\tau(x) \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$ for all $x \in X$, and $\Phi$ is continuous by [15, Theorem 4.2]. Now, according to [15, Theorem 4.3], $\varphi$ is a Lipschitz homeomorphism. Moreover, as

$$\tau(x)(a) = \Phi(1_X \cdot a)(x) \quad (x \in X, a \in \mathcal{B}(\mathcal{H})), $$

for every $x, y \in X$ and $a \in \mathcal{B}(\mathcal{H})$, we have

$$\|(\tau(x) - \tau(y))(a)\| = \|\Phi(1_X \cdot a)(x) - \Phi(1_X \cdot a)(y)\|$$

$$\leq L_a(\Phi(1_X \cdot a))d(x, y)^\alpha \leq \|\Phi(1_X \cdot a)\|_\alpha d(x, y)^\alpha \leq \|\Phi\| \|a\| d(x, y)^\alpha.$$
Henceforward \( \|\tau(x) - \tau(y)\| \leq \|\Phi\|d(x, y)^\alpha \) for all \( x, y \in X \), and thus \( \tau \) is a Lipschitz map from \( (X, d^\alpha) \) into \( \text{Aut}(\mathcal{B}(\mathcal{H})) \).

To prove the uniqueness, assume that there are a Lipschitz map \( \tau' \) from \( (X, d^\alpha) \) into \( \text{Aut}(\mathcal{B}(\mathcal{H})) \) and a Lipschitz homeomorphism \( \varphi' : X \to X \) such that \( \Phi(f)(x) = \tau'(x)(f(\varphi'(x))) \) for all \( x \in X \) and \( f \in \text{lip}_\alpha(X, \mathcal{B}(\mathcal{H})) \). For any \( x \in X \) and \( a \in \mathcal{B}(\mathcal{H}) \), it is clear that \( \tau'(x)(a) = \Phi(1_X \cdot a)(x) = \tau(x)(a) \) and thus \( \tau' = \tau \).

Therefore, given any \( x \in X \), we have \( \tau(x)(f(\varphi'(x))) = \tau'(x)(f(\varphi'(x))) \) for all \( f \in \text{lip}_\alpha(X, \mathcal{B}(\mathcal{H})) \), which yields \( f(\varphi'(x)) = f(\varphi(x)) \) for all \( f \in \text{lip}_\alpha(X, \mathcal{B}(\mathcal{H})) \). Since \( \text{lip}_\alpha(X, \mathcal{B}(\mathcal{H})) \) separates the points of \( X \), we infer that \( \varphi'(x) = \varphi(x) \). This holds for every \( x \in X \), and so we conclude that \( \varphi' = \varphi \).

We finish the proof by characterizing the \(*\)-automorphisms of \( \text{lip}_\alpha(X, \mathcal{B}(\mathcal{H})) \). Let \( \Phi : \text{lip}_\alpha(X, \mathcal{B}(\mathcal{H})) \to \text{lip}_\alpha(X, \mathcal{B}(\mathcal{H})) \) be an automorphism and let \( \tau, \varphi \) be the maps that permit us to express \( \Phi \) in the form (1). Suppose first that \( \Phi \) preserves the involution in \( \text{lip}_\alpha(X, \mathcal{B}(\mathcal{H})) \). Then, given \( x \in X \), we have

\[
\tau(x)(a^*) = \Phi(1_X \cdot a^*)(x) = \Phi((1_X \cdot a)^*)(x) = (\Phi(1_X \cdot a)^*)(x) = (\Phi(1_X \cdot a)(x))^* = (\tau(x)(a))^*
\]

for all \( a \in \mathcal{B}(\mathcal{H}) \), and therefore \( \tau(x) \) is a \(*\)-automorphism. Conversely, assume that \( \tau(x) \) is a \(*\)-automorphism for every \( x \in X \). Given \( f \in \text{lip}_\alpha(X, \mathcal{B}(\mathcal{H})) \), we have

\[
\Phi(f^*)(x) = \tau(x)(f^*(\varphi(x))) = \tau(x)((f(\varphi(x))^*) = (\tau(x)(f(\varphi(x))))^* = (\Phi(f)(x))^* = (\Phi(f))^*(x)
\]

for every \( x \in X \), and so \( \Phi \) preserves the involution. \( \square \)

The following result may be proved in the same way as Theorem 4.2. We only need some facts from [1] on biseparating linear maps between spaces \( \text{Lip}(X, E) \).

**Theorem 4.3.** Let \( X \) be a compact metric space, and let \( \mathcal{H} \) be a complex infinite-dimensional Hilbert space. A map \( \Phi : \text{Lip}(X, \mathcal{B}(\mathcal{H})) \to \text{Lip}(X, \mathcal{B}(\mathcal{H})) \) is an automorphism if and only if there exist a unique Lipschitz map \( \tau : X \to \text{Aut}(\mathcal{B}(\mathcal{H})) \) and a unique Lipschitz homeomorphism \( \varphi : X \to X \) such that \( \Phi \) is of the form

\[
\Phi(f)(x) = \tau(x)(f(\varphi(x))) \quad (f \in \text{Lip}(X, \mathcal{B}(\mathcal{H})), \ x \in X).
\]

Moreover, if \( \Phi : \text{Lip}(X, \mathcal{B}(\mathcal{H})) \to \text{Lip}(X, \mathcal{B}(\mathcal{H})) \) is an automorphism and \( \tau : X \to \text{Aut}(\mathcal{B}(\mathcal{H})) \) is the map given above, then \( \Phi \) is \(*\)-preserving if and only if \( \tau(x) \) is \(*\)-preserving for every \( x \in X \).
Proof. Just the “only if” part deserves some comment. Let \( \Phi \) be an automorphism of \( \text{Lip}(X, \mathcal{B}(\mathcal{H})) \). By Corollary 3.4, \( \Phi \) is biseparating. From [1, Theorem 3.1], there are a map \( \tau : X \to \mathcal{L}(\mathcal{B}(\mathcal{H}))^{-1} \) and a Lipschitz homeomorphism \( \varphi : X \to X \) so that

\[
\Phi(f)(x) = \tau(x)(f(\varphi(x))), \quad \forall f \in \text{Lip}(X, \mathcal{B}(\mathcal{H})), \forall x \in X.
\]

Since \( \Phi \) is a homomorphism, \( \tau(x) \) is multiplicative and thus continuous, for all \( x \in X \). Equivalently, the set \( Y_\alpha := \{ x \in X : \tau(x) \text{ is discontinuous} \} \) is empty and therefore \( \Phi \) is continuous by [1, Theorem 3.4]. A glance at the comments preceding [1, Proposition 3.2] reveals that

\[
\|\tau(x) - \tau(y)\| \leq \|\Phi\| d(x, y), \quad \forall x, y \in X,
\]

and thus \( \tau : X \to \text{Aut}(\mathcal{B}(\mathcal{H})) \) is Lipschitz. The uniqueness of \( \tau \) and \( \varphi \) is proved similarly as in Theorem 4.2.

For the proof of our results we also need the following well-known facts on the general form of the automorphisms of \( \text{Lip}(X) \) and \( \text{lip}_\alpha(X) \) with \( 0 < \alpha < 1 \).

**Theorem 4.4.** Let \( X \) be a compact metric space.

(1) [26, Corollary 5.2] A map \( \Phi : \text{Lip}(X) \to \text{Lip}(X) \) is an automorphism if and only if there exists a Lipschitz homeomorphism \( \varphi : X \to X \) such that \( \Phi(f) = f \circ \varphi \) for every \( f \in \text{Lip}(X) \).

(2) [15, Corollary 5.3] Given \( \alpha \in (0, 1) \), a map \( \Phi : \text{lip}_\alpha(X) \to \text{lip}_\alpha(X) \) is an automorphism if and only if \( \Phi \) is of the form \( \Phi(f) = f \circ \varphi \) for all \( f \in \text{lip}_\alpha(X) \), where \( \varphi \) is a Lipschitz homeomorphism of \( X \).

**5. Algebraic reflexivity of the automorphism group of \( F_\alpha(X) \)**

In this section we prove that \( \text{Aut}(F_\alpha(X)) \) is algebraically reflexive. In view of Theorem 4.4, note that \( \text{Aut}(F_\alpha(X)) = \text{Aut}^*(F_\alpha(X)) \).

Let us recall that for a compact metric space \( X \), Sherbert proved in [26, Theorem 5.1] that a map \( \Phi : \text{Lip}(X) \to \text{Lip}(X) \) is a unital homomorphism if and only if there exists a Lipschitz map \( \varphi : X \to X \) such that \( \Phi(f) = f \circ \varphi \) for every \( f \in \text{Lip}(X) \). By using the same idea of the proof of this statement, we can see that an analogous result holds for unital endomorphisms of \( \text{lip}_\alpha(X) \), with \( 0 < \alpha < 1 \).
**Theorem 5.1.** Let $X$ be a compact metric space. Then the automorphism group of $F_\alpha(X)$ is algebraically reflexive.

**Proof.** Let $\Phi$ be a local automorphism of $F_\alpha(X)$. Then, for each $f \in F_\alpha(X)$, there exists an automorphism $\Phi_f$ of $F_\alpha(X)$ so that $\Phi(f) = \Phi_f(f)$. This implies that $\Phi$ is injective. As $\Phi_f(1_X) = 1_X$, for every $f \in F_\alpha(X)$, it follows that $\Phi(1_X) = \Phi_f(1_X) = 1_X$. By Theorem 4.4, there exists a Lipschitz homeomorphism $\varphi_f : X \to X$ such that

$$\Phi_f(g)(z) = g(\varphi_f(z)) \quad (g \in F_\alpha(X), \ z \in X).$$

In particular,

$$\Phi(f)(z) = \Phi_f(f)(z) = f(\varphi_f(z)) \quad (z \in X).$$

Fix $x \in X$, and define the unital linear functional $\Phi_x : F_\alpha(X) \to \mathbb{C}$ by

$$\Phi_x(f) = \Phi(f)(x), \quad \forall f \in F_\alpha(X).$$

Let $f \in F_\alpha(X)^{-1}$. Since $\Phi_x(f) = \Phi(f)(x) = f(\varphi_f(x))$, we have $\Phi_x(f) \neq 0$. By the Gleason–Kahane–Żelazko theorem [13], [17], we infer that $\Phi_x$ is multiplicative. Hence $\Phi$ is a homomorphism, that is, there exists a Lipschitz map $\varphi : X \to X$ such that

$$\Phi(f)(z) = f(\varphi(z)) \quad (f \in F_\alpha(X), \ z \in X).$$

(2)

We claim that $\varphi$ is onto. Suppose, to the contrary, that there exists $x \in X \setminus \varphi(X)$. Then $d(x, \varphi(X)) > 0$ since $\varphi(X)$ is closed. For $\delta = d(x, \varphi(X))$, the Lipschitz map $h_{x,\delta} \in \text{Lip}(X) \subset F_\alpha(X)$ satisfies $h_{x,\delta}(\varphi(X)) = \{0\}$. By (2), $\Phi(h_{x,\delta}) = 0$, but $h_{x,\delta}(x) = 1$, which contradicts the fact that $\Phi$ is linear and injective.

To show that $\varphi$ is injective, let $x, y \in X$ be such that $\varphi(x) = \varphi(y)$. Define $h : X \to \mathbb{R}$ by

$$h(z) = d(z, \varphi(x)), \quad \forall z \in X.$$  

Clearly, $h$ belongs to $\text{Lip}(X)$, and $h(z) = 0$ if and only if $z = \varphi(x)$. Since $\Phi$ is a local automorphism of $F_\alpha(X)$,

$$\Phi(h)(z) = h(\varphi_h(z)) \quad (z \in X),$$

(3)

where $\varphi_h$ is a Lipschitz homeomorphism of $X$. From (2) and (3), it follows

$$h(\varphi_h(x)) = \Phi(h)(x) = h(\varphi(x)) = 0, \quad h(\varphi_h(y)) = \Phi(h)(y) = h(\varphi(y)) = 0.$$  

This implies that $\varphi_h(x) = \varphi_h(y) = \varphi(x)$, and as $\varphi_h$ is injective, we get $x = y$. 


By taking into account Theorem 4.4, it remains to show that \( \varphi^{-1} : X \to X \) is Lipschitz to ensure that \( \Phi \) is an automorphism of \( F_{\alpha}(X) \). In order to prove this, we follow the argument used by Botelho and Jamison in [4, Theorem 2.1]. Assume that \( \varphi^{-1} \) is not Lipschitz. Then there exist sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \), with \( x_n \neq y_n \) for all \( n \), such that
\[
\lim_{n \to \infty} \frac{d(\varphi(x_n), \varphi(y_n))}{d(x_n, y_n)} = 0.
\]

Let \( \tilde{X} = \{(x, y) \in X^2 : x \neq y\} \), and let \( F : \tilde{X} \to \mathbb{R} \) be defined by
\[
F(x, y) = \frac{d(\varphi(x), \varphi(y))}{d(x, y)}.
\]

Denote by \( \beta\tilde{X} \) the Stone-Čech compactification of \( \tilde{X} \), and by \( \beta F \) the unique continuous extension of \( F \) to \( \beta\tilde{X} \). By the compactness of \( \beta\tilde{X} \), there exists a subnet \( \{(x_i, y_i)\} \) converging to \( \xi \in \beta\tilde{X} \). By using the continuity of \( \beta F \), we have \( \beta F(\xi) = 0 \). Moreover, \( \xi \notin \tilde{X} \) since \( F(x, y) \neq 0 \) for all \( (x, y) \in \tilde{X} \). As \( X \) is compact, taking subnets if necessary, we may assume that \( \{x_i\} \) converges to some \( x \in X \) and \( y_i \neq x \) for all \( i \). Define \( k : X \to \mathbb{R} \) by
\[
k(z) = d(z, \varphi(x)), \quad \forall z \in X.
\]

Since \( \Phi \) is a local automorphism of \( F_{\alpha}(X) \), we get
\[
\Phi(k)(z) = k(\varphi_k(z)) \quad (z \in X),
\]
for some Lipschitz homeomorphism \( \varphi_k : X \to X \). By applying (2) and (4), we have
\[
d(\varphi(z), \varphi(x)) = d(\varphi_k(z), \varphi(x)) \quad (z \in X).
\]

In particular, \( \varphi_k(x) = \varphi(x) \) and thus
\[
d(\varphi(z), \varphi(x)) = d(\varphi_k(z), \varphi_k(x)) \quad (z \in X).
\]

Therefore
\[
\frac{d(\varphi(y_i), \varphi(x))}{d(y_i, x)} = \frac{d(\varphi_k(y_i), \varphi_k(x))}{d(y_i, x)} \geq \frac{1}{L(\varphi_k^{-1})} > 0
\]
for all \( i \). If we use that
\[
|B \varphi(y_i, x) - \beta F(\xi)| \leq |B \varphi(y_i, x) - \beta F(y_i, x_i)| + |\beta F(y_i, x_i) - \beta F(\xi)|
\]
for all \( i \) and the uniform continuity of \( \beta F \), it follows that \( \{B \varphi(y_i, x)\} \) converges to \( \beta F(\xi) \). Hence \( \beta F(\xi) \geq 1/L(\varphi_k^{-1}) \), a contradiction. This proves that \( \varphi^{-1} \) is Lipschitz, as desired.
\[\square\]
6. Algebraic reflexivity of the *-automorphism group of \( F_\alpha(X, \mathcal{B}(\mathcal{H})) \)

Our aim is to prove that the *-automorphism group of \( F_\alpha(X, \mathcal{B}(\mathcal{H})) \) is algebraically reflexive whenever \( \mathcal{H} \) is a separable complex infinite-dimensional Hilbert space. We will use the following three lemmas. The first two appear essentially in the manuscript by GYÖRY and MOLNÁR [23].

We begin by showing that the set of all scalar multiples of *-automorphisms on \( \mathcal{B}(\mathcal{H}) \) is algebraically reflexive.

**Lemma 6.1.** Let \( \mathcal{H} \) be a separable complex infinite-dimensional Hilbert space. Let \( \Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) be a continuous linear map with the property that for each \( a \in \mathcal{B}(\mathcal{H}) \), there exist \( \lambda_\alpha \in \mathbb{C} \) and \( \tau_\alpha \in \text{Aut}^*(\mathcal{B}(\mathcal{H})) \) such that \( \Psi(a) = \lambda_\alpha \tau_\alpha(a) \). Then there exist \( \lambda \in \mathbb{C} \) and \( \tau \in \text{Aut}^*(\mathcal{B}(\mathcal{H})) \) such that \( \Psi(a) = \lambda \tau(a) \) for every \( a \in \mathcal{B}(\mathcal{H}) \).

**Proof.** Since the *-automorphisms of \( \mathcal{B}(\mathcal{H}) \) are both automorphisms and surjective linear isometries, [23, Lemmas 2.3 and 2.4] ensure that there exist \( \lambda_1, \lambda_2 \in \mathbb{C} \), \( \tau_1 \in \text{Aut}(\mathcal{B}(\mathcal{H})) \) and \( \tau_2 \in \text{Iso}(\mathcal{B}(\mathcal{H})) \) such that \( \Psi = \lambda_1 \tau_1 \) and \( \Psi = \lambda_2 \tau_2 \). If \( \lambda_1 = 0 \), then \( \Psi = 0 = 0I_{\mathcal{H}} \). So we can assume that \( \lambda_1 \neq 0 \). Therefore \( \tau_1 = (\lambda_2/\lambda_1) \tau_2 \). Since \( \tau_1 \) is unital, it follows that \( |\lambda_2/\lambda_1| = 1 \), and so \( \tau_1 \in \text{Iso}(\mathcal{B}(\mathcal{H})) \). According to [18, Lemma 8], \( \tau_1 \) is a *-automorphism of \( \mathcal{B}(\mathcal{H}) \), which proves the lemma.

**Lemma 6.2.** [23, Lemma 2.2]. Let \( \mathcal{H} \) be a separable complex infinite-dimensional Hilbert space. Let \( \tau, \tau_1, \tau_2 \) be in \( \text{Aut}^*(\mathcal{B}(\mathcal{H})) \), and let \( \lambda \) and \( 0 \neq \lambda_1, \lambda_2 \) be in \( \mathbb{C} \) satisfying that \( \lambda \tau(a) = \lambda_1 \tau_1(a) + \lambda_2 \tau_2(a) \) for every \( a \in \mathcal{B}(\mathcal{H}) \). Then \( \tau_1 = \tau_2 \).

**Lemma 6.3.** Let \( X \) be a compact metric space and let \( E \) be a Banach space. Then \( F_\alpha(X, E) \) is the uniformly closed linear span of the set of functions \( \{ g : e : g \in F_\alpha(X), e \in E \} \).

**Proof.** Let \( f \in F_\alpha(X, E) \) and \( \epsilon > 0 \). For every \( x \in X \) the set

\[
U_x = \left\{ y \in X : \|f(y) - f(x)\| < \frac{\epsilon}{2} \right\}
\]

is open in \( X \). Since \( X = \bigcup_{x \in X} U_x \) and \( X \) is compact, there exist \( x_1, \ldots, x_n \in X \) such that \( X = \bigcup_{k=1}^n U_{x_k} \). Let \( \{g_1, \ldots, g_n\} \subset F_\alpha(X) \) be a partition of unity on \( X \) subordinate to the open covering \( \{U_{x_1}, \ldots, U_{x_n}\} \) (see, for example, [14, Lemma 2.2]). Thus, \( g_1, \ldots, g_n \) are functions in \( F_\alpha(X) \) from \( X \) into \([0,1]\) such that \( \sum_{k=1}^n g_k = 1_X \) and \( \text{supp}(g_k) \subset U_{x_k} \) for every \( k = 1, \ldots, n \). Here \( \text{supp}(g_k) \) denotes the closure of the cozero set of \( g_k \).
Let $\Phi$ be a local automorphism (of $F_\alpha(X,B(H))$), that is, $\Phi$ is a continuous linear map satisfying that for every $f \in F_\alpha(X,B(H))$, there is $\Phi_f \in \text{Aut}^*(F_\alpha(X,B(H)))$ such that $\Phi(f) = \Phi_f(f)$. In light of Theorems 4.2 and 4.3, for every $f \in F_\alpha(X,B(H))$, there are a Lipschitz map $\tau_f$ from $(X,d^\alpha)$ into $\text{Aut}^*(B(H))$ and a Lipschitz homeomorphism $\varphi_f: X \to X$ such that

$$\Phi(f)(x) = \tau_f(x)(f(\varphi_f(x))) \quad (x \in X).$$

(5)

Since $\tau_f(x)$ is a linear isometry for every $x \in X$, we have

$$\|\Phi(f)(x)\| = \|\tau_f(x)(f(\varphi_f(x)))\| = \|f(\varphi_f(x))\|$$

for all $x \in X$, and hence $\|\Phi(f)\|_\infty = \|f\|_\infty$. Consequently, $\Phi$ preserves the supremum norm.

Moreover, for every $g \in F_\alpha(X)$ there are a unique Lipschitz map $\tau_g I_H$ from $(X,d^\alpha)$ into $\text{Aut}^*(B(H))$ and a unique Lipschitz homeomorphism $\varphi_g I_H : X \to X$ such that

$$\Phi(g \cdot I_H)(x) = \tau_g I_H(x)(g \cdot I_H(\varphi_g I_H(x))) = (g(\varphi_g I_H(x)) I_H) I_H \quad (x \in X).$$

(6)

Let $\Psi : F_\alpha(X) \to F_\alpha(X)$ be the map given by $\Psi(g) = g \circ \varphi_g I_H$ for all $g \in F_\alpha(X)$. By the uniqueness of $\varphi_g I_H$ and (6), $\Psi$ is well-defined and, clearly, it is linear and continuous. Notice that $\Psi$ is a local $^*$-automorphism of $F_\alpha(X)$, and since $\text{Aut}^*(F_\alpha(X))$ is algebraically reflexive by Theorem 5.1, we deduce that
there is a Lipschitz homeomorphism \( \varphi: X \to X \) such that \( \Psi(g) = g \circ \varphi \) for all \( g \in F_\alpha(X) \). Then we can rewrite (6) as

\[
\Phi(g \cdot I_H)(x) = g(\varphi(x))I_H \quad (g \in F_\alpha(X), \ x \in X).
\]

(7)

Fix a function \( g \in F_\alpha(X)^{-1} \) and a point \( x \in X \), and consider \( \Phi_{g,x}: \mathcal{B}(H) \to \mathcal{B}(H) \) defined by

\[
\Phi_{g,x}(a) = \Phi(g \cdot a)(x) \quad (a \in \mathcal{B}(H)).
\]

(8)

Clearly, \( \Phi_{g,x} \) is linear and continuous. Since \( \Phi \) is a local \(^*\)-automorphism of \( F_\alpha(X, \mathcal{B}(H)) \), from Theorems 4.2 and 4.3, for each \( a \in \mathcal{B}(H) \) there exist a Lipschitz homeomorphism \( \varphi_a \) of \( X \), a complex number \( g(\varphi_a(x)) \) and an \(^*\)-automorphism \( \tau_a(x)(a) \) of \( \mathcal{B}(H) \) such that

\[
\Phi_{g,x}(a) = \Phi(g \cdot a)(x) = \tau_a(x)(g \cdot a(\varphi_a(x))) = g(\varphi_a(x))\tau_a(x)(a).
\]

Then, by Lemma 6.1, there are \( \lambda_{g,x} \in \mathbb{C} \) and \( \tau_{g,x} \in \text{Aut}^*(\mathcal{B}(H)) \) for which

\[
\Phi_{g,x}(a) = \lambda_{g,x} \tau_{g,x}(a) \quad (a \in \mathcal{B}(H)).
\]

(9)

By using (7) and taking \( a = I_H \) in (8) and (9), we deduce that

\[
g(\varphi(x))I_H = \Phi(g \cdot I_H)(x) = \lambda_{g,x} \tau_{g,x}(I_H) = \lambda_{g,x} I_H,
\]

(10)

and thus \( g(\varphi(x)) = \lambda_{g,x} \). Now from (9), we obtain

\[
\Phi(g \cdot a)(x) = g(\varphi(x))\tau_{g,x}(a) \quad (a \in \mathcal{B}(H)).
\]

Since \( g \) and \( x \) are arbitrary, we have proved that

\[
\Phi(g \cdot a)(x) = g(\varphi(x))\tau_{g,x}(a) \quad (g \in F_\alpha(X)^{-1}, \ x \in X, \ a \in \mathcal{B}(H)).
\]

(11)

Now let \( x \in X \) and \( g_1, g_2 \in F_\alpha(X)^{-1} \). By (11) and (5), we get that

\[
g_1(\varphi(x))\tau_{g_1,x}(a) + g_2(\varphi(x))\tau_{g_2,x}(a) = \Phi(g_1 \cdot a)(x) + \Phi(g_2 \cdot a)(x)
\]

\[
= \Phi((g_1 + g_2) \cdot a)(x) = (g_1 + g_2)(\varphi_{(g_1 + g_2)})(a) = \tau_{(g_1 + g_2)}(a)(x)(a)
\]

for every \( a \in \mathcal{B}(H) \). By Lemma 6.2, it follows that \( \tau_{g_1,x} = \tau_{g_2,x} \). Therefore \( \tau: X \to \text{Aut}^*(\mathcal{B}(H)) \) given by \( \tau(x) = \tau_{g,x} \) for some \( g \in F_\alpha(X)^{-1} \) is well-defined. From (11) we infer

\[
\Phi(g \cdot a)(x) = \tau(x)(g \cdot a(\varphi(x))) \quad (g \in F_\alpha(X)^{-1}, \ x \in X, \ a \in \mathcal{B}(H)).
\]

(12)
To see that $\tau$ is a Lipschitz map from $(X, d^\alpha)$ into $\text{Aut}^*(B(\mathcal{H}))$, let $x, y \in X$. If we set $g = 1_X$ in (12), we have

$$
\| (\tau(x) - \tau(y))(a) \| = \| \Phi(1_X \cdot a)(x) - \Phi(1_X \cdot a)(y) \|
\leq L_\alpha(\Phi(1_X \cdot a))d(x, y)^\alpha \leq \| \Phi \| \| a \| d(x, y)^\alpha
$$

for all $a \in B(\mathcal{H})$, and thus $\| \tau(x) - \tau(y) \| \leq \| \Phi \| d(x, y)^\alpha$.

Since every function in $F_\alpha(X)$ can be expressed as a linear combination of functions in $F_\alpha(X) - 1$, from (12) we deduce

$$
\Phi(g \cdot a)(x) = \tau(x)(g \cdot a(\varphi(x))) \quad (g \in F_\alpha(X), \ x \in X, \ a \in B(\mathcal{H})). \tag{13}
$$

As $\Phi$ is linear and continuous for the supremum norm, Lemma 6.3 together with (13) yield

$$
\Phi(f)(x) = \tau(x)(f(\varphi(x))) \quad (f \in F_\alpha(X, B(\mathcal{H})), \ x \in X).
$$

In view of Theorems 4.2 and 4.3, $\Phi$ is a *-automorphism of $F_\alpha(X, B(\mathcal{H}))$, and the proof is complete. \qed

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