A normality relationship between two families
and its applications

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Abstract. Let \( k \) be a positive integer, and let \( F \) be a family of meromorphic
functions defined in a domain \( D \subset \mathbb{C} \), all of whose zeros have multiplicity at least \( k \),
and there exists \( M > 0 \) such that \( |f(z)| \leq M \) whenever \( f(z) = 0 \) for \( f \in F \).
If \( F_k = \{ f^{(k)} : f \in F \} \) is normal, then \( F \) is also normal in \( D \). Some applications of this
result are given.

1. Introduction

Let \( D \) be a domain in \( \mathbb{C} \), and \( F \) be a family of meromorphic functions defined
on \( D \). \( F \) is said to be normal on \( D \), in the sense of Montel, if for any sequence
\( \{ f_n \} \in F \) there exists a subsequence \( \{ f_{n_j} \} \), such that \( \{ f_{n_j} \} \) converges spherically
locally uniformly on \( D \), to a meromorphic function or \( \infty \) (see [6], [9], [12]).

Let \( k \) be a positive integer. Consider the family \( F_k \) consisting of \( k \)th deriva-
tive functions of all \( f \in F \), that is, \( F_k = \{ f^{(k)} : f \in F, z \in D \} \). It is natural to
consider the normality relation between these two families. However, the following
examples show that there seems no direct relation between \( F \) and \( F_k \).

Example 1. Let \( \Delta = \{ z : |z| < 1 \} \), and \( F = \{ f_n(z) = n(z^2 - n^2) : n = 1, 2, \ldots \} \).
Then \( F_1 = \{ f'_n(z) = 2nz : n = 1, 2, \ldots \} \). For each \( z \in \Delta \),

\[
f''_n(z) = \frac{|2nz|}{1 + |n(z^2 - n^2)|^2} \leq \frac{2n}{1 + (n^3 - n)^2} \to 0
\]

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as \( n \to \infty \), where \( f_n^\#(z) = |f_n'(z)|/(1 + |f_n(z)|^2) \) is the spherical derivative of \( f_n \). By Marty’s criterion, \( \mathcal{F} \) is normal in \( \Delta \). But it is easy to see that \( \mathcal{F}_1 \) is not normal in \( \Delta \).

**Example 2.** Let \( \Delta = \{ z : |z| < 1 \} \), and \( \mathcal{F} = \{ f_n(z) = nz : n = 1, 2, \ldots \} \). Then \( \mathcal{F}_1 = \{ f_n'(z) = n : n = 1, 2, \ldots \} \). Clearly, \( \mathcal{F}_1 \) is normal in \( \Delta \); but \( \mathcal{F} \) is not normal in \( \Delta \).

In 1996, CHEN and LAPPAN [2] first gave an interesting normality relation between \( \mathcal{F} \) and \( \mathcal{F}_k \) under an additional condition, as follows.

**Theorem A** ([2, Corollary 4]). Let \( k \) be a positive integer, and let \( \mathcal{F} \) be a family of meromorphic functions defined in a domain \( D \), all of whose zeros have multiplicity at least \( k+1 \). If \( \mathcal{F}_k = \{ f^{(k)} : f \in \mathcal{F} \} \) is normal, then \( \mathcal{F} \) is also normal in \( D \).

In this paper, by using a different method from that in [2], we first give an extension to the above result, as follows.

**Theorem 1.** Let \( k \) be a positive integer, and let \( \mathcal{F} \) be a family of meromorphic functions defined in a domain \( D \), all of whose zeros have multiplicity at least \( k \), and there exists \( M > 0 \) such that \( |f^{(k)}(z)| \leq M \) whenever \( f(z) = 0 \), \( f \in \mathcal{F} \). If \( \mathcal{F}_k = \{ f^{(k)} : f \in \mathcal{F} \} \) is normal, then \( \mathcal{F} \) is also normal in \( D \).

**Remark 1.** Theorem 1 is sharp, which can also be shown by Example 2.

The above normality relation between \( \mathcal{F} \) and \( \mathcal{F}_k \) is indeed useful to study normal families. In section 3, we shall give some applications of Theorem 1.

## 2. Proof of Theorem 1

We need the following well-known PANG–ZALCMAN lemma, which is the local version of [8, Lemma 2](cf. [13, pp. 216–217]).

**Lemma 1.** Let \( k \) be a positive integer and let \( \mathcal{F} \) be a family of functions meromorphic in a domain \( D \), all of whose zeros have multiplicity at least \( k \), and suppose that there exists \( A \geq 1 \) such that \( |f^{(k)}(z)| \leq A \) whenever \( f(z) = 0 \), \( f \in \mathcal{F} \). Then if \( \mathcal{F} \) is not normal at \( z_0 \in D \), there exist, for each \( \alpha, 0 \leq \alpha \leq k \),

- (a) points \( z_n \in D, z_n \to z_0 \),
- (b) positive numbers \( \rho_n \to 0 \), and
- (c) functions \( f_n \in \mathcal{F} \)
such that \( g_n(\zeta) = \rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) \to g(\zeta) \) locally uniformly with respect to the spherical metric, where \( g \) is a nonconstant meromorphic function in \( \mathbb{C} \), all of whose zeros have multiplicity at least \( k \), such that \( g^#(\zeta) \leq g^#(0) = kA + 1 \).

**Proof of Theorem 1.** Suppose that \( \mathcal{F} \) is not normal at \( z_0 \in D \). By Lemma 1, there exist functions \( f_n \in \mathcal{F} \), points \( z_n \to z_0 \) and positive numbers \( \rho_n \to 0 \), such that

\[
g_n(\zeta) = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^k} \to g(\zeta)
\]

converges spherically uniformly on compact subsets of \( \mathbb{C} \), where \( g(\zeta) \) is a nonconstant meromorphic function in \( \mathbb{C} \), all of whose zeros have multiplicity at least \( k \), and \( g^#(\zeta) \leq g^#(0) = kM + 1 \). (Without loss of generality, we assume that \( M > 1 \)).

From (1), we have

\[
g_n^{(k)}(\zeta) = f_n^{(k)}(z_n + \rho_n \zeta) \to g^{(k)}(\zeta)
\]

converges uniformly on compact subsets of \( \mathbb{C} \) disjoint from the poles of \( g \). Suppose that \( g(\zeta_0) = 0 \), by Hurwitz’s theorem, there exist \( \zeta_n \to \zeta_0 \) such that \( f_n(z_n + \rho_n \zeta_n) = 0 \). By the assumption of Theorem 1, we have \( |f_n^{(k)}(z_n + \rho_n \zeta_n)| \leq M \). Now, it follows from (2) that \( |g^{(k)}(\zeta_0)| \leq M \). This proves that \( |g^{(k)}| \leq M \) whenever \( g = 0 \).

We claim that \( g \) cannot be a polynomial of degree less than \( k + 1 \). Indeed, \( g \) cannot be a polynomial of degree less than \( k \) since all zeros of \( g \) have multiplicity at least \( k \). Now assume that \( g \) is a polynomial of degree \( k \). It follows that \( g \) has the form

\[
g(\zeta) = \frac{A}{k!}(\zeta - \alpha)^k
\]

where \( A, \alpha \) are complex numbers. Since \( g = 0 \Rightarrow |g^{(k)}| \leq M \), we see that \( |A| \leq M \). Calculating \( g^#(0) \), we get

\[
g^#(0) = \frac{|A| |\alpha|^{k-1}}{1 + \left(\frac{|A| |\alpha|^k}{k!}\right)^2} = \frac{k}{|\alpha|} \cdot \frac{|A| |\alpha|^k}{1 + \left(\frac{|A| |\alpha|^k}{k!}\right)^2}.
\]

From the middle expression, we see that \( g^#(0) \leq |A| \) if \( |\alpha| \leq 1 \), and from the expression on the right we see that \( g^#(0) < k/2 \) if \( |\alpha| > 1 \). But these contradict the fact that \( g^#(0) = kM + 1 \) and \( |A| \leq M \).

Hence, there exist a point \( \zeta_0 \) and \( M_1 > 0 \) such that

\[
M_1^{-1} \leq |g^{(j)}(\zeta_0)| \leq M_1, \quad \text{for } j = k, k + 1.
\]
It follows that $(2M_1)^{-1} \leq |g_n^{(j)}(\zeta_0)| \leq 2M_1(j = k, k + 1)$ for sufficiently large $n$. From (2), $g_n^{(k)}(\zeta_0) = f_n^{(k)}(z_n + \rho_n \zeta_0)$, and then $|f_n^{(k)}(z_n + \rho_n \zeta_0)| \leq 2M_1$ for sufficiently large $n$. So we have

$$(2M_1)^{-1} \leq |g_n^{(k+1)}(\zeta_0)| = \rho_n |f_n^{(k+1)}(z_n + \rho_n \zeta_0)| \leq \rho_n (1 + 4M_2) \frac{|f_n^{(k+1)}(z_n + \rho_n \zeta_0)|}{1 + |f_n^{(k)}(z_n + \rho_n \zeta_0)|^2},$$

for sufficiently large $n$.

On the other hand, by Marty’s criterion, the normality of the family $F_k$ implies that for each compact subset $K \subset D$, there exists a positive number $M_2$ such that

$$\frac{|f^{(k+1)}(z)|}{1 + |f^{(k)}(z)|^2} \leq M_2$$

for each $f \in F$ and $z \in K$. Then, for sufficiently large $n$, we have

$$\frac{|f_n^{(k+1)}(z_n + \rho_n \zeta_0)|}{1 + |f_n^{(k)}(z_n + \rho_n \zeta_0)|^2} \leq M_2.$$

Substituting (5) in (4), we obtain

$$(2M_1)^{-1} \leq |g_n^{(k+1)}(\zeta_0)| \leq \rho_n (1 + 4M_2^2) M_2 \to 0,$$

as $n \to \infty$, a contradiction. Theorem 1 is thus proved.

3. Some applications of Theorem 1

In this section, we shall give some applications of Theorem 1.

Recently, Chang [1] proved the following result, which improve and generalize the related results due to Pang and Zalcman [8], Fang and Zalcman [5].

**Theorem B** ([1, Theorem 1]). Let $F$ be a family of meromorphic functions defined in a domain $D$, let $a, b$ be two nonzero complex numbers such that $a/b \notin \mathbb{N} \setminus \{1\}$. If, for each $f \in F$, $f = a \Rightarrow f'(z) = a$, and $f'(z) = b \Rightarrow f''(z) = b$ in $D$, then $F$ is normal.

There is an example [1, Example 1], which shows that the condition ‘$a/b \notin \mathbb{N} \setminus \{1\}$’ in Theorem B is necessary. Chang proved another result without the condition ‘$a/b \notin \mathbb{N} \setminus \{1\}$’, as follows.
Theorem C ([1, Theorem 2]). Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain $D$, let $a, b$ be two nonzero complex numbers. If, for each $f \in \mathcal{F}$, $f = a \Rightarrow f'(z) = a, f''(z) \neq b$ in $D$, then $\mathcal{F}$ is normal.

Remark 2. Chang also gave another example [1, Example 2] to show that the condition $f''(z) \neq b$ in Theorem C can not be omitted. However, it is easy to see that $f''(z) \neq b$ in Theorem C is not necessary for the case $a = b(\neq 0)$. Indeed, $f = a \Rightarrow f'(z) = a$ and $f''(z) \neq b$ yield that $f \neq a$ and $f' \neq a$ since $a = b$, then Gu’s normal criterion [3] implies that $\mathcal{F}$ is normal. We also find that ‘$a$ is nonzero’ in Theorem C can be removed. In fact, if $a = 0$ and $b \neq 0$, noting that $f' \neq b$ and $f'' \neq b$, Gu’s normal criterion asserts that $\mathcal{F}_1 = \{f' : f \in \mathcal{F}\}$ is normal in $D$. Since $f = 0 \Rightarrow f' = 0$, we conclude from Theorem 1 that $\mathcal{F}$ is also normal in $D$.

Here, by using Theorem 1 and some known results, we can prove the following results, which improve and generalize Theorem C much more.

Theorem 2. Let $a, b, c$ be three complex numbers with $c \neq 0$, $k, l$ be two positive integers, and let $\mathcal{F}$ be a family of meromorphic functions defined in a domain $D$. Suppose that, for each $f \in \mathcal{F}$ and $z \in D$,

1. all zeros of $f - a$ have multiplicity at least $k$, and there exists $M > 0$ such that $f = a \Rightarrow |f^{(k)}| \leq M$;
2. all zeros of $f^{(k)} - b$ have multiplicity at least $l + 1$, and $f^{(k+l)} \neq c$.

Then $\mathcal{F}$ is normal in $D$.

Let $k = l = 1$ and $b = c$ in Theorem 2, we have

Corollary 1. Let $a, b$ be two complex numbers with $b \neq 0$, and let $\mathcal{F}$ be a family of meromorphic functions defined in a domain $D$. Suppose that, for each $f \in \mathcal{F}$ and $z \in D$,

1. there exists $M > 0$ such that $f = a \Rightarrow |f'| \leq M$;
2. all zeros of $f' - b$ have multiplicity at least 2, and $f'' \neq b$.

Then $\mathcal{F}$ is normal in $D$.

Obviously, the above results improve and generalize Theorem C.

Next we give some more general extensions of Theorem C by extending constants ‘$a, b, c$’ in Theorem 2 to functions ‘$a(z), b(z), c(z)$’.

Theorem 3. Let $k, l$ be two positive integers, $D$ be a domain in $\mathbb{C}$, let $a(z)$, $b(z)$ be two holomorphic functions in $D$, and $c(z)$ be a meromorphic function in $D$ such that $c(z) \neq \infty$ and $c(z) \neq b'(z)$, and let $\mathcal{F}$ be a family of meromorphic functions defined in $D$. Suppose that, for each $f \in \mathcal{F}$ and $z \in D$,
(i) all zeros of \( f(z) - a(z) \) have multiplicity at least \( k \), and there exists \( M > 0 \) such that \( f(z) = a(z) \Rightarrow |f^{(k)}| \leq M \);
(ii) all zeros of \( f^{(k)}(z) - b(z) \) have multiplicity at least \( k \), and \( f^{(k+1)}(z) \neq c(z) \).

Then \( \mathcal{F} \) is normal in \( D \).

**Theorem 4.** Let \( k, l \geq 2 \) be two positive integers, \( D \) be a domain in \( \mathbb{C} \), let \( a(z), b(z) \) be two holomorphic functions in \( D \), and \( c(z) \) be a meromorphic function in \( D \) such that \( c(z) \neq \infty \) and \( c(z) \neq b^{(l)}(z) \), and let \( \mathcal{F} \) be a family of meromorphic functions defined in \( D \). Suppose that, for each \( f \in \mathcal{F} \) and \( z \in D \),

(i) all zeros of \( f(z) - a(z) \) have multiplicity at least \( k \), and there exists \( M > 0 \) such that \( f(z) = a(z) \Rightarrow |f^{(k)}| \leq M \);
(ii) all zeros of \( f^{(k)}(z) - b(z) \) have multiplicity at least \( l + 1 \), and \( f^{(k+1)}(z) \neq c(z) \).

Then \( \mathcal{F} \) is normal in \( D \).

**Remark 3.** If \( k = 1 \), the condition ‘all zeros of \( f - a \) or \( (f - a(z)) \) have multiplicity at least \( k \)’ in Theorem 2–4 holds naturally, and then can be removed.

**Remark 4.** The condition \( c \neq 0 \) in Theorem 2 (\( b \neq 0 \) in Corollary 1), \( c(z) \neq b'(z) \) in Theorem 3, and \( c(z) \neq b^{(l)}(z) \) in Theorem 4 can not be omitted, as is shown by the following examples.

**Example 3.** Let \( \Delta = \{ z : |z| < 1 \} \), \( a \neq 0 \) and \( b = c = 0 \), and let \( \mathcal{F} = \{ f_n(z) = e^{az} + a : n = 1, 2, \ldots ; z \in \Delta \} \). Obviously, \( f_n(z) \neq a \), thus \( f(z) = a \Rightarrow f^{(1)} = a \);
\( f'_n(z) = ne^{az} \neq 0 \), and \( f''_n(z) = n^2e^{az} \neq 0 \). Then all conditions excepting \( c \neq 0 \) (or \( c \neq 0 \)) of Theorem 2 (Corollary 1) are satisfied. But \( \mathcal{F} \) is not normal in \( \Delta \).

**Example 4.** Let \( \Delta = \{ z : |z| < 1 \} \), \( a(z) = b(z) = c(z) = e^z \), and let \( \mathcal{F} = \{ f_n(z) = e^{az} + e^z : n = 1, 2, \ldots ; z \in \Delta \} \). It is easy to see that all conditions excepting \( c(z) \neq b'(z) \) (\( c(z) \neq b^{(l)}(z) \)) of Theorem 3–4 are satisfied. But \( \mathcal{F} \) is not normal in \( \Delta \).

**Remark 5.** Example 4 also shows that ‘nonzero constants \( a, b \)’ in Theorem B can not be replaced two nonconstant functions (even for non-vanishing holomorphic functions).

To prove the above theorems, we need some known results.

**Lemma 2** ([10, Theorem 5]). Let \( k \) be a positive integer, and let \( \mathcal{F} \) be a family of meromorphic functions defined in a domain \( D \), all of whose poles are multiple and whose zeros all have multiplicity at least \( k + 1 \). If, for each \( f \in \mathcal{F} \), \( f^{(k)}(z) \neq 1 \) in \( D \), then \( \mathcal{F} \) is normal in \( D \).
Lemma 3 ([7, Theorem 1.3], cf. [11, Theorem 2]). Let \( F \) be a family of meromorphic functions defined in a domain \( D \), all of whose poles are multiple and whose zeros all have multiplicity at least 3, and let \( \psi(z)(\neq 0, \infty) \) be a function meromorphic in \( D \). If, for each \( f \in F \) and for each \( z \in D \), \( f'(z) \neq \psi(z) \), then \( F \) is normal in \( D \).

Lemma 4 ([14, Theorem 2]). Let \( k \geq 2 \) be an integer, \( F \) be a family of meromorphic functions defined in a domain \( D \), all of whose poles are multiple and whose zeros all have multiplicity at least \( k + 1 \), and let \( \psi(z)(\neq 0, \infty) \) be a function meromorphic in \( D \). If, for each \( f \in F \) and for each \( z \in D \), \( f^{(k)}(z) \neq \psi(z) \), then \( F \) is normal in \( D \).

Proof of Theorem 2. Let \( G = \{g = f^{(k)} - b : f \in F\} \). Obviously, the poles of \( g \) have multiplicity at least \( k + 1 \geq 2 \). By the assumptions of theorem, for each \( g \in G \), all zeros of \( g \) have multiplicity at least \( l + 1 \), and \( g^{(l)} = f^{(k+l)} \neq c \). Lemma 2 implies that \( G \) is normal in \( D \). Hence, the family \( H_k = ((f-a)^{k} : f \in F, z \in D) \) is also normal in \( D \), where \( H = \{f - a : f \in F\} \). Noting condition (1), by Theorem 1, we get that \( H \) is normal, and then \( F \) is normal in \( D \). Theorem 2 is proved.

Proof of Theorem 3. Since normality is a locally property, we only need to prove \( F \) is normal at each point in \( D \).

Let \( z_0 \in D \), then there exists \( \delta > 0 \) such that \( D_\delta(z_0) \subset D \), where \( D_\delta(z_0) = \{z : |z - z_0| \leq \delta\} \). Let \( G = \{g(z) = f^{(k)}(z) - b(z) : f \in F\} \). Clearly, all poles of \( g \in G \) are multiple. By the hypotheses of the theorem, for each \( g \in G \), all zeros of \( g \) have multiplicity at least 3. Noting that \( b(z) \) is holomorphic and \( f^{(k+1)}(z) \neq c(z) \), we have \( g' = f^{(k+1)}(z) - b'(z) \neq c(z) - b'(z) (\neq 0) \). Then, by Lemma 3, \( G \) is normal in \( D \), and then in \( D_\delta(z_0) = \{z : |z - z_0| < \delta\} \). It follows that the family \( H_k = \{(f(z) - a(z))^{(k)} : f \in F\} \) is normal in \( D_\delta(z_0) \), where \( H = \{h = f(z) - a(z) : f \in F\} \). By the hypotheses of the theorem, for each \( h \in H \), all zeros of \( h \) have multiplicity at least \( k \). Moreover, if \( h(z) = 0 \), that is, \( f(z) = a(z) \), then \( |f^{(k)}(z)| \leq M \), and thus

\[
|h^{(k)}(z)| \leq M + |a^{(k)}(z)|.
\]

Noting that \( a(z) \) is holomorphic in \( D \), there exists \( M_1 > 0 \) such that \( |a^{(k)}(z)| \leq M_1 \) in \( D_\delta(z_0) \), and then in \( D_\delta(z_0) \). We get that \( h(z) = 0 \Rightarrow |h^{(k)}(z)| \leq M_2 \) for \( z \in D_\delta(z_0) \), where \( M_2 = M + M_1 \). By Theorem 1, \( H \) is normal in \( D_\delta(z_0) \). It follows that \( F \) is normal in \( D_\delta(z_0) \), and this means that \( F \) is normal at \( z_0 \). Theorem 3 is thus proved.
Proof of Theorem 4. Using the same argument as in Theorem 3 and Lemma 4, we can prove Theorem 4. We here omit the details. □

Next we give another application of Theorem 1. In [4], Fang and Chang gave an extension to Gu’s normal criterion in some sense, by allowing \( f^{(k)} - 1 \) have zeros but restricting the zeros of \( f^{(k)} \), as follows.

**Theorem D** ([4, Theorem 1]). Let \( \mathcal{F} \) be a family of meromorphic functions defined in a domain \( D \), and let \( k \) be a positive integer. If, for each \( f \in \mathcal{F}, \ f \neq 0 \), \( f^{(k)} \neq 0 \) and the zeros of \( f^{(k)} - 1 \) have multiplicity at least \( (k + 2)/k \), then \( \mathcal{F} \) is normal.

Here, we can prove the following extension of Theorem D.

**Theorem 5.** Let \( k, l_1, l_2 \) be three positive integers (\( l_1, l_2 \) can be \( \infty \)) with \( 1/l_1 + 1/l_2 < k/(k + 1) \), and let \( \mathcal{F} \) be a family of meromorphic functions defined in a domain \( D \). Suppose that, for each \( f \in \mathcal{F} \) and \( z \in D \),

1. all zeros of \( f \) have multiplicity at least \( k \) and there exists \( M > 0 \) such that \( |f^{(k)}(z)| \leq M \) whenever \( f(z) = 0 \);
2. all zeros of \( f^{(k)} \) have multiplicity at least \( l_1 \); and
3. all zeros of \( f^{(k)} - 1 \) have multiplicity at least \( l_2 \).

Then \( \mathcal{F} \) is normal in \( D \).

**Remark 6.** We should indicate that Theorem 5 can be followed from [4, Theorem 2] if condition (1) is replaced by a stronger condition “all zeros of \( f \) have multiplicity at least \( k + 1 \)”. However, the method in [4] does not work here, and our proof is very simple.

To prove Theorem 5, we need the following classical result due to Bloch and Valiron, which can be found in [6], [9], [12].

**Lemma 5.** Let \( a_1, a_2, \ldots, a_q \) be \( q \) distinct complex numbers, and \( l_1, l_2, \ldots, l_q \) be positive integers (may equal to \( \infty \)) with \( \sum_{i=1}^{q}(1-1/l_i) > 2 \). Let \( \mathcal{F} \) be a family of meromorphic functions defined in a domain \( D \). If, for each \( f \in \mathcal{F} \), the zeros of \( f - a_i \) have multiplicity at least \( l_i \) \( (i = 1, 2, \ldots, q) \) in \( D \), then \( \mathcal{F} \) is normal in \( D \).

**Proof of Theorem 5.** Obviously, the poles of \( f^{(k)} \) have multiplicity at least \( k + 1 \). Since

\[
\frac{1}{l_1} + \frac{1}{l_2} < \frac{k}{k + 1},
\]

we have

\[
\left(1 - \frac{1}{l_1}\right) + \left(1 - \frac{1}{l_2}\right) + \left(1 - \frac{1}{k + 1}\right) > 2.
\]
Let $q = 3$, $a_1 = 0$, $a_2 = 1$ and $a_3 = \infty$, applying Lemma 6 for $F_k = \{f^{(k)} : f \in F\}$, we know that $F_k$ is normal in $D$. Noting condition (1), Theorem 1 implies that $F$ is also normal in $D$. Theorem 5 is proved.

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