Local distribution of the parts of unequal partitions in arithmetic progressions II

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1. Introduction

This paper contains the main parts of the proofs of the results announced in [6]. We recall below some notations and the main result of this paper but we recommend to the reader to study first [6]. Let $d \in \mathbb{N}^*$, $\mathcal{D}$ a non-empty subset of $\{1, \ldots, d\}$ and $\mathcal{D}^c = \{1, \ldots, d\} \setminus \mathcal{D}$ its complement. Let $\mathcal{R}_D = \{N_r : r \in \mathcal{D}\}$ be a multiset of $|\mathcal{D}|$ non-negative integers. The main goal of our work is to obtain an asymptotic formula for $\Pi^*_d(n, \mathcal{R}_D)$, the number of unequal partitions of $n$ with exactly $N_r$ parts congruent to $r$ modulo $d$ for all $r \in \mathcal{D}$. We adopt the convention $\Pi^*_d(0, \mathcal{R}_D) = 1$ if $\mathcal{R}_D = \{0, \ldots, 0\}$ and 0 otherwise.

Recall that if $n \geq 1$ and $\Pi^*_d(n, \mathcal{R}_D) \geq 1$ then $n$ satisfies

$$n \equiv R_D \pmod{\delta}, \quad (1.1)$$

where $R_D = \sum_{r \in \mathcal{D}} rN_r$ and $\delta$ is the g. c. d. of the elements of $\mathcal{D}^c \cup \{d\}$. In the introduction of [6], we observed that the $N_r, r \in \mathcal{D}$ may be expected to be close to $k_0$ with

$$k_0 := \frac{2 \sqrt{3} \log 2 \sqrt{n}}{\pi \frac{\sqrt{n}}{d}}. \quad (1.2)$$

More precisely we suppose that for all $r \in \mathcal{D}$ we have

$$|N_r - k_0| \leq \frac{n^{\frac{1}{4}} \sqrt{\log n}}{d^{1/3} |\mathcal{D}|^{2/3} w(n)}, \quad (1.3)$$

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where \( w(n) \) is a non-decreasing function such that \( w(n) \to \infty \) if \( n \to \infty \). Let us recall the main result of [6].

**Theorem 1.1.** Let \( \varepsilon > 0 \). The following two propositions hold.

(i) Let \( d \leq n^{1/4-\varepsilon} \), \( \mathcal{D} = \{1, \ldots, d\} \) and \( n \equiv R_{\mathcal{D}} \) (mod \( d \)). Let \( \mathcal{R} = R_{\mathcal{D}} = \{N_1, \ldots, N_d\} \) be a multiset of integers satisfying (1.3). Then we have

\[
\Pi^*_d(n, R_{\mathcal{D}}) = (1 + o(1))q(n) \frac{d}{\sqrt{1 - \frac{12(\log 2)^2}{\pi^2}}} \left(1 - \frac{d}{2\sqrt{3n}}\right)^{d/2}
\times \exp \left\{ - \frac{2\sqrt{3}\log^2 2}{\pi(1 - \frac{12(\log 2)^2}{\pi^2})\sqrt{n}} \left( \sum_{r=1}^{d} (N_r - k_0)^2 \right) - \frac{\pi d}{2\sqrt{3n}} \sum_{r=1}^{d} (N_r - k_0)^2 \right\}.
\]

(ii) We suppose now that \( d \leq n^{1/6-\varepsilon} \) and \( \mathcal{D} \subset \{1, \ldots, d\} \). Then under (1.1) and (1.3) we have

\[
\Pi^*_d(n, R_{\mathcal{D}}) = q(n) \frac{\delta(1 + o(1))}{\sqrt{1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}}} \left( \frac{d}{2\sqrt{3n}} \right)^{|\mathcal{D}|/2}
\times \exp \left( - \frac{2\sqrt{3}(\log 2)^2}{\pi(1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2})\sqrt{n}} \left( \sum_{r \in \mathcal{D}} (N_r - k_0)^2 \right) - \frac{\pi d}{2\sqrt{3n}} \sum_{r \in \mathcal{D}} (N_r - k_0)^2 \right).
\]

First we complete the proof of Theorem 1.1 in the case \( \mathcal{D} = \{1, \ldots, d\} \), after we will handle the complementary case when \( \mathcal{D}^c \neq \emptyset \). The last sections are devoted to the proofs of the different corollaries of [6].

### 2. The term \( S_2 \)

We begin to assume that

\[
d \leq n^{1/2-\varepsilon}
\]

with some fixed positive \( \varepsilon \) and

\[
|k - k_0| = o \left( \frac{\sqrt{n}}{d} \right).
\]

Let

\[
x_0 := \frac{\pi}{2\sqrt{3n}}, \quad t := dx_0.
\]

Then

\[
k_0t = k_0dx_0 = \log 2.
\]
We also suppose that

\[ |N_r - k_0| = o \left( \frac{\sqrt{n}}{d} \right) \]  \hspace{1cm} (r = 1, \ldots, d). \tag{2.5} \]

In Section 4 of [6] we proved that as \( n \to \infty \) then we have

\[ \prod_{r=1}^{d} g_{N_r}(dx_0) = \exp \left( \frac{\pi^2}{12x_0} + \frac{(\log 2)^2}{2x_0} + o(\sqrt{n}) \right). \tag{2.6} \]

According to the notations of [6] Sections 3 and 4, we have

\[ |S_2| \leq \frac{d}{2\pi} \int_{3\pi x_0 \leq |y| \leq \pi/d} \left\{ \prod_{r=1}^{d} |g_{N_r}(d(x_0 + iy))| \right\} \exp((n - R - Q)x_0) dy. \]

The main part of this section is the following lemma.

**Lemma 2.1.** Under the notations and hypotheses (1.2), (2.1), (2.2), (2.3), (2.4), and \( 3\pi x_0 \leq |y| \leq \pi/d \), we have

\[ |g_k(d(x_0 + iy))| \leq g_k(dx_0) \exp \left( -1 + o(1) \right). \]

**Proof.** This time we start out from the first expression of \( g_k \) and develop the logarithms:

\[
g_k(w) = \exp \left( \sum_{\nu=1}^{k} \log \frac{1}{1 - \exp(-\nu w)} \right) = \exp \left( \sum_{\nu=1}^{k} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-\nu mw) \right)
\]

\[
= \exp \left( \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\nu=1}^{k} \exp(-\nu mw) \right)
\]

\[
= \exp \left( \sum_{\nu=1}^{k} \exp(-\nu w) + \sum_{m=2}^{\infty} \frac{1}{m} \sum_{\nu=1}^{k} \exp(-\nu mw) \right). \tag{2.7} \]

We take the moduli

\[
|g_k(w)| \leq \exp \left( \left| \sum_{\nu=1}^{k} \exp(-\nu w) \right| + \sum_{m=2}^{\infty} \frac{1}{m} \sum_{\nu=1}^{k} \exp(-\nu mt) \right)
\]

\[
= g_k(t) \exp \left( \left| \sum_{\nu=1}^{k} \exp(-\nu w) \right| - \sum_{\nu=1}^{k} \exp(-\nu t) \right)
\]
\[ g_k(t) \exp \left( \frac{1 - \exp(-kw)}{|\exp(w) - 1|} - \frac{1 - \exp(-kt)}{|\exp(t) - 1|} \right) \]

\[ \leq g_k(t) \exp \left( \frac{1 + \exp(-kt)}{|\exp(w) - 1|} - \frac{1 - \exp(-kt)}{|\exp(t) - 1|} \right). \]

When \(|\Im w| \leq \pi\),

\[ |\exp(w) - 1|^2 = (|\exp(t) - 1|^2 + 4e^t(\sin(b/2))^2 \geq 4(\sin(b/2))^2 \geq \frac{4|b|^2}{\pi^2}, \]

thus

\[ |g_k(w)| \leq g_k(t) \exp \left( \frac{1 + \exp(-kt)}{\frac{2}{3}|\Im w|} - \frac{1 - \exp(-kt)}{|\exp(t) - 1|} \right) \]

if \(|\Im w| \leq \pi\). Therefore, \(3\pi x_0 \leq |y| \leq \pi/d\) implies that

\[ |g_k(dx_0 + iy)| \leq g_k(dx_0) \exp \left( \frac{1 + \exp(-kdx_0)}{\frac{2}{6}d|y|} - \frac{1 - \exp(-kdx_0)}{|\exp(dx_0) - 1|} \right) \]

\[ \leq g_k(dx_0) \exp \left( \frac{1 + \exp(-kdx_0)}{6dx_0} - \frac{1 - \exp(-kdx_0)}{|\exp(dx_0) - 1|} \right). \]

By (1.2), (2.1), (2.2), (2.3), (2.4),

\[ |g_k(dx_0 + iy)| \leq g_k(dx_0) \exp \left( \frac{\frac{3}{2} + o(1)}{6dx_0} - \frac{\frac{1}{2} + o(1)}{|\exp(dx_0) - 1|} \right) \]

\[ = g_k(dx_0) \exp \left( \frac{\frac{3}{2} + o(1)}{6dx_0} - \frac{\frac{1}{2} + o(1)}{dx_0} + O(1) \right) = g_k(dx_0) \exp \left( -\frac{1 + o(1)}{4dx_0} \right). \]

This ends the proof of Lemma 2.1. \(\square\)

By (2.5) we obtain for \(S_2\),

\[ |S_2| \leq \left\{ \prod_{r=1}^{d} g_{\lambda_r}(dx_0) \right\} \exp \left( -\frac{1 + o(1)}{4x_0} \right) \exp((n - R - Q)x_0) \]

\[ = \exp \left( \frac{\pi \sqrt{u}}{\sqrt{3}} - \frac{\sqrt{3u}}{2\pi} + o(\sqrt{u}) \right), \quad (2.8) \]

by (2.6) and according to the estimates of \(Q\) and \(R\) obtained in Sections 2 and 4 of [6].
3. The term $S_1$

Next, we will try to give a similar and simple estimation for $n^{-\frac{5}{8} + \varepsilon} \leq |y| \leq 3\pi x_0$.

**Lemma 3.1.** (i) Under the notations and hypotheses (1.2), (2.1), (2.2), (2.3), (2.4), for $n^{-\frac{5}{8} + \varepsilon} \leq |y| \leq 3\pi x_0$ we have

$$|g_k(d(x_0 + iy))| \leq g_k(dx_0) \exp \left( -\frac{\sqrt{3}n^{1/4+2\varepsilon}}{27\pi^3d} \right).$$

(ii) Under the notations and hypotheses (1.2), (2.1), (2.2), (2.3), (2.4), for $n^{-\frac{5}{8} + \varepsilon} \leq |y| \leq n^{-\frac{5}{8} + \varepsilon}$ we have

$$|g_k(d(x_0 + iy))| \leq g_k(dx_0) \exp \left( -\frac{\sqrt{3}n^{2\varepsilon/3}}{27\pi^3d} \right).$$

**Proof.** First we prove (i). We suppose that $n^{-\frac{5}{8} + \varepsilon} \leq |y| \leq 3\pi x_0$. By (2.7) we have

$$g_k(w) = \exp \left( \sum_{\nu=1}^{k} \exp(-\nu w) + \sum_{m=2}^{\infty} \frac{1}{m} \sum_{\nu=1}^{k} \exp(-\nu mw) \right).$$

We study again $|g_k(w)|$:

$$|g_k(w)| \leq \exp \left( \Re \left( \sum_{\nu=1}^{k} \exp(-\nu w) \right) + \sum_{m=2}^{\infty} \frac{1}{m} \exp(-\nu mt) \right)$$

$$= g_k(t) \exp \left( \sum_{\nu=1}^{k} \Re \left( \exp(-\nu w) \right) - \sum_{\nu=1}^{k} \exp(-\nu t) \right)$$

$$= g_k(t) \exp \left( \sum_{\nu=1}^{k} \exp(-\nu t)(\Re \exp(-\nu b) - 1) \right)$$

$$= g_k(t) \exp \left( \sum_{\nu=1}^{k} \exp(-\nu t)(\cos(\nu b) - 1) \right)$$

$$= g_k(t) \exp \left( -2 \sum_{\nu=1}^{k} \exp(-\nu t) \sin^2 \left( \frac{\nu|b|}{2} \right) \right).$$

Let $K_0 := \left\lfloor \frac{\sqrt{n}}{3\log 2} \right\rfloor$. If $k = k_0 + o\left(\frac{\sqrt{n}}{n}\right)$ then $k > K_0$ for $n$ large enough:

$$|g_k(w)| \leq g_k(t) \exp \left( -2 \sum_{\nu=1}^{K_0} \exp(-\nu t) \sin^2 \left( \frac{\nu|b|}{2} \right) \right)$$

$$\leq g_k(t) \exp \left( -2 \sum_{\nu=1}^{K_0} \exp(-k_0 t) \sin^2 \left( \frac{\nu|b|}{2} \right) \right) = g_k(t) \exp \left( -\sum_{\nu=1}^{K_0} \sin^2 \left( \frac{\nu|b|}{2} \right) \right).$$
Writing \( w = d(x_0 + iy) \) and using the inequality \( |\sin t| \geq \frac{2|t|}{\pi} \) for \(|t| \leq \pi/2\), we obtain

\[
|g_k(d(x_0 + iy))| \leq g_k(dx_0) \exp \left( - \sum_{\nu=1}^{K_0} \left( \frac{\nu d|y|}{\pi} \right)^2 \right)
\]

since

\[
\frac{\nu d|y|}{\pi} \leq K_0 d3\pi x_0 = \frac{k_0 d^{3/2}}{2 \log 2} \leq \frac{\pi}{2}.
\]

Since \( \sum_{\nu=1}^{K_0} \nu^2 \geq \frac{K_0^3}{3} \), we have

\[
|g_k(d(x_0 + iy))| \leq g_k(dx_0) \exp \left( - \frac{\sqrt{3}n^{1/4+2\varepsilon}}{\pi^{3/2}d} \right) \exp((n - R - Q)x_0).
\]

(ii) can be obtained similarly.

By (2.5),

\[
\frac{d}{2\pi} \int_{n^{-\frac{3}{4} + \varepsilon} \leq |y| \leq 3\pi x_0} \left\{ \prod_{r=1}^{d} g_{N_r}(d(x_0 + iy)) \right\} \exp((n - R - Q)(x_0 + iy))dy \leq \frac{d}{2\pi} \int_{n^{-\frac{3}{4} + \varepsilon} \leq |y| \leq 3\pi x_0} \left| \prod_{r=1}^{d} g_{N_r}(d(x_0 + iy)) \right| \exp((n - R - Q)x_0)dy
\]

\[
\leq \left\{ \prod_{r=1}^{d} g_{N_r}(dx_0) \right\} \exp \left( - \frac{\sqrt{3}n^{1/4+2\varepsilon}}{\pi^{3/2}d} \right) \exp((n - R - Q)x_0).
\]

We have to stop here since the previously error term \( o(\sqrt{n}) \) is rough. Otherwise the above proof can be applied, e.g., for \( n^{-\frac{3}{4} + \varepsilon} \leq |y| \leq n^{-\frac{3}{4} + \varepsilon} \) and results that

\[
\frac{d}{2\pi} \int_{n^{-\frac{3}{4} + \varepsilon} \leq |y| \leq n^{-\frac{3}{4} + \varepsilon}} \left\{ \prod_{r=1}^{d} g_{N_r}(d(x_0 + iy)) \right\} \exp((n - R - Q)(x_0 + iy))dy
\]

\[
\leq \left\{ \prod_{r=1}^{d} g_{N_r}(dx_0) \right\} \exp \left( - \frac{\sqrt{3}n^{2\varepsilon/3}}{\pi^{5/3}} \right) \exp((n - R - Q)x_0).
\]

Finally we obtain for \( S_1 \):

\[
|S_1| \ll \left\{ \prod_{r=1}^{d} g_{N_r}(dx_0) \right\} \exp \left( - \frac{\sqrt{3}n^{2\varepsilon/3}}{\pi^{5/3}} \right) \exp((n - R - Q)x_0). \tag{3.1}
\]
4. The function $g_k$ in the range $|y| < y_1$

Let $|y| \leq y_1 = n^{-\frac{1}{2} + \varepsilon}$, $w = t + ib = dx_0 + idy$. Now $\frac{|b|}{t} = O(n^{-\frac{1}{2} + \varepsilon})$.

In this section we work with a general subset $D \subset \{1, \ldots, d\}$. Instead of (2.2) and (2.5), we suppose that

$$|k - k_0| \leq n^{-\frac{2}{3}} \sqrt{\log n} \frac{d^{1/3} |D|^{2/3} w(n)}{d^{1/3} |D|^{2/3} w(n)} \quad (r \in D) \quad (4.1)$$

where $w(n)$ is a non-decreasing function such that $w(n) \to \infty$ if $n \to \infty$. The aim of this paragraph is to obtain an asymptotic formula for $g_k(x_0 + iy)$ for $|y| \leq y_1$.

Instead of (2.1), we suppose that $d \leq n^{1 - \frac{2}{3}\varepsilon}$. (4.2)

Thus (4.1) implies (2.2) and (2.5). We will prove the following Lemma:

**Lemma 4.1.** Under (4.1) we have

$$g_k(w) = f(w) \exp \left\{ -\frac{C_2}{t} + (k - k_0) \log 2 - \frac{(k - k_0)^2 t}{2} + \log 2 \right\} \left\{ + ib \left( \frac{C_2}{t^2} + \frac{k_0 \log 2}{t} - k_0 (k - k_0) \right) + t^2 \left( \frac{C_2}{t^3} + \frac{k_0 \log 2}{t^2} + \frac{k_0^2}{2t} \right) + O \left( \frac{n^{-\varepsilon}}{|D|} \right) \right\}.$$

We have again by Lemma 4.1 of [6]

$$g_k(w) = f(w) \exp \left\{ -\frac{1}{w} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmw) + \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmw) + O(k^{-1}) \right\}.$$

By (1.2) and (2.2),

$$\frac{1}{k} = O \left( \frac{d}{\sqrt{n}} \right) = O \left( \frac{1}{d \sqrt{n}} \right).$$

Then $\frac{1}{t} = O(n^{-\frac{2}{3}\varepsilon}/d)$ and $\frac{|b|}{t} = o(n^{-\varepsilon} d^{-1})$. Since now $|y| \leq y_1$, it is possible to replace $w$ by $t$ in the different $\exp(-kmw)$, in cost of an admissible error term:

**Lemma 4.2.** (i) We have

$$\sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmw) = \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmw) + O \left( \frac{|b|}{t} \right).$$

(ii) We have

$$\frac{1}{w} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmw) = \frac{1}{w} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmw) - \frac{kib}{w} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmw)$$

$$- \frac{k^2 b^2}{2w} \sum_{m=1}^{\infty} \exp(-kmw) + O \left( \frac{k^3 |b|^3}{|w|} \right) \sum_{m=1}^{\infty} m \exp(-kmw).$$
Proof. By standard approximations we have
\[
\sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmw) = \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmw) \exp(-kmt)
\]
\[
= \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) + \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt)O(km|b|)
\]
\[
= \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) + O \left( \frac{|b|}{t} \right) = \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) + O \left( \frac{|b|}{t} \right).
\]
This proves (i). Next we prove (ii). We have
\[
\frac{1}{w} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmw) = \frac{1}{w} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmt) \exp(-ikmb)
\]
\[
= \frac{1}{w} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmt) \left\{ 1 - ikmb - \frac{(kmb)^2}{2} + O((km|b|)^3) \right\}.
\]
It remains to develop to end the proof of Lemma 4.2.

We also have
\[
\sum_{m=1}^{\infty} \frac{m \exp(-kmt)}{(1 - \exp(-kmt))^2} = \frac{1}{(k \exp(-|b|))} \leq \frac{1}{(kt)^2},
\]
since for \( u > 0, \)
\[
e^u - 1 = u \sum_{n=0}^{\infty} \frac{u^n}{(n+1)!} > u \sum_{n=0}^{\infty} \frac{u^n}{n!2^n} = ue^u,
\]
thus \((1 - e^{-u}) > we^{-u/2} and (e^u - 1)(1 - e^{-u}) > u^2.\)

This gives for \( g_k(w):\)
\[
g_k(w) = f(w) \exp \left\{ - \frac{1}{w} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmt) + \frac{ikb}{w} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) + \frac{k^2b^2}{2w} \sum_{m=1}^{\infty} \exp(-kmt) + \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) + \frac{n^e}{|b|d} \right\} + O \left( \frac{|b|^3}{t^3} \right).
\]

The next step of the proof of Lemma 4.1 consists of “replacing” \( \frac{1}{w} \) by \( \frac{1}{t} \) and computing the terms arisen by this manipulation. We use the formula
\[
\frac{1}{w} = \frac{1}{t(1 - (\frac{-i}{t} \frac{b}{t})} = \frac{1}{t} \left( 1 - \frac{b}{t} - \frac{b^2}{t^2} + O \left( \frac{|b|^3}{t^3} \right) \right).
\]
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This gives for $g_k(w)$

$$g_k(w) = f(w) \exp \left\{ -\frac{1}{t} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmt) + \frac{ib}{t^2} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) ight.$$ 

$$+ \frac{k^2}{t^3} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmt) + O \left( \frac{|b|^3}{t^4} \right) + \frac{ikb}{t} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt)$$

$$+ \frac{kb^2}{t^2} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) + O \left( \frac{k^2 |b|^3}{t^4 (e^{kt} - 1)} \right)$$

$$+ \frac{k^2 b^2}{2t} \sum_{m=1}^{\infty} \exp(-kmt) + O \left( \frac{k^3 |b|^3}{t^5} \right) \right\}.$$

We collect the terms with $ib$, the terms with $b^2$:

$$g_k(w) = f(w) \exp \left\{ -\frac{1}{t} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmt) + \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) ight.$$ 

$$+ \frac{ib}{t^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmt) + \frac{k}{t} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt)$$

$$+ \frac{k^2}{t^3} \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(-kmt) + \frac{k^3}{t^4} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) + \frac{k^2 b^2}{2t} \sum_{m=1}^{\infty} \exp(-kmt)$$

$$+ \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-kmt) + o \left( \frac{n^{-\varepsilon}}{d} \right) + O \left( \frac{k |b|^3}{t^4} \right) \right\}.$$ 

Now we compute the different summations over $m$. By (4.1), $\exp(-kmt)$ is close to $\exp(-k_0mt)$ if $m$ is not too large, but we have again some computations to do to control this approximation. For $s = 0, 1, 2$:

$$\sum_{m=1}^{\infty} \frac{1}{m^s} \exp(-kmt) = \sum_{m=1}^{\infty} \frac{1}{m^s} \exp(-(k-k_0)mt).$$

Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=1}^{\infty} \frac{x^n}{n!}$, we have

$$\left| e^x - \sum_{n=0}^{M} \frac{x^n}{n!} \right| \leq |x|^{M+1} \sum_{n=M+1}^{\infty} \frac{|x|^{n-M-1}}{n!} \leq |x|^{M+1} e^{\left| x\right|}.$$
Thus we obtain
\[
\sum_{m=1}^{\infty} \frac{1}{m^s} \exp(-kmt) = \sum_{m=1}^{\infty} \frac{1}{m^s} \exp(-k_0mt) \left\{ 1 - (k - k_0)mt + \frac{1}{2}(k - k_0)^2 m^2 t^2 \right. \\
- \frac{1}{6}(k - k_0)^3 m^3 t^3 + O(|k - k_0|^4 m^4 t^4 \exp(|k - k_0|mt)).
\]

By (2.2), \( \exp(|k - k_0|mt) \leq \exp(\frac{m k_0 t}{2}) \). Next we use the fact that \( k_0 t = \log 2 \) :
\[
\sum_{m=1}^{\infty} \frac{1}{m^s} \exp(-kmt) = \sum_{m=1}^{\infty} \frac{1}{m^s 2^m} \left( 1 - (k - k_0)mt + \frac{1}{2}(k - k_0)^2 m^2 t^2 \right. \\
- \frac{1}{6}(k - k_0)^3 m^3 t^3 + O(|k - k_0|^4 m^4 t^4 2^{m/2}) \\
\left. - \frac{(k - k_0)^3}{6} t^4 \sum_{m=1}^{\infty} \frac{2^{-m}}{m^5} + O\left(|k - k_0|^4 t^4 \sum_{m=1}^{\infty} \frac{2^{-m/2}}{m^{5/2}} \right) \right).
\]

We obtain for the function \( g_k \)
\[
g_k(w) = f(w) \exp \left\{ -\frac{1}{t} \sum_{m=1}^{\infty} \frac{1}{m^2 2^m} + (k - k_0) \sum_{m=1}^{\infty} \frac{1}{m 2^m} - \frac{(k - k_0)^2 t}{2} \sum_{m=1}^{\infty} 2^{-m} \right. \\
+ O(|k - k_0|^2 t^2) + \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m 2^m} + O(|k - k_0| t) \\
+ ib \left[ \frac{1}{t^2} \sum_{m=1}^{\infty} \frac{1}{m^2 2^m} - \frac{(k - k_0)^2 t}{2} \right. \sum_{m=1}^{\infty} \frac{1}{m 2^m} + O(|k - k_0| t^2) \\
\left. + \frac{k^2 t}{2} \sum_{m=1}^{\infty} \frac{1}{m^2 2^m} - k(k - k_0) \sum_{m=1}^{\infty} 2^{-m} + O(kt|k - k_0|^2) \right) \\
+ b^2 \left[ \frac{1}{t^3} \sum_{m=1}^{\infty} \frac{1}{m^2 2^m} + O\left( \frac{|k - k_0|}{t^2} \right) + \frac{k^2 t}{2} \sum_{m=1}^{\infty} \frac{1}{m^2 2^m} + O\left( \frac{|k - k_0|}{t^2} \right) \right] \\
\left. + \frac{k^2}{2t} \sum_{m=1}^{\infty} 2^{-m} + O(kt^2|k - k_0|) \right) + o(n^{-\frac{2}{3}} d^{-1}) + O\left( \left( k + \frac{1}{t} \right) \frac{|b|^3}{t^3} \right). \]

Next we compute the different sums on \( m \):
\[
g_k(w) = f(w) \exp \left\{ -\frac{C_2}{t} + (k - k_0) \log 2 - \frac{(k - k_0)^2 t}{2} + \frac{1}{2} \log 2 \right\}
\]
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\[ + ib \left( \frac{C_2}{t^2} + \frac{k - k_0}{t} \log 2 + \frac{k \log 2}{t} - k(k - k_0) \right) + b^2 \left( \frac{C_2}{t^3} + \frac{k}{t^2} \log 2 + \frac{k^2}{2t} \right) + O \left( |k - k_0|^2 t^2 + |k - k_0| t + |k - k_0|^2 |b| + \frac{|k - k_0| b^2}{t^2} \right) + o \left( \frac{n^{-\varepsilon}}{d^2} \right) + O \left( (k + 1) \frac{|b|^3}{t^3} \right). \]

Then by (4.1) the above error terms give \( o(n^{-\varepsilon} |D|^{-1}) \) and we can replace \(-k(k - k_0)\) with \(-k_0(k - k_0)\) in the coefficient of \(ib\) and analogously in that of \(b^2\). Finally,

\[ g_k(w) = f(w) \exp \left\{ - \frac{C_2}{t} + (k - k_0) \log 2 - \frac{(k - k_0)^2 t}{2} + \log 2 \right\} + ib \left( \frac{C_2}{t^2} + \frac{k_0 \log 2}{t} - k_0(k - k_0) \right) + b^2 \left( \frac{C_2}{t^3} + \frac{k_0 \log 2}{t^2} + \frac{k_0^2}{2t} \right) + o \left( \frac{n^{-\varepsilon}}{|D|} \right), \]

as claimed in Lemma 4.1.

\[ \square \]

5. The term \( S_0 \), end of the proof of Theorem 1.1

in the case \( D = \{1, \ldots, d\} \)

As a special case of Lemma 4.1 applied with \( D = \{1, \ldots, d\} \), we remark that

\[ g_k(dx_0) = f(dx_0) \exp \left\{ - \frac{C_2}{d} + (k - k_0) \log 2 - \frac{(k - k_0)^2 dx_0}{2} + \frac{\log 2}{2} \right\} + o \left( \frac{n^{-\varepsilon}}{d^2} \right), \]

and

\[ \prod_{r=1}^{d} g(N_r, dx_0) = f^d(dx_0) \exp \left\{ - \frac{C_2}{x_0} + \sum_{r=1}^{d} (N_r - k_0) \log 2 \right. \]

\[ \left. - \frac{dx_0}{2} \sum_{r=1}^{d} (N_r - k_0)^2 + \frac{d \log 2}{2} + o(n^{-\varepsilon}) \right\}. \] (5.1)

Since \( f(w) = \exp \left( \frac{\pi^2}{6w} + \frac{1}{2} \log \frac{w}{2\pi} + O(|w|) \right) \) for \( w \to 0 \) in \( |\arg w| \leq \kappa < \pi/2 \) and \( \Re w > 0 \), we have for \( |y| \leq y_1 \leq n^{-\frac{1}{2}} + \varepsilon \):

\[ f(dx_0 + iy) = \exp \left( \frac{\pi^2}{6(dx_0 + iy)} + \frac{1}{2} \log \left( \frac{d(dx_0 + iy)}{2\pi} \right) + O(dx_0) \right), \]

\[ f^d(dx_0 + iy) = \exp \left( \frac{\pi^2}{6(dx_0 + iy)} + \frac{d}{2} \log \left( \frac{d(dx_0 + iy)}{2\pi} \right) + O(d^2 dx_0) \right). \]
As a special case, we obtained that

\[
\frac{d}{dx_0} \left( \exp \left( \frac{\pi^2}{6x_0} \left( 1 - \frac{iy}{x_0} - \frac{y^2}{x_0^2} + O \left( \frac{|y|^3}{x_0^3} \right) \right) + \frac{d}{2} \log \left( \frac{dx_0}{2\pi} \right) \right) \right)
\]

\[
= \exp \left( \frac{\pi^2}{6x_0} \left( -iy \frac{\pi^2}{6x_0^2} - \frac{\pi^2y^2}{6x_0^2} + \frac{d}{2} \log \left( \frac{dx_0}{2\pi} \right) \right) + \frac{\epsilon}{6} \right).
\]

We obtain for the integrand

\[
P := \left\{ \frac{d}{dx_0} g_{N_r}(d(x_0 + iy)) \right\} \exp((n - R - Q)(x_0 + iy))
\]

\[
= \exp \left( \frac{\pi^2}{6x_0} - iy \frac{\pi^2}{6x_0^2} - \frac{\pi^2y^2}{6x_0^2} + \frac{d}{2} \log \left( \frac{dx_0}{2\pi} \right) + o(n^{-\epsilon}) \right)
\]

\[
+ \frac{d}{2} \log 2 + id \left( \frac{C_2}{x_0} + k_0 \log 2 \frac{dx_0}{x_0} - k_0 \sum_{r=1}^{d} (N_r - k_0) \right)
\]

\[
+ d^2 y^2 \left( \frac{C_2}{dx_0^2} + \frac{k_0 log 2}{x_0} + \frac{k_0^2}{2x_0} \right) + o(n^{-\epsilon}) \}.
\]

We collect terms in \( iy, y^2 \):

\[
P = \exp \left( \frac{\pi^2}{6x_0} + \frac{d}{2} \log \left( \frac{dx_0}{2\pi} \right) + (n - R - Q)x_0 \right)
\]

\[
- \frac{C_2}{x_0} + \sum_{r=1}^{d} (N_r - k_0) \log 2 - \frac{1}{2} \sum_{r=1}^{d} (N_r - k_0)^2 dx_0 + \frac{d \log 2}{2}
\]

\[
+ iy \left( n - R - Q - \frac{\pi^2}{6x_0} + \frac{C_2}{x_0^2} + \frac{dk_0 log 2}{x_0} - dk_0 \sum_{r=1}^{d} (N_r - k_0) \right)
\]

\[
+ iy \left( \frac{\pi^2}{6x_0^2} + \frac{C_2}{x_0^2} + \frac{dk_0 log 2}{x_0^2} + \frac{d^2 k_0^2}{2x_0} \right) + o(n^{-\epsilon}) \}. \]

As a special case, we obtained that

\[
\left\{ \frac{d}{dx_0} g_{N_r}(dx_0) \right\} \exp((n - R - Q)x_0) = \exp \left( \frac{\pi^2}{6x_0} + \frac{d}{2} \log \left( \frac{dx_0}{2\pi} \right) + (n - R - Q)x_0 \right)
\]

\[
- \frac{C_2}{x_0} + \sum_{r=1}^{d} (N_r - k_0) \log 2 - \frac{1}{2} \sum_{r=1}^{d} (N_r - k_0)^2 dx_0 + \frac{d \log 2}{2} + o(n^{-\epsilon}) \}. \quad (5.2)
\]
Consequently,
\[
\left\{ \prod_{r=1}^{d} g_{N_r}(d(x_0 + iy)) \right\} \exp((n - R - Q)(x_0 + iy)) = \left\{ \prod_{r=1}^{d} g_{N_r}(dx_0) \right\} \exp((n - R - Q)x_0)
\]
\[
\times \exp \left\{ i y \left(n - R - Q - \frac{\pi^2}{6x_0^2} + \frac{1}{x_0} \sum_{m=1}^{\infty} \frac{1}{m^2} 2^m + \frac{dk_0 \log 2}{x_0} - d \sum_{r=1}^{d} (N_r - k_0) \right) + y^2 \left(- \frac{\pi^2}{6x_0^2} + \frac{1}{x_0} \sum_{m=1}^{\infty} \frac{1}{m^2} 2^m + \frac{dk_0 \log 2}{x_0^2} + \frac{d^2 k_0^2}{2x_0^2}\right) + o(n^{-\varepsilon}) \right\}, (5.3)
\]

The coefficient of \(y^2\) in (5.3) is
\[
\left. \frac{1}{x_0^3} \left(- \frac{\pi^2}{6} + \frac{\pi^2}{12} - \frac{\log^2 2}{2} - \frac{1}{2} \frac{d^2 k_0^2 x_0^2}{2} \right) \right|_{x_0 = 0} = 2 \frac{2 \sqrt{3} \pi}{9} \left(1 - \frac{12 \log^2 2}{\pi^2}\right),
\]
where \(\frac{12 \log^2 2}{\pi^2} < 0.49 < \frac{6}{\pi^2} < 1.\)

The coefficient of \(iy\) in (5.3) is
\[
\left. n - R - Q - \frac{1}{x_0^2} \left( \frac{\pi^2}{6} - \frac{1}{2} \frac{d^2 k_0^2}{2} \right) - \frac{dk_0 \log 2}{x_0} - d \sum_{r=1}^{d} (N_r - k_0) \right|_{x_0 = 0} = -R + Q + \frac{1}{x_0^2} \frac{\log^2 2}{2} - \frac{1}{2} \frac{d^2 k_0^2}{2} - \frac{1}{2} \frac{dk_0}{x_0} \sum_{r=1}^{d} (N_r - k_0) = -2dk_0 \sum_{r=1}^{d} (N_r - k_0) + O(n^{\frac{3}{4} - 2\varepsilon}).
\]

Since \(|y|O(n^{\frac{3}{4} - 2\varepsilon}) = o(n^{-\varepsilon})\) we infer from (5.3) that
\[
\left\{ \prod_{r=1}^{d} g_{N_r}(d(x_0 + iy)) \right\} \exp((n - R - Q)(x_0 + iy)) = \left\{ \prod_{r=1}^{d} g_{N_r}(dx_0) \right\} \exp((n - R - Q)x_0)
\]
\[
\times \exp \left\{ - i y 2dk_0 \sum_{r=1}^{d} (N_r - k_0) + y^2 \frac{2 \sqrt{3} \pi}{\pi} \left(1 - \frac{12 \log^2 2}{\pi^2}\right) + o(n^{-\varepsilon}) \right\}, (5.4)
\]

Let \(A = \frac{2 \sqrt{3} \pi}{\pi} (1 - \frac{12 \log^2 2}{\pi^2}), (A > 0)\) and \(B = 2dk_0 \sum_{r=1}^{d} (N_r - k_0)\). Then, from (5.4)
\[
S_0 = \frac{d}{2\pi} \int_{-y_1}^{y_1} \left\{ \prod_{r=1}^{d} g_{N_r}(d(x_0 + iy)) \right\} \exp((n - R - Q)(x_0 + iy))dy
\]
\[ \frac{d}{2\pi} \left\{ \prod_{r=1}^{d} g_{\chi_r}(dx_0) \right\} \exp((n - R - Q)x_0) \int_{-y_1}^{y_1} \exp(-iyB - Ay^2 + o(n^{-\varepsilon}))dy. \]

**Lemma 5.1.** We have:

\[ \int_{-y_1}^{y_1} \exp(-iyB - Ay^2 + o(n^{-\varepsilon}))dy = \sqrt{\frac{\pi}{A}} \exp \left( -\frac{B^2}{4A} \right) \left\{ 1 + o(n^{-\varepsilon}) \exp \left( \frac{B^2}{4A} \right) \right\}. \quad (5.5) \]

**Proof.** These are standard manipulations on Gaussian integrals thus we won’t write all the details. Let \( I_{AB}(y_1) \) be the integral of the left hand side of (5.5). Since for \(|y| \leq y_1\), \( \exp(-iyB - Ay^2 + o(n^{-\varepsilon})) = (1 + o(n^{-\varepsilon})) \exp(-iyB - Ay^2) \), we have:

\[ I_{AB}(y_1) = \int_{-\infty}^{+\infty} \exp(-iyB - Ay^2)dy \]
\[ + O \left( \int_{y_1}^{+\infty} \exp(-Ay^2)dy \right) + o(n^{-\varepsilon}) \int_{-\infty}^{+\infty} \exp(-Ay^2)dy. \]

The main term is a Gaussian integral:

\[ \int_{-\infty}^{+\infty} \exp(-iyB - Ay^2)dy = \sqrt{\frac{\pi}{A}} \exp \left( -\frac{B^2}{4A} \right). \]

For the error terms we have

\[ \int_{y_1}^{+\infty} \exp(-Ay^2)dy \ll \frac{1}{Ay_1} \exp(-Ay_1^2) \quad \text{and} \quad \int_{-\infty}^{+\infty} \exp(-Ay^2) \ll \frac{1}{\sqrt{A}}. \]

the Lemma follows.

Furthermore

\[ \frac{B^2}{4A} = O \left( n^{-\frac{3}{2}} \left( \sqrt{md} n^{\frac{1}{2}} \frac{\sqrt{\log n}}{dw(n)} \right)^2 \right) = o(\log n). \]

Thus the error term in Lemma 5.1 is:

\[ o(n^{-\varepsilon}) \exp \left( \frac{B^2}{4A} \right) = o(1). \]
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Therefore

\[ S_0 = (1 + o(1)) \sqrt{\frac{\pi}{4}} \exp \left( -\frac{B^2}{4A} \right) \frac{d}{2\pi} \exp \left( \prod_{r=1}^{d} g_{N_r}(dx_0) \right) \exp((n - R - Q)x_0). \]

Adding the estimates for the trivial parts (see (2.8), (3.1)) we obtain that for \( n \equiv R \pmod{d} \),

\[ \Pi'_d(n, R) = \left\{ \prod_{r=1}^{d} g_{N_r}(dx_0) \right\} \exp((n - R - Q)x_0) \]

\[ \times \left\{ (1 + o(1)) \sqrt{\frac{\pi}{4}} \exp \left( -\frac{B^2}{4A} \right) + O \left( \frac{d}{2\pi} \exp \left( -\frac{1 + o(1)}{4x_0} \right) \right) \right\} \]

\[ + O \left( \frac{d}{\sqrt{2\pi}} \exp \left( -\frac{\sqrt{3}}{\pi^{5/3}} n^{2/3} \right) \right) \]

\[ = (1 + o(1)) \sqrt{\frac{\pi}{4}} \exp \left( -\frac{B^2}{4A} \right) \frac{d}{2\pi} \exp \left( \prod_{r=1}^{d} g_{N_r}(dx_0) \right) \exp((n - R - Q)x_0). \] (5.6)

To end the proof, it remains to insert the classical formula

\[ q(n) = (1 + o(1)) \frac{1}{4 \cdot 3^{1/4} n^{3/4}} \exp \left( \frac{\pi \sqrt{n}}{\sqrt{3}} \right), \] (5.7)

and our previous results on the \( g_{N_r}(dx_0) \) (see (5.2)), and to do the convenient computations. We obtain

\[ \Pi'_d(n, R) = (1 + o(1))\frac{1}{4 \cdot 3^{1/4} n^{3/4}} \exp \left( \frac{\pi \sqrt{n}}{\sqrt{3}} \right) \exp \left( -\frac{B^2}{4A} \right) \]

\[ + \frac{d}{2\sqrt{23^{1/4}}} \left( \frac{d}{2\sqrt{3n}} \right)^{d/2} \exp \left( -\frac{B^2}{4A} \right) \]

\[ + \sum_{r=1}^{d} (N_r - k_0) \log 2 - x_0 \left( R + Q - \frac{\log^2 2}{2x_0^2} \right) - \frac{1}{2} \sum_{r=1}^{d} (N_r - k_0)^2 + o(\sqrt{n}). \] (5.8)

By formulae (2.11) and (2.12) of [6] and by (2.4), we have:

\[ R + Q - \frac{\log^2 2}{2x_0^2} = \frac{d k_0}{2} + d k_0 \sum_{r=1}^{d} (N_r - k_0) + \frac{1}{2} \sum_{r=1}^{d} (N_r - k_0)^2 + o(\sqrt{n}). \]

Thus the argument of the exponential in (5.8) is

\[ \exp \left( -\frac{B^2}{4A} + \ldots \right) = \exp \left( -\frac{B^2}{4A} - \frac{\log 2}{2} - dx_0 \sum_{r=1}^{d} (N_r - k_0)^2 + o(1) \right). \]

Inserting this in (5.8) ends the proof of Theorem 1.1 for \( D = \{1, \ldots, d\} \). □
6. First steps of the proof of Theorem 1.1 for $D \neq \{1, \ldots, d\}$

Like in Section 3 of [6], we apply Lemma 2.1 of [6], the Cauchy formula and write $z = x_0 + iy$:

$$\Pi^*_D(n, \mathcal{R}_D) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \prod_{r \in D^c} h_r(x_0 + iy) \right\} \times \left\{ \prod_{r \in D} g_{N_r}(d(x_0 + iy)) \right\} \exp \left( (n - \mathcal{R}_D - Q_D)(x_0 + iy) \right) dy,$$

(6.1)

with

$$h_r(z) = \prod_{j=0}^{\infty} (1 + \exp(-r + jd)z).$$

When $D^c \neq \emptyset$ and $D^c \neq \{d\}$, the functions $h_r$ are not $2\pi/d$-periodic but we still split the integral in intervals of length $2\pi/d$ in order to use our previous work on the functions $g_k$.

Next we do some convenient change of variables:

$$\Pi^*_D(n, \mathcal{R}_D) = \frac{1}{2\pi} \sum_{|\lambda| \leq \lfloor \frac{d-1}{2} \rfloor} \int_{-\frac{\pi}{d} + 2\lambda \pi}^{\frac{\pi}{d} + 2\lambda \pi} \cdots + B,$$

with

$$B = \begin{cases} 0 & \text{if } d \text{ is odd} \\ \int_{-\frac{\pi}{d}}^{-\frac{\pi}{d} + \frac{\pi}{2}} \cdots + \int_{\frac{\pi}{d} - \frac{\pi}{2}}^{\frac{\pi}{d}} \cdots = \int_{\frac{\pi}{d} - \frac{\pi}{2}}^{\frac{\pi}{d}} \text{ if } d \text{ is even.} \end{cases}$$

Next we do some convenient change of variables:

$$\Pi^*_D(n, \mathcal{R}_D) = \frac{1}{2\pi} \sum_{-\frac{\pi}{d} \leq \lambda \leq \frac{\pi}{d} - \frac{\pi}{2}} \int_{-\frac{\pi}{d} + \frac{\pi}{2}}^{\frac{\pi}{d} - \frac{\pi}{2}} \left\{ \prod_{r \in D^c} h_r \left( x_0 + iy + i \frac{2\lambda \pi}{d} \right) \right\} \left\{ \prod_{r \in D} g_{N_r}(d(x_0 + iy)) \right\} \times \exp \left( (n - \mathcal{R}_D - Q_D)(x_0 + iy) + (n - \mathcal{R}_D) \frac{2i \lambda \pi}{d} \right) dy = \sum_{-\frac{\pi}{d} < \lambda \leq \frac{\pi}{d}} S(\lambda),$$

say. We will write the splitting

$$S(\lambda) = S_0(\lambda) + S_1(\lambda) + S_2(\lambda),$$

where in $S_0(\lambda)$ the range of integration for $y$ is $|y| \leq y_1$, in $S_1(\lambda)$ it is for $y_1 \leq |y| \leq y_2$ (with $y_2 = 3\pi x_0$) and in $S_2(\lambda)$ we take $y_2 \leq |y| \leq \frac{\pi}{d}$ (cf. (4.11) of [6]).
7. Upper bounds of $S_1(\lambda)$ and $S_2(\lambda)$

To obtain a convenient upper bound of these terms we first remark that

$$\left| h_r \left( x_0 + iy + \frac{2i\pi \lambda}{d} \right) \right| \leq h_r(x_0). \quad (7.1)$$

Let $j \in \mathbb{N}$. To prove (7.1) it is enough to prove that each $T_j \leq 1$ with

$$T_j := \frac{|1 + \exp \left( -(r + jd)(x_0 + iy + \frac{2i\pi \lambda}{d}) \right)|}{|1 + \exp(-(r + jd)x_0)|}.$$

By a simple computation we have

$$T_j^2 = 1 - \frac{4 \exp(-x_0(r + jd)) \sin^2 \left( \frac{y}{2}(r + jd) + \frac{\pi \lambda}{d} \right)}{\left(1 + \exp(-x_0(r + jd))\right)^2} \leq 1.$$

By Lemma 2.1 we have

$$|S_2(\lambda)| \leq \left\{ \prod_{r \in \mathcal{D}} g_{N_r}(dx_0) \right\} \left\{ \prod_{r \in \mathcal{D}^c} h_r(x_0) \right\} \times \exp((n - R_{\mathcal{D}} - Q_{\mathcal{D}})x_0) \exp \left( \frac{|\mathcal{D}|(-1 + o(1))}{4dx_0} \right).$$

By Lemma 3.1 we also have

$$|S_1(\lambda)| \leq \left\{ \prod_{r \in \mathcal{D}} g_{N_r}(dx_0) \right\} \left\{ \prod_{r \in \mathcal{D}^c} h_r(x_0) \right\} \times \exp((n - R_{\mathcal{D}} - Q_{\mathcal{D}})x_0) \exp \left( -\frac{|\mathcal{D}|\sqrt{3n^{2/3}}}{27\pi^3d} \right).$$

If $|\mathcal{D}|$ is small, i.e., if $|\mathcal{D}| \leq dn^{-\varepsilon/3}$ this last estimate for $S_1(\lambda)$ is not sufficient. However, by Lemma 3.1 (i), the contribution of the range $n^{-5/8+\varepsilon} \leq |y| \leq 3\pi x_0$ to $S_1(\lambda)$ is

$$\leq \left\{ \prod_{r \in \mathcal{D}} g_{N_r}(dx_0) \right\} \left\{ \prod_{r \in \mathcal{D}^c} h_r(x_0) \right\} \exp((n - R_{\mathcal{D}} - Q_{\mathcal{D}})x_0) \exp \left( -\frac{|\mathcal{D}|\sqrt{3n^{1/4+2\varepsilon}}}{27\pi^3d} \right).$$

Thus it remains to handle the case $|\mathcal{D}| \leq dn^{-\varepsilon/3}$ in the range $y_1 \leq |y| < n^{-5/8+\varepsilon}$. 
• First we study the case $\lambda = 0$. We use a similar argument as in the proof of Lemma 3.1.

For any $1 \leq r \leq d$, let $J_r$ denote the set of the integers $j$ such that

$$\frac{1}{2x_0} \leq r + jd \leq \frac{1}{x_0}. \quad (7.2)$$

Then for $j \in J_r$ we have

$$\frac{\exp(-x_0(r + jd))}{(1 + \exp(-x_0(r + jd)))^2} \geq \frac{1}{e(1 + e^{-1/2})^2}. \quad (7.3)$$

This gives for the correspondent $T_j$:

$$T_j^2 \leq 1 - \frac{4}{e(1 + e^{-1/2})^2} \sin^2 \left(\frac{y}{2}(r + jd)\right).$$

Next, quite like in the proof of Lemma 3.1, we have for $n$ large enough

$$|h_r(x_0 + iy)| \leq h_r(x_0) \left(1 - \frac{4}{e(1 + e^{-1/2})^2} \frac{y^2}{4\pi^2 x_0^2} \right)^{|J_r|/2} \leq h_r(x_0) \exp \left(-\frac{y^2}{48\pi^2 x_0^3} \right) \leq h_r(x_0) \exp \left(-\frac{\sqrt{3}n^{2/3}}{2\pi^3 d} \right).$$

This upper bound combined with Lemma 3.1 is sufficient to obtain

$$|S_1(0)| \leq \left\{ \prod_{r \in D} g_{N_r}(dx_0) \right\} \left\{ \prod_{r \in D^c} h_r(x_0) \right\} \exp((n-R_D-Q_D)x_0) \exp \left(-\frac{\sqrt{3}n^{2/3}}{27\pi^5} \right).$$

• Now we suppose that $\lambda \neq 0$. We write $\frac{1}{d} = \frac{1}{d'}$ with $(\lambda', d') = 1$ and $d' > 0$.

First we suppose that $d' \geq n^{1/4}$. Since $(\lambda', d') = 1$, there are $d' + O(1)$ integers $r_0 \in \{1, \ldots, d'\}$ such that $\frac{\lambda r_0}{d'} \mod 1 \in \left[\frac{1}{4}, \frac{1}{2}\right]$. Thus there are $\frac{d'}{4} + O\left(\frac{d'}{\pi^2}\right)$ integers $r \in \{1, \ldots, d\}$ such that $\frac{\lambda r}{d'} \mod 1 \in \left[\frac{1}{4}, \frac{1}{2}\right]$ (again for $n$ large enough).

Since $|D| \leq dn^{-e/3} < \frac{d}{4} + O\left(\frac{d}{\pi^2}\right)$ for $n$ large enough, there exists $r_1 \in D^c$ such that $\frac{\lambda r_1}{d'} \mod 1 \in \left[\frac{1}{4}, \frac{1}{2}\right]$. For $j \in J_{r_1}$, $|(r_1 + jd)y| \ll n^{-1/8 + \epsilon}$. Thus

$$\sin^2 \left(\frac{(r_1 + jd)y}{2} + \frac{\lambda r_1 x_0}{d'}\right) \geq \sin^2 \frac{\pi}{4} = \frac{1}{4}.$$

This gives

$$\prod_{j \in J_{r_1}} |T_j(r_1)|^2 \leq \left(\frac{11}{12}\right)^{|J_{r_1}|} \leq \exp \left(-\frac{1}{30dx_0} \right),$$

which is a sufficient upper bound.
Now we suppose that $2 \leq d' < n^{\varepsilon/4}$. There are $d - \frac{d}{d'}$ integers $r$ such that $d' \nmid r$. For these integers $r$ and $j \in J_r$, we have

$$\sin^2 \left( \frac{(r + jd)y}{2} + \frac{\pi \lambda r}{d} \right) \geq \sin^2 \left( \frac{\pi}{2d'} \right) \geq \frac{1}{d^2}.$$

Since $|D| \leq dn^{-\varepsilon/3}$, there are at least $d/3$ such integers $r \in D$ such that $d' \nmid r$. Thus we have:

$$\prod_{r \in D^c} \left| h_r(x_0 + iy + \frac{2i\pi \lambda}{d}) \right| \leq \prod_{r \in D^c} \prod_{j \in J_r} \left( 1 - \frac{1}{6d^2} \right) \leq \exp \left( -\frac{d}{48d^3x_0} \right),$$

which is a sufficient upper bound when $d \leq n^{1/4 - 2\varepsilon}$.

8. The terms $S_0(\lambda)$ for $\lambda \neq 0$

We have to consider the integrals

$$S_0(\lambda) = \frac{1}{2\pi} \int_{|y| \leq y_1} \left\{ \prod_{r \in D^c} h_r \left( x_0 + iy + \frac{2i\pi \lambda}{d} \right) \right\} \left\{ \prod_{r \in D} g_{N_r}(d(x_0 + iy)) \right\} \times \exp \left( (n - R_D - Q_D)(x_0 + iy + \frac{2i\pi \lambda}{d}) \right) dy.$$

- First we suppose that there exists $r \in D^c$ such that $\lambda r \not\equiv 0 \pmod{d}$. In the previous section we remarked that

$$\left| h_r \left( x_0 + iy + \frac{2i\pi \lambda}{d} \right) \right| = h_r(x_0) \prod_{j=0}^{\infty} \left( 1 - \frac{4\exp(-x_0(r + jd))\sin^2 \left( \frac{y}{2}(r + jd) + \frac{\pi \lambda r}{d} \right)}{(1 + \exp(-x_0(r + jd))^2) \right)^{1/2}.$$

We work again with the sets $J_r$ defined by (7.2) in the previous section. For $j \in J_r$, $\lambda r \equiv a \pmod{d}$, $1 \leq |a| \leq d/2$ and $|y| \leq y_1$ we have:

$$\sin^2 \left( \frac{\pi |a|}{d} + \frac{y}{2}(r + jd) \right) \geq \sin^2 \left( \frac{\pi |a|}{d} - \frac{y_1}{x_0} \right) \geq \sin^2 \left( \frac{\pi}{d} - \frac{y_1}{x_0} \right) \geq \frac{3}{d^2},$$
for \( n \) large enough (we recall that \( d \leq n^{1/4-2\varepsilon} \)).

Since \( |J_r| = \frac{1}{2d x_0} + O(1) \geq \frac{1}{3dx_0} \), we have:

\[
\left| h_r\left(x_0 + iy + \frac{2i\pi \lambda}{d}\right)\right| \leq h_r(x_0) \prod_{j \in J_r} \left(1 - \frac{12}{ed^2(1 + e^{-1/2})^2}\right)^{1/2}
\leq h_r(x_0) \exp\left(\frac{1}{6dx_0} \log\left(1 - \frac{12}{ed^2(1 + e^{-1/2})^2}\right)\right) \leq h_r(x_0) \exp\left(-\frac{2}{ed^4x_0(1 + e^{-1/2})^2}\right).
\]

Thus if there exists \( r_0 \in D^c \) such that \( \lambda r_0 \not\equiv 0 \pmod{d} \) then

\[
|S_0(\lambda)| \leq \left\{ \prod_{r \in D} g_{N_r}(dx_0) \right\} \left\{ \prod_{r \in D^c} h_r(x_0) \right\} \exp((n - R_D - Q_D)x_0) \times \exp\left(-\frac{2}{ed^4x_0(1 + e^{-1/2})^2}\right). \quad (8.1)
\]

This upper bound is sufficient only for \( d \leq n^{1/6-\varepsilon} \).

- We suppose now that \( \lambda r \equiv 0 \pmod{d} \) for all \( r \in D^c \). \hspace{1cm} (8.2)

If \( D^c = \{d\} \) then by (1.1), \( n - R_D - Q_D \equiv 0 \pmod{d} \) and \( S_0(\lambda) = S_0(0) \) for all \( \lambda \).

In the general case, if (8.2) holds then we must have \( \frac{d}{(\lambda, \delta)} |r| \). Thus \( \frac{d}{(\lambda, \delta)} |\delta \) and again by (1.1), we have \( n - R_D - Q_D \equiv 0 \pmod{\frac{d}{(\lambda, \delta)}} \). Thus \( \exp\left((n - R_D - Q_D)x_0\right) = 1 \) and \( S_0(\lambda) = S_0(0) \).

For \( D^c \) given, there exists \( \delta \) integers \( \lambda \) modulo \( d \) such that \( r\lambda \equiv 0 \pmod{d} \) for all \( r \in D^c \).

We summarize these observations in the following lemma.

**Lemma 8.1.** For \( d \leq n^{1/6-\varepsilon} \) we have:

\[
\sum_{-\frac{d}{2} < \lambda \leq \frac{d}{2}} S_0(\lambda) = \delta S_0(0)(1 + o(1)).
\]

9. **The function \( h_r^{-1} \) in the range \(|y| \leq y_1\)**

The generating function associated to unequal partitions is \( h(z) = \prod_{j=1}^{\infty} (1 + z^j) \).

For \( S_0(0) \) it remains to handle the integral

\[
\frac{1}{2\pi} \int_{|y| \leq y_1} h(\exp(-(x_0 + iy))) \left\{ \prod_{r \in D} g_{N_r}(dx_0 + iy) h_r(x_0 + iy) \right\} \exp((n - R_D - Q_D)(x_0 + iy)) dy.
\]
In this section we will state an asymptotic estimate related to $h_r^{-1}$ in the range $|y| \leq y_1$.

The following method looks like the general method of Meinardus [7] for studying generating functions associated to partitions (this method is presented in details in the book of Andrews [1]). In fact we were also inspired by the chapter on the application of saddle point method to the partitions function in the Master course of Tenenbaum [8].

In Lemma 4.1, we obtained an estimation of $g_k(dw)$ in function of $f(dw)$.

This leads us to try in fact to obtain an estimation of $U_r(z) := f(dz)h_r^{-1}(z)$ for $|y| \leq y_1$ instead of $h_r^{-1}(z)$. This change will make some computations easier.

Thus we consider

$$U_r(z) = \left\{ \prod_{j=1}^{\infty} (1 - \exp(-jdz))^{-1} \right\} \left\{ \prod_{j=0}^{\infty} (1 + \exp(-(r + jd)z))^{-1} \right\}.$$

The main result of this section is the following lemma.

**Lemma 9.1.** Let $\eta > 0$. For $|y| \leq y_1$ and $1 \leq r \leq d$, we have

$$U_r(z) = \exp \left( \frac{\pi^2}{12d^2z} + \left( \frac{r}{d} - 1 \right) \log 2 + \frac{1}{2} \log \left( \frac{dz}{\pi} \right) + O(d^{1+\eta|z|}) + O(d^{-1}n^{-2\varepsilon}) \right).$$

In the next section we will apply this lemma with $\eta > 0$ small enough such that $d^{2+\eta}|z| \leq n^{-\varepsilon}$.

**Proof.** If $r = d$, there are quite no work to do since

$$U_d(z) = \prod_{j=1}^{\infty} (1 + \exp(-jdz))^{-1} \left( 1 - \exp(-2jdz) \right)^{-1} = \prod_{j=1}^{\infty} (1 - \exp(-2jdz))^{-1} = f(2dz).$$

Thus

$$U_d(z) = \exp \left( \frac{\pi^2}{12d^2z} + \frac{1}{2} \log \left( \frac{2dz}{\pi} \right) + O(d|z|) \right).$$

Now we suppose that $r \neq d$. We prefer to work with

$$\tilde{U}_r(z) := \prod_{j=1}^{\infty} (1 - \exp(-jdz))^{-1} (1 + \exp(-(r + jd)z))^{-1} = U_r(z)(1 + \exp(-rz)) = U_r(z)u_r(z).$$

We easily see that

$$u_r(z) = (2 + O(r|z|)). \quad (9.1)$$

Let $F(v, s) = \sum_{k=1}^{\infty} \frac{\exp(-kv)}{k^s}$. If $\Re v > 0$, then $s \mapsto F(v, s)$ is analytic on $\mathbb{C}$. 


The first step of the proof is the following result.

**Lemma 9.2.** For $r \neq d$, $\eta > 0$, $z = x_0 + iy$ with $|y| \leq y_1$, we have

$$
\log(\tilde{U}_r(z)) = \frac{1}{dz} \left( \frac{\pi^2}{6} + F(rz + i\pi, 2) \right) + \frac{1}{2} \log \left( \frac{dz}{2\pi} \right) \\
- \frac{1}{2} F(rz + i\pi, 1) + dz \left( -\frac{1}{2} + F(z + i\pi, 0) \right) + O(|z|^{-\eta}d^{1+\eta}).
$$

The last term $dz \left( -\frac{1}{2} + F(z + i\pi, 0) \right)$ in the above formula could be removed because it is $O(|z|^{-\eta}d^{1+\eta})$. The following proof uses Mellin formula and looks like the general method of Meinardus for studying generating functions associated to partitions.

We begin by some standard manipulations

$$
\tilde{U}_r(z) = \exp \left\{ -\sum_{j=1}^{\infty} \log(1 + \exp(-(r + jd)z)) + \log(1 - \exp(-zd)) \right\} \\
= \exp \left\{ \sum_{j, k \geq 1} \frac{\exp(-jkdz)}{k} \left( (-1)^k \exp(-krz) + 1 \right) \right\}.
$$

Let us write

$$
\log(\tilde{U}_r(z)) = \sum_{m=1}^{\infty} \frac{\beta(m)}{m} \exp(-mdz),
$$

with

$$
\beta(m) = \sum_{j=1}^{m} j((-1)^k \exp(-krz) + 1).
$$

By the Mellin transform formula we have:

$$
\log(\tilde{U}_r(z)) = \sum_{m \geq 1} \frac{\beta(m)}{m} \cdot \frac{1}{2i\pi} \int_{2-i\infty}^{2+i\infty} \Gamma(s) \left( \frac{\zeta(s)}{(mdz)^s} \right) ds.
$$

But the Dirichlet series is

$$
\sum_{m \geq 1} \frac{\beta(m)}{m^{s+1}} = \sum_{j \geq 1} \frac{1}{j} \sum_{k \geq 1} \frac{(1 + (-1)^k \exp(-krz))}{k^{s+1}} \\
= \zeta(s)\zeta(s+1) + F(rz + i\pi, s+1)).
$$

Thus we have

$$
\log(\tilde{U}_r(z)) = \frac{1}{2i\pi} \int_{2-i\infty}^{2+i\infty} \frac{\Gamma(s)\zeta(s)(\zeta(s+1) + F(rz + i\pi, s+1))}{(dz)^s} ds.
$$
Let $\eta \in [0, 1]$. We move the integral until the line $\Re s = -1 - \eta$. This gives

$$\log(\tilde{U}_r(z)) = \frac{1}{2\pi i} \int_{-1-\eta-i\infty}^{-1-\eta+i\infty} \frac{\Gamma(s)\zeta(s)(\zeta(s+1) + F(rz + i\pi, s + 1))}{(dz)^s} ds$$

$$+ \text{Res}(1) + \text{Res}(0) + \text{Res}(-1) + E,$$

where $E$ is the error term arising from the horizontal branches.

For $\Re s \in [-1 - \eta, 2]$ we have:

$$F(zr + i\pi, s + 1) = \sum_{\ell \geq 1} e^{-2\ell rz} \left( \frac{1}{(2\ell + 1)^{s+1}} - \frac{e^{rz}}{(2\ell - 1)^{s+1}} \right)$$

$$= \sum_{\ell \geq 1} e^{-2\ell rz} \left( \frac{1}{(2\ell + 1)^{s+1}} - \frac{1}{(2\ell - 1)^{s+1}} \right) + O\left( (rz)^{1-\eta} \right).$$

When $\ell$ is small, the term $e^{-\ell rz}$ is $O(1)$. Thus for $-1 - \eta \leq \Re s \leq 2$, we have:

$$\sum_{\ell \geq 1} e^{-2\ell rz} \ll \sum_{\ell \leq 100/rx_0} \ell^0 + \sum_{\ell > 100/rx_0} \ell^0 e^{-2\ell rx_0} \ll (rx_0)^{-1-\eta},$$

since for the second sum we have

$$\sum_{\ell > 100/rx_0} \ell^0 e^{-2\ell rx_0} \ll \sum_{\ell > 100/rx_0} \ell e^{-2\ell rx_0} \ll (rx_0)^{1-\eta} \ll (1 - \exp(-2\ell rx_0))^2.$$

For the other term we have

$$\left| \sum_{\ell \geq 1} e^{-2\ell rz} \left( \frac{1}{(2\ell)^{s+1}} - \frac{1}{(2\ell - 1)^{s+1}} \right) \right| \ll (1 + |s|) \sum_{\ell \geq 1} e^{-2\ell rz} \frac{1}{(\ell^{1-\eta})} \ll \frac{1 + |s|}{(rx_0)^{\eta}}.$$

Thus for $-1 - \eta \leq \Re s \leq 2$, we have

$$|F(zr + i\pi, s + 1)| \ll |s|(rx_0)^{-1-\eta} + rz|(rx_0)^{-1-\eta}. \quad (9.2)$$

We will also use the following classical results for the functions $\zeta$ and $\Gamma$ in vertical strips:

- there exists $H > 0$ such that $|\zeta(s)| \ll |3m s|^H$ for $-3 \leq \Re s \leq 3$ (in fact more generally for $\Re s \in [\sigma_1, \sigma_2]$ and $|3m s| \geq 1$ (cf. [2] Theorem 12.23 p. 270 for a more precise formulation);
− for $-3 \leq \Re s \leq 3$ (or $\Re s \in [\sigma_1, \sigma_2]$) and $|\Im s| \rightarrow +\infty$, ([9] Corollaire II.0.13 p. 182) we have

$$\Gamma(s) = (1 + O(|\Im s|^{-1})) \sqrt{2\pi} |\Im s|^{\Re s - \frac{1}{2}} e^{-\pi |\Im s|/2} e^{\alpha(s)},$$

with $\alpha(s) = (\Im s) \log |\Im s| - \Im s + \frac{1}{2} \pi (\Re s - \frac{1}{2}) \sgn(\Im s)$.

With this two formulae and by (9.2) we easily see that

$$\lim_{T \rightarrow +\infty} \int_{-1 - \eta \pm iT}^{2 \pm iT} \frac{\Gamma(s) \zeta(s)(\zeta(s+1) + F(rz + i\pi, s + 1))}{(dz)^s} ds = 0,$$  \hspace{1cm} (9.3)

and

$$\left| \int_{\Re s = -1 - \eta} \frac{\Gamma(s) \zeta(s)(\zeta(s+1) + F(rz + i\pi, s + 1))}{(dz)^s} ds \right| \ll s^{-\eta} d^{1+\eta}|z|. \hspace{1cm} (9.4)$$

Now we compute the different residues:

We have

$$\text{Res}(1) = \frac{\Gamma(1)(\zeta(2) + F(rz + i\pi, 2))}{dz} = \frac{1}{dz} \left( \frac{\pi^2}{6} + F(rz + i\pi, 2) \right).$$  \hspace{1cm} (9.5)

In $s = 0$, we have two poles, one from $\Gamma$, the other from the function $s \mapsto \zeta(s+1)$. We use the well known results $\Gamma'(1) = -\gamma$, $\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|)$.

Thus $\text{Res}(0)$ is the coefficient in $s^{-1}$ in the following formula:

$$\left( \frac{1}{s} - \gamma s + O(|s|^2) \right) \left( \zeta(0) + s \zeta'(0) + O(|s|^2) \right)$$

$$\times \left[ \frac{1}{s} + \gamma + O(|s|) + F(rz + i\pi, 1) \right] (1 - s \log(dz) + O(|s|^2)). \hspace{1cm} (9.6)$$

Since $\zeta(0) = -\frac{1}{2}$, $\zeta'(0) = -\frac{1}{2} \log(2\pi)$, we find

$$\text{Res}(0) = \frac{1}{2} \left( \log \left( \frac{dz}{2\pi} \right) - F(rz + i\pi, 1) \right).$$  \hspace{1cm} (9.7)

In $s = -1$, $\Gamma$ has a simple pole with residue $-1$ thus we have:

$$\text{Res}(-1) = -\zeta(-1)(\zeta(0) + F(rz + i\pi, 0))dz.$$

Since $\zeta(-1) = -\frac{1}{12}$, we obtain

$$\text{Res}(-1) = \frac{dz}{12} \left( -\frac{1}{2} + F(rz + i\pi, 0) \right). \hspace{1cm} (9.8)$$
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Formulae (9.5), (9.7), (9.8), (9.3), (9.4) end the proof of Lemma 9.2.

Now we have to study the terms \( F(s, s) \) with \( s = 0, 1, 2 \). First we have

\[
F(rz + i\pi, 0) = \sum_{m=1}^{\infty} (-1)^m \exp(-mrz) = -\frac{1}{1 + \exp(rz)} = -\frac{1}{2} + O(rn^{-1/2}),
\]

for \(|y| \leq y_1\). For the contribution of \( F(rz + i\pi, 1) \) we have

\[
\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{2m} \exp(-mrz) = \frac{1}{2} \log \left( 1 + \exp(-rz) \right) = \frac{1}{2} \log 2 + O(r|z|). \quad (9.9)
\]

The contribution of \( F(rz + i\pi, 2) \) is more difficult to handle because of the factor \((dz)^{-1}\). We will prove the following lemma

**Lemma 9.3.** For \(|y| \leq y_1\), we have for \( d \leq n^{1/4-2\varepsilon} \)

\[
\frac{1}{dz} F(rz + i\pi, 2) = -\frac{\pi^2}{12dz} + \frac{r}{d} \log 2 + O(d^{-1}n^{-2\varepsilon}).
\]

**Proof.** Let \( M \) be an odd integer. We have

\[
\frac{1}{dz} F(rz + i\pi, 2) = \frac{1}{dz} \sum_{m=1}^{M} \frac{(-1)^m \exp(-mrz)}{m^2} + \sum_{m=M+1}^{\infty} \frac{(-1)^m \exp(-mrz)}{dzm^2} = T_1 + T_2,
\]

say. We begin with \( T_2 \). Since the sum is absolutely convergent, we can regroup the terms \( m = 2\ell \) with the terms \( m = 2\ell + 1 \)

\[
|T_2| \leq \frac{1}{dz} \sum_{\ell = M+1}^{\infty} \exp(-2\ell x_0) \left| \frac{1}{4\ell^2} \exp(-rz) \right| \leq \frac{1}{dz} \sum_{\ell = M+1}^{\infty} \frac{1+O(r|z|)}{4\ell^2 + (2\ell + 1)^2} \leq \frac{1}{dz} \frac{4\ell + 1}{4\ell^2(2\ell + 1)^2} + O\left( \frac{r|z|}{(2\ell + 1)^2} \right) \ll \frac{1}{dz} M^2 + \frac{r}{dM}.
\]

This leads us to choose \( M \) the smallest odd integer \( \geq n^{1/4+\varepsilon} \). For \( T_1 \) we use the fact that \( \exp(-mrz) \) is near 1 if \( M \) is not too large. Let \( J \in \mathbb{N} \) to be specified later:

\[
T_1 = \frac{1}{dz} \sum_{m=1}^{M} \frac{(-1)^m}{m^2} (1 - mrz + \sum_{j=2}^{J-1} \frac{(-1)^j m^j r^j z^j}{j!}) + O(m^J r^J |z|^J)).
\]
By the same type of argument as for $T_2$ (which are standard manipulations on alternating series) we show that

$$
\frac{1}{dz} \sum_{m=1}^{M} \frac{(-1)^m}{m^2} = \frac{1}{dz} \sum_{m=1}^{\infty} \frac{(-1)^m}{m^2} + O\left(\frac{1}{d|z|M^2}\right) = -\frac{\pi^2}{12dz} + O\left(\frac{1}{d|z|M^2}\right),
$$

and

$$
\frac{r}{d} \sum_{m=1}^{M} \frac{(-1)^m}{m} = \frac{r}{d} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} + O\left(\frac{r}{dM}\right) = -\frac{r}{d} \log 2 + O\left(\frac{r}{dM}\right).
$$

Next, for each $2 \leq j \leq J - 1$ we have (recall that $r \leq d \leq n^{1/4 - 2\varepsilon}$)

$$
\frac{r^j|z|^{j-1}}{d} \left| \sum_{m=1}^{M} (-1)^m m^{j-2} \right| \ll \frac{r^j|z|^{j-1} M^{j-2}}{d} \ll n^{-(2+j)\varepsilon}.
$$

Finally for the last error term we have:

$$
r^j|z|^{j-1} d^{-1} \sum_{m=1}^{M} m^{j-2} \ll (M|z|)^{j-1} r^j d^{-1} \ll n^{j-\varepsilon} d^{-1},
$$

which is small enough for $J = \lceil \frac{5}{2} \rceil$.

This ends the proof of Lemma 9.3.

If we insert (9.9) and Lemma 9.3 in Lemma 9.2 and don’t forget (9.1) we obtain Lemma 9.1 in the case $r \neq d$. □

10. The term $S_0(0)$, end of the proof of Theorem 1.1

Recall that

$$
h(\exp(-z)) = \exp\left(\frac{\pi^2}{12z} - \frac{1}{2} \log 2 + O(|z|)\right),
$$

if $z \to 0$ with $|\arg z| \leq \kappa < \pi/2$. By Lemma 9.1 we have for $|y| \leq y_1$:

$$
h(\exp(-z)) \prod_{r \in D} U_r(z) = \exp\left(\frac{\pi^2}{12z} \left(1 + \frac{|D|}{d}\right) + \frac{|D|}{2} \log z - \frac{\log 2}{2}
\right.
\left. + \sum_{r \in D} \left(\frac{r}{d} - 1\right) \log 2 + \frac{|D|}{2} \log \left(\frac{d}{\pi}\right) + O(n^{-\varepsilon})\right).
$$
As in the case \( \mathcal{D} = \{1, \ldots, d\} \) we insert above the two formulæ:

\[
1/z = 1/x_0 - iy/x_0^2 + O \left( |y|^3/x_0 \right) \quad \text{and} \quad \log z = \log x_0 + O \left( |y|/x_0 \right),
\]

and we apply Lemma 4.1:

\[
S_0(0) = \frac{1}{2\pi} \int_{-y_1}^{y_1} \exp(C_\mathcal{D} + iy\tilde{B}_\mathcal{D} - y^2 A_\mathcal{D})dy,
\]

with

\[
C_\mathcal{D} = \frac{\pi^2}{12x_0} \left( 1 + \frac{|\mathcal{D}|}{d} \right) \log x_0 - \frac{\log 2}{2} + \sum_{r \in \mathcal{D}} \left( \frac{r}{d} - 1 \right) \log 2 + \frac{|\mathcal{D}|}{2} \log \left( \frac{d}{\pi} \right)
\]
\[
- \frac{C_2}{dx_0} + \frac{|\mathcal{D}|}{2} \log 2 + \sum_{r \in \mathcal{D}} (N_r - k_0) \log 2 - \frac{dx_0}{2} \sum_{r \in \mathcal{D}} (N_r - k_0)^2
\]
\[
+ (n - R_\mathcal{D} - Q_\mathcal{D})x_0 + O(n^{-\varepsilon}), \quad (10.1)
\]

\[
\tilde{B}_\mathcal{D} = -\frac{\pi^2}{12x_0^3} \left( 1 + \frac{|\mathcal{D}|}{d} \right) \log x_0 - \frac{C_2}{dx_0} + \frac{|\mathcal{D}|}{2} \log 2 - \frac{dx_0}{2} \sum_{r \in \mathcal{D}} (N_r - k_0)
\]
\[
+ n - R_\mathcal{D} - Q_\mathcal{D}, \quad (10.2)
\]

\[
A_\mathcal{D} = \frac{\pi^2}{12x_0} \left( 1 + \frac{|\mathcal{D}|}{d} \right) - \frac{C_2}{dx_0} - \frac{|\mathcal{D}|}{2} \log 2 - \frac{dx_0}{2} \sum_{r \in \mathcal{D}} (N_r - k_0) - \frac{d|\mathcal{D}|k_0^2}{2x_0}. \quad (10.3)
\]

Since \( k_0 dx_0 = \log 2 \), and \( C_2 = \frac{\pi^2}{12} - \frac{(\log 2)^2}{2} \), \( A_\mathcal{D}, \tilde{B}_\mathcal{D} \) and \( C_\mathcal{D} \) may be simplified:

\[
A_\mathcal{D} = \frac{\pi^2}{12x_0} - \frac{|\mathcal{D}|(\log 2)^2}{dx_0} = \frac{2\sqrt{3}n^2}{\pi} \left( 1 - \frac{12|\mathcal{D}|(\log 2)^2}{dx_0^2} \right).
\]

Recall that \( k_0 = \frac{2\sqrt{\log n}}{\pi d} \) and \( |N_r - k_0| \leq \frac{n^{1/3}\sqrt{\log n}}{d^{1/3}|\mathcal{D}|^{2/3}w(n)} \).

By Lemma 2.2 of [6], for \( |y| \leq y_1 \), we have

\[
y\tilde{B}_\mathcal{D} = -2yk_0 \sum_{r \in \mathcal{D}} (N_r - k_0) + O(n^{-\varepsilon/3}) = yB_\mathcal{D} + O(n^{-\varepsilon/3}),
\]

say.

We end the computations as in the case \( \mathcal{D} = \{1, \ldots, d\} \). Finally we obtain

\[
\Pi_2(n, R_\mathcal{D}) = (1 + o(1)) \frac{\delta}{2\pi} \sqrt{\frac{\pi}{A_\mathcal{D}}} \exp \left( C_\mathcal{D} - \frac{B^2_{\mathcal{D}}}{4A_\mathcal{D}} \right). \quad (10.4)
\]
To simplify $C_D$ we need some more precise estimations of $R_D$ and $Q_D$:

$$x_0 R_D = x_0 k_0 \sum_{r \in D} r + O(d^{2/3} |D|^{1/3} n^{-1/4+\varepsilon}),$$

$$Q_D x_0 = \frac{dx_0}{2} \sum_{r \in D} (N_r - k_0)^2 + k_0 dx_0 \sum_{r \in D} (N_r - k_0)$$

$$+ \frac{x_0 d k_0^2 |D|}{2} - \frac{d |D| k_0 x_0}{2} + O(d n^{-1/4+\varepsilon}).$$

It remains to insert these different formulae in (10.4) to finish the proof of Theorem 1.1.

11. Local stability of $\Pi^*_d(n, R_D)$

In this section we settle a result analogous to Corollary 9.1 of [5]. If $R_D = \{N_r : r \in D\}$ and $R^*_D = \{N^*_r : r \in D\}$ are two sets and such that $N^*_r$ is near $N_r$ on average then in the estimation of $\Pi^*_d(n, R_D)$ given by Theorem 1.1, we may replace the $N_r$ by $N^*_r$. Like in [5] this corollary will be useful for the proofs of the different corollaries announced in the introduction of [6].

**Corollary 11.1.** Let $0 < \varepsilon < 10^{-2}$, $n \geq n_0$, $d^3 |D| \leq n^{1/2-3\varepsilon}$ and two sets $R_D = \{N_r : r \in D\} \in \mathbb{Z}^{|D|}$, $R^*_D = \{N^*_r : r \in D\} \in \mathbb{R}^{|D|}$ such that

(i) (1.1) is satisfied for $R_D$;
(ii) $|N_r - k_0| \leq \frac{n^{1/4} \log n}{d^{3/4} |D|^{3/4} w(n)}$ for all $r \in D$ where $w(n)$ is a non-decreasing function such that $\lim_{u \to \infty} w(u) = \infty$;
(iii) $\sum_{r \in D} |N_r - N^*_r| \leq \delta + |D| - 1$, $\sum_{r \in D} |N_r - N^*_r|^2 \leq \delta^2 + |D| - 1$.

Then we have

$$\Pi^*_d(n, R_D) = q(n) \frac{\delta(1+o(1))}{\sqrt{1 - \frac{12 |D| (\log 2)^2}{d \pi^2}}} \left( \frac{d}{2 \sqrt{3n}} \right)^{|D|/2}$$

$$\times \exp \left( - \frac{2 \sqrt{3} (\log 2)^2}{\pi (1 - \frac{12 |D| (\log 2)^2}{d \pi^2}) \sqrt{n}} \left( \sum_{r \in D} (N_r - k_0)^2 \right)^2 - \frac{\pi d}{2 \sqrt{3n}} \sum_{r \in D} (N^*_r - k_0)^2 \right).$$

**Proof.** By (iii), $\sum_{r \in D} |N_r - N^*_r| \leq 2d$ and we have

$$\left| \sum_{r \in D} (N_r - k_0)^2 - \left( \sum_{r \in D} (N^*_r - k_0) \right)^2 \right|$$
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\[
\leq \left( \sum_{r \in \mathcal{D}} |N_r - N_r^*| \right) \left( 2 \sum_{r \in \mathcal{D}} |N_r - k_0| + \sum_{r \in \mathcal{D}} |N_r - N_r^*| \right)
\]
\[
= O \left( d^{2/3} |\mathcal{D}|^{1/3} \frac{n^{1/4} \sqrt{\log n}}{w(n)} + d^2 \right) = o(n^{1/2}).
\]

Similarly we have using also \( \delta \leq |\mathcal{D}| + 1 \) (since \( \delta \leq \min_{a \in \mathcal{D}} a \) if \( \mathcal{D} \neq \emptyset \))

\[
\left| \sum_{r \in \mathcal{D}} (N_r - k_0)^2 - \sum_{r \in \mathcal{D}} (N_r^* - k_0)^2 \right| \leq \sum_{r \in \mathcal{D}} |N_r - N_r^*| (2|N_r - k_0| + |N_r - N_r^*|)
\]
\[
\ll (\delta + |\mathcal{D}| - 1) |\mathcal{D}|^{-2/3} d^{-1/3} n^{1/4} \sqrt{\log n} \frac{d}{w(n)} + \delta^2 + |\mathcal{D}|
\]
\[
\ll |\mathcal{D}|^{1/3} d^{-1/3} n^{1/4} \sqrt{\log n} \frac{d}{w(n)} + \delta^2 + |\mathcal{D}| = o \left( \frac{\sqrt{n}}{d} \right).
\]

This ends the proof of Corollary 11.1.

12. On the normal order of the numbers of parts:
proof of Corollary 1.2 of [6]

Let \( C_\mathcal{D} = \left[ \frac{2\sqrt{3} \log 2}{\pi} \frac{\sqrt{n}}{d}, \frac{n^{1/4} \sqrt{\log n}}{d^{1/3} |\mathcal{D}|^{2/3} w(n)} \right] \) and \( D_\mathcal{D} = \left[ \frac{2\sqrt{3} \log 2}{\pi} \frac{\sqrt{n}}{d}, \frac{n^{1/4} \sqrt{\log n}}{d^{1/3} w(n)} \right] \).

To prove Corollary 1.2 of [6], it is sufficient to show that

\[
S^* := \sum_{N_r \in [C_\mathcal{D}, D_\mathcal{D}], R_\mathcal{D} \equiv n \mod \delta} \Pi_r^* = q(n)(1 + o(1)).
\]

(12.1)

As in [5] p. 82, we have to remove the dependence between \( n \) and the \( N_r \) given by the congruence condition modulo \( \delta \). If \( 1 \notin \mathcal{D} \) then \( \delta = 1 \), there are no difficulty and we take \( N_r^* = N_r \) for all \( r \in \mathcal{D} \). If \( 1 \in \mathcal{D} \) we write \( N_1^* = \lfloor N_1 / \delta \rfloor \delta \) and \( N_r^* = N_r \) for \( r \in \mathcal{D} \setminus \{1\} \). Now we will suppose that \( 1 \in \mathcal{D} \), the proof of the other case being similar. Next we apply Corollary 11.1 with these two sets:

\[
S^* = (1 + o(1)) q(n) \frac{\delta}{\sqrt{1 - \frac{12 |\mathcal{D}| \log 2}{d \pi^2} \frac{d}{2 \sqrt{3} n}}} |\mathcal{D}|^{1/2}
\]
\[
\times \sum_{C_\mathcal{D} \leq N_1 \delta \leq D_\mathcal{D}} \sum_{N_r \in [C_\mathcal{D}, D_\mathcal{D}], R_\mathcal{D} \equiv n \mod \delta} \exp \left( - \frac{2 \sqrt{3} (\log 2)^2}{\pi (1 - \frac{12 |\mathcal{D}| \log 2}{d \pi^2}) \sqrt{n}} \left( \delta N_1 - k_0 + \sum_{r \in \mathcal{D} \setminus \{1\}} (N_r - k_0)^2 \right) \right)
\]
− \frac{\pi d}{2\sqrt{3n}} \left( (\delta N_1 - k_0)^2 + \sum_{r \in D \setminus \{1\}} (N_r - k_0)^2 \right).

We apply again Corollary 11.1 with \( N_i^* = t_i \) so that \( |t_i - N_i| \leq 1 \) for \( i \in D \), we easily see that

\[
|\delta t_1' - \delta N_1| + \sum_{r \in D \setminus \{1\}} (t_r - N_r)^2 \leq \delta^2 + |D| - 1.
\]

Thus we have (after replacing \( \delta t_1' \) by \( t_1 \) in the integral):

\[
S^* = (1 + o(1))q(n) \frac{1}{\sqrt{1 - \frac{12D(|\log 2)|^2}{d\pi^2}}} \times \left( \frac{d}{2\sqrt{3n}} \right)^{|D|/2} \prod_{r \in D} \int_{C_r} \cdots \int_{C_r} \exp \left( -\frac{2\sqrt{3}(|\log 2)|^2}{\pi(1 - \frac{12D(|\log 2)|^2}{d\pi^2})\sqrt{n}} \right) \left( \sum_{r \in D}(t_r - k_0)^2 \right) \prod_{r \in D} dt_r.
\]

**Lemma 12.1.** We have

\[
S^* = (1 + o(1))q(n) \frac{1}{\sqrt{1 - \frac{12D(|\log 2)|^2}{d\pi^2}}} \times \left( \frac{d}{2\sqrt{3n}} \right)^{|D|/2} \prod_{r \in D} \int_{C_r} \cdots \int_{C_r} \exp \left( -\frac{2\sqrt{3}(|\log 2)|^2}{\pi(1 - \frac{12D(|\log 2)|^2}{d\pi^2})\sqrt{n}} \right) \left( \sum_{r \in D}(t_r - k_0)^2 \right) \prod_{r \in D} dt_r.
\]

**Proof.** We have to consider the contribution of terms of type \( \int_{D_D} \int_{C_D} \cdots \int_{C_D} \exp \left( -\frac{\pi d}{2\sqrt{3n}} \left( \sum_{r \in D}(t_r - k_0)^2 \right) \right) \prod_{r \in D} dt_r. \)

Clearly it is less than

\[
\int_{D_D} \int_{C_D} \cdots \int_{C_D} \exp \left( -\frac{\pi d}{2\sqrt{3n}} \left( \sum_{r \in D}(t_r - k_0)^2 \right) \right) \prod_{r \in D} dt_r.
\]

But

\[
\int_{C_D} \exp \left( -\frac{\pi d}{2\sqrt{3n}} (t_r - k_0)^2 \right) dt_r \leq \int_{-\infty}^{+\infty} \exp \left( -\frac{\pi d t_r^2}{2\sqrt{3n}} \right) dt_r = \sqrt{\frac{2\sqrt{3n}}{d}},
\]
and
\[ \int_{D^*} \exp \left( - \frac{\pi d}{2 \sqrt{3n}} (t_r - k_0)^2 \right) dt_r \leq \int_{D^*} \exp \left( - \frac{\pi d}{2 \sqrt{3n}} (D_D - k_0)(t_r - k_0) \right) dt_r \]
\[ = \frac{2 \sqrt{3n}}{\pi d(D_D - k_0)} \exp \left( - \frac{\pi d}{2 \sqrt{3n}} (D_D - k_0)^2 \right). \]

We have \( d^{1/3}|D|^{2/3} = d^{1/6}|D|^{1/2} \leq d^{1/6}|D|^{1/2} = (d^3|D|)^{1/2} \leq n^{1/4 - \varepsilon} \). Thus the contribution of some term of type \( \int_{D^*} \int_{C^*} \ldots \int_{D^*} \ldots \to S^* \) is
\[ O\left( q(n) \frac{w(n)|D|^{2/3}}{d^{1/6} \sqrt{\log n}} \exp \left( - \frac{\pi d^{1/3} \log n}{2 \sqrt{3}|D|^{4/3} w^2(n)} \right) \right). \]

Since there are \( 2|D| \) such error terms we obtain Lemma 12.1. It remains to compute the main term to finish the proof of Corollary 1.2 of [6]. We remark that
\[ \exp \left( - \frac{2 \sqrt{3}(\log 2)^2}{\pi (1 - \frac{12|D|(|\log 2|)^2}{d \varepsilon^2}) \sqrt{n}} \left( \sum_{r \in D} (t_r - k_0) \right)^2 - \frac{\pi d}{2 \sqrt{3n}} \left( \sum_{r \in D} (t_r - k_0)^2 \right) \right) \]
\[ = \exp \left( - \frac{1}{2} T^t MT \right), \tag{12.2} \]
with \( T = \begin{pmatrix} t_1^* - k_0 & \cdots & t_D^* - k_0 \end{pmatrix} \in \mathbb{R}^{|D|} \) and \( M \) is the symmetric matrix \( M = (m_{ij})_{1 \leq i, j \leq |D|} \) defined by
\[ m_{ii} = \frac{2 \sqrt{3}(\log 2)^2}{\pi (1 - \frac{12|D|(|\log 2|)^2}{d \varepsilon^2}) \sqrt{n}} + \frac{\pi d}{2 \sqrt{3n}} =: V + U \quad (1 \leq i \leq |D|) \]
\[ m_{ij} = m_{ji} = \frac{2 \sqrt{3}(\log 2)^2}{\pi (1 - \frac{12|D|(|\log 2|)^2}{d \varepsilon^2}) \sqrt{n}} =: V \quad (1 \leq i < j \leq |D|). \]

A classical result on determinant announces that
\[ \det M = 2^{|D|}|U|^{(V)} (U + |D|V) = \left( \frac{\pi d}{\sqrt{3n}} \right)^{|D|} \left( 1 - \frac{12|D|(|\log 2|)^2}{d \varepsilon^2} \right)^{-1}. \]

Thus the function (12.2) is proportional to the density of the law of a Gaussian vector with covariance matrix \( M^{-1} \). We deduce that
\[ \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \exp \left( - \frac{2 \sqrt{3}(\log 2)^2}{\pi (1 - \frac{12|D|(|\log 2|)^2}{d \varepsilon^2}) \sqrt{n}} \left( \sum_{r \in D} (t_r - k_0) \right)^2 \right) \]
\[ - \frac{\pi d}{2 \sqrt{3n}} \left( \sum_{r \in D} (t_r - k_0)^2 \right) \prod_{r \in D} dt_r = (2\pi)^{|D|/2} \sqrt{\det M^{-1}}. \]

This ends the proof of Corollary 1.2 of [6]. \( \square \)
13. Unequal partitions with equilibrated residue classes:
proof of Corollary 1.3 of [6]

For $1 \leq a < b \leq d$, let $E^*(a, b)$ denote the number of unequal partitions of $n$ such that $N_a = N_b$:

$$E^*(a, b) = \sum_{n \equiv aN_a + bN_b \pmod{\delta}} \Pi_d^*(n, R_{\{a,b\}}) = \sum_{N_a=N_b}^{N_a=N_c} \Pi_d^*(n, R_{\{a,b\}}) + o(q(n)),$$

by Corollary 1.2 of [6] applied with $w(n) = 2^{-2/3} \log \log n$,

$$C = \left[ \frac{2\sqrt{3} \log 2 \sqrt{n}}{\pi d^2} - \frac{n^{1/4} \sqrt{\log n}}{d^{1/3} \log \log n} \right] d \quad \text{and} \quad D = \left[ \frac{2\sqrt{3} \log 2 \sqrt{n}}{\pi d^2} + \frac{n^{1/4} \sqrt{\log n}}{d^{1/3} \log \log n} \right] d.$$

Next we apply Theorem 1.1 and Corollary 11.1. Here again we have to remove the condition $n \equiv aN_a + bN_b \pmod{\delta}$ otherwise $N_a, N_b, n$ wouldn’t be independent.

If $d \geq 5$ then $\delta = 1$. For $2 \leq d \leq 4$, the problematic cases are $(a, b, d) \in \{ (1, 2, 2), (1, 2, 3), (1, 3, 4) \}$. We will handle these cases later. Suppose now that $\delta = 1$.

$$E^*(a, b) = q(n) \frac{1}{\sqrt{1 - \frac{24(\log 2)^2}{d\pi^2}}} \left( \frac{d}{2\sqrt{3n}} \right)$$

$$\times \int_C^D \exp \left[ - \left( \frac{8\sqrt{3} \log 2^2}{\pi (1 - \frac{24(\log 2)^2}{d\pi^2}) \sqrt{n}} + \frac{\pi d}{\sqrt{3n}} \right) (t - k_0)^2 \right] dt + o(q(n))$$

$$= \sqrt{\frac{d}{4\sqrt{3n}}} q(n) + o(q(n)).$$

As in the proof of Corollary 1.2 of [6] we show that the contributions of $\int_{D}^{\infty}$ and $\int_{-\infty}^{C}$ are small enough. When $\delta > 1$, as explained in [5] we fixe some congruence conditions on $N_a$ and $N_b$ and next do quite the same computations.

14. Comparison between the number of parts in two residue classes:
proof of Corollary 1.4 of [6]

We proceed in the same way as in the previous section. Le $\Delta \in \{0, 1\}$. We have to estimate

$$T^*(a, b) = \sum_{n \equiv R_{\{a,b\}} \pmod{\delta}} \Pi_d^*(n, R_{\{a,b\}}).$$
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We take $\Delta = 0$ to examine the unequal partitions with $N_a \geq N_b$ and $\Delta = 1$ for the unequal partitions with $N_a > N_b$. In the same way as in the proof of Corollary 1.3 of [6], we obtain

\[
T^*(a, b) = q(n) \frac{1}{\sqrt{1 - \frac{24(\log 2)^2}{d\pi^2}}} \left( \frac{d}{2\sqrt{3n}} \right) \times \int_{-\infty}^{\infty} \int_{t_b}^{\infty} \exp \left[ - \frac{2\sqrt{3}(\log 2)^2}{\pi \left(1 - \frac{24(\log 2)^2}{d\pi^2}\right) \sqrt{n}} (t_a + t_b - 2k_b)^2 \right] \\
- \frac{\pi d}{2\sqrt{3n}} ((t_a - k_0)^2 + (t_b - k_0)^2) \, dt_a \, dt_b + o(q(n)).
\]

In the same way we find a similar formula for $T^*(b, a)$. This proves that $T^*(a, b) = q(n)/2 + o(q(n))$.

15. On the $d$ regularity of the unequal partitions:
proof of Corollary 1.5 of [6]

Now we suppose that $d$ is fixed in order to use Corollary 1.3 of [6] uniformly. We now study for $\Delta = 0$ or $1$ :

\[
W^*(a) := \sum_{\substack{N_1, \ldots, N_d \geq 1 \mod d \\text{ such that } N_a \geq \Delta + \max_{b \neq a} N_b \mod d}} \Pi_\Delta(n, x).
\]

We proceed as [5] Sections 12, 13 and like the previous paragraphs of this present paper:

\[
W^*(a) = o(q(n)) + \frac{d}{\sqrt{1 - \frac{12(\log 2)^2}{\pi^2}}} \left( \frac{d}{2\sqrt{3n}} \right)^{d/2} \int \ldots \int_{t_a \geq \max_{b \neq a} t_b} \int_{t_{b}^{\infty}} \ldots \int \ldots \int_1 f(t_1, \ldots, t_d) \, dt_1 \ldots dt_d,
\]

with

\[
f(t_1, \ldots, t_d) = \exp \left\{ - \frac{2\sqrt{3}\log^2 2}{\pi \left(1 - \frac{12(\log 2)^2}{\pi^2}\right) \sqrt{n}} \left( \sum_{r=1}^{d} (t_r - k_0) \right)^2 \right\} - \frac{\pi d}{2\sqrt{3n}} \sum_{r=1}^{d} (t_r - k_0)^2 \right\}.
\]

By Corollary 1.3 of [6] applied $d$ times (it is why $d$ is fixed), we have

\[
\sum_{a=1}^{d} W^*(a) = q(n) + o(q(n)).
\]
Since \( f(t_1, \ldots, t_d) \) is symmetrical the above terms are asymptotically equal:

\[
W^*(a) = \left( \frac{1}{d} + o(1) \right) q(n),
\]
as it was conjectured in [4] p. 334 and in the introduction of [3].

The proof of the second assertion of Corollary 1.5 of [6] is similar. We have only to replace the condition \( N_a \geq \max_{b \neq a} N_b \) by \( N_{\sigma(1)} \geq N_{\sigma(2)} \geq \cdots \geq N_{\sigma(d)} \).

References