The convergence of the sequences coding the ground model reals

By MILOŠ S. KURILIĆ (Novi Sad) and ALEKSANDAR PAVLOVIĆ (Novi Sad)

Abstract. We investigate the convergence $\lambda_1$ on a complete Boolean algebra $\mathbb{B}$ defined in the following way: a sequence $x = \langle x_n : n \in \omega \rangle$ in $\mathbb{B}$ converges to the point $\limsup x$ of $\mathbb{B}$, if in each generic extension $V_\mathbb{B}[G]$ the real coded by the name $\tau_x = \{(\bar{n}, x_n) : n \in \omega\}$ belongs to the ground model $V$; otherwise, $x$ has no limit points. It is shown that $\lambda_1$ generates the same topology as the convergence $\bar{\lambda}_4$, generalizing the sequential convergence on the Aleksandrov cube and that for a c.B.a. $\mathbb{B}$ the following conditions are equivalent: (1) The algebra $\mathbb{B}$ is $(\omega, 2)$-distributive; (2) The (L2)-closure of $\lambda_1$, $\bar{\lambda}_1$, is a topological convergence; (3) $\bar{\lambda}_1 = \bar{\lambda}_4$; (4) $\lambda_1 = \lambda_4$; and, for the algebras satisfying $\text{hcc}(\mathbb{B}) > \mathfrak{c}$, (5) $\lambda_1$ is a weakly topological convergence. Also, it is shown that the convergence $\bar{\lambda}_1$ is not weakly topological, if forcing by $\mathbb{B}$ produces splitting reals.

1. Preliminaries

Topologies and convergence structures on Boolean algebras as well as the interplay between the topological, algebraic and forcing-related properties of Boolean algebras are extensively investigated. The results concerning this interplay are useful because, for example, they enable us to attack algebraic problems by topological methods (see e.g. [3]) or topological problems using the techniques of forcing [7].

In this paper we investigate the convergence $\lambda_1$ on an arbitrary complete...
Boolean algebra $\mathcal{B}$ defined in the following way: a sequence $x = (x_n : n \in \omega)$ converges to $\limsup x$, if $1 \vdash \tau_x \in V$, where $\tau_x = \{ (\tilde{n}, x_n) : n \in \omega \}$ and, otherwise, $x$ has no limit points. In addition, we compare this convergence with some convergences considered in [8]. One of them is the algebraic convergence [11], [1] related to the von Neumann and the Maharam problem and generalizing the convergence on the Cantor cube; another one is a generalization of the convergence on the Alexandrov cube considered in [9].

Our notation is mainly standard. So, $\omega$ denotes the set of natural numbers, $Y^X$ the set of all functions $f : X \to Y$ and $\omega^\omega$ the set of all strictly increasing functions from $\omega$ into $\omega$. A sequence in a set $X$ is each function $x : \omega \to X$; instead of $x(n)$ we usually write $x_n$ and also $x = (x_n : n \in \omega)$. The constant sequence $\langle a, a, a, \ldots \rangle$ is denoted by $\langle a \rangle$. If $f \in \omega^\omega$, the sequence $y = x \circ f$ is said to be a subsequence of the sequence $x$ and we write $y \prec x$.

If $(X, O)$ is a topological space, a point $a \in X$ is said to be a limit point of a sequence $x \in X^\omega$ (we will write: $x \xrightarrow{O} a$) iff each neighborhood $U$ of $a$ contains all but finitely many members of the sequence. A space $(X, O)$ is called sequential iff a set $A \subseteq X$ is closed whenever it contains each limit of each sequence in $A$.

If $X$ is a non-empty set, each mapping $\lambda : X^\omega \to P(X)$ is a convergence on $X$ and the mapping $u_\lambda : P(X) \to P(X)$, defined by $u_\lambda(A) = \bigcup_{x \in A^\omega} \lambda(x)$, is called the operator of sequential closure determined by $\lambda$. If $\lambda_1$ is another convergence on $X$, then we will write $\lambda \leq \lambda_1$ iff $\lambda(x) \subseteq \lambda_1(x)$, for each sequence $x \in X^\omega$. Clearly, $\leq$ is a partial order on the set $\operatorname{Conv}(X) = \{ \lambda : \lambda$ is a convergence on $X \}$. If $(X, O)$ is a topological space, then the mapping $\lim_O : X^\omega \to P(X)$ defined by $\lim_O(x) = \{ a \in X : x \xrightarrow{O} a \}$ is the convergence on $X$ determined by the topology $O$ and for the operator $\lambda = \lim_O$ we have (see [2])

(L1) $\forall a \in X \ a \in \lambda(\langle a \rangle)$;
(L2) $\forall x \in X^\omega \ \forall y \prec x \ \lambda(x) \subseteq \lambda(y)$;
(L3) $\forall x \in X^\omega \ \forall a \in X \ (\forall y \prec x \exists z \prec y \ a \in \lambda(z)) \Rightarrow a \in \lambda(x)$.

A convergence $\lambda : X^\omega \to P(X)$ is called a topological convergence iff there is a topology $O$ on $X$ such that $\lambda = \lim_O$. The following fact (see, for example, [8]) shows that each convergence has a minimal topological extension and connects topological and convergence structures.

**Fact 1.1.** Let $\lambda : X^\omega \to P(X)$ be a convergence on a non-empty set $X$. Then

(a) There is the maximal topology $O_\lambda$ on $X$ satisfying $\lambda \leq \lim_{O_\lambda}$;

(b) $O_\lambda = \{ O \subseteq X : \forall x \in X^\omega \ (O \cap \lambda(x) \neq \emptyset \Rightarrow \exists n_0 \in \omega \ \forall n \geq n_0 \ x_n \in O) \}$. 


The convergence of the sequences coding the ground model reals 279

(c) \( \langle X, \mathcal{O}_\lambda \rangle \) is a sequential space;
(d) \( \mathcal{O}_\lambda = \{ X \setminus F : F \subseteq X \land u_\lambda(F) = F \} \), if \( \lambda \) satisfies (L1) and (L2);
(e) \( \lim_{\mathcal{O}_\lambda} = \min \{ \lambda' \in \text{Conv}(X) : \lambda' \text{ is topological and } \lambda \leq \lambda' \} \);
(f) \( \mathcal{O}_{\lim_{\mathcal{O}_\lambda}} = \mathcal{O}_\lambda \);
(g) If \( \lambda_1 : X^w \to P(X) \) and \( \lambda_1 \leq \lambda_0 \), then \( \mathcal{O}_\lambda \subseteq \mathcal{O}_{\lambda_1} \);
(h) \( \lambda \) is a topological convergence iff \( \lambda = \lim_{\mathcal{O}_\lambda} \).

In our proofs we will mainly use the technique of forcing (see [4]). So, if \( \mathcal{B} \) is a complete Boolean algebra belonging to the ground model \( V \) of ZFC, \( V^\mathcal{B} \) will be the class of \( \mathcal{B} \)-names. For a formula \( \varphi(v_0, \ldots, v_n) \) and \( \tau_0, \ldots, \tau_n \in V^\mathcal{B} \) the corresponding Boolean value will be denoted by \( \| \varphi(\tau_0, \ldots, \tau_n) \|. \) If \( G \) is a \( \mathcal{B} \)-generic filter over \( V \) and \( \tau \in V^\mathcal{B} \), the \( G \)-evaluation of \( \tau \) will be denoted by \( \tau_G \).

For \( A \in V \), the corresponding \( \mathcal{B} \)-name will be \( \dot{A} = \{ (a, 1) : a \in A \} \).

Subsets of \( \omega \) are called \emph{reals} and can be coded by convenient names. Namely, each real belonging to a generic extension has a nice name of the form \( \tau_x = \{ \langle \dot{n}, x_n \rangle : n \in \omega \} \), where \( x_n = \| \dot{n} \in \tau \| \), for each \( n \in \omega \).

A real \( r \in [\omega]^w \cap V^\mathcal{B} [G] \) will be called: new iff \( r \notin V \); old iff \( r \in V \); dependent iff there is \( A \in [\omega]^w \cap V \) such that \( A \subset r \) or \( A \subset \omega \setminus r \); independent or a splitting real iff it is not dependent [6]; supported iff there is \( A \in [\omega]^w \cap V \) such that \( A \subset r \); unsupported if it is not supported [5]. Using the elementary properties of forcing it is easy to prove the following two facts (see [9])

**Fact 1.2.** Let \( x = \langle x_n : n \in \omega \rangle \) be a sequence in a complete Boolean algebra \( \mathcal{B} \) and \( \tau_x = \{ \langle \dot{n}, x_n \rangle : n \in \omega \} \) the corresponding \( \mathcal{B} \)-name for a subset of \( \omega \). Then

(a) \( \| \tau_x \in \omega \| = \bigwedge_{n \in \omega} x_n \);
(b) \( \| \tau_x \text{ is cofinite} \| = \bigvee_{k \in \omega} \bigwedge_{n \geq k} x_n = (\lim \inf x) \);
(c) \( \| \tau_x \text{ is old infinite} \| = \bigvee_{A \in [\omega]^w} \bigwedge_{n \in \omega} x_n^{\lambda \leq (n)} \); where \( x_n^1 = x_n, x_n^0 = x'_n \).
(d) \( \| \tau_x \text{ is supported} \| = \bigvee_{A \in [\omega]^w} \bigwedge_{n \in A} x_n \);
(e) \( \| \tau_x \text{ is dependent} \| = \bigvee_{A \in [\omega]^w} \bigwedge_{n \in A} x_n \lor \bigwedge_{n \in A} x'_n \);
(f) \( \| \tau_x \text{ is infinite} \| = \bigwedge_{k \in \omega} \bigvee_{n \geq k} x_n = (\lim \sup x) \);
(g) \( \| \tau_x = \omega \| \leq \| \tau_x \text{ is cofinite} \| \leq \| \tau_x \text{ is old infinite} \| \leq \| \tau_x \text{ is supported} \| \leq \| \tau_x \text{ is infinite dependent} \| \leq \| \tau_x \text{ is infinite} \| \).

**Proof.** We prove (c) and the rest of the proof is similar.

\[ \| \tau_x \text{ is old infinite} \| = \| \exists A \in (\omega)^{\omega} \forall \forall n \in A (n \in \tau_x) \land \forall n \in \omega \setminus A (n \notin \tau_x) \| = \bigvee_{A \in [\omega]^w} \bigwedge_{n \in A} x_n \land \bigwedge_{n \in \omega \setminus A} x'_n = \bigvee_{A \in [\omega]^w} \bigwedge_{n \in \omega} x_n^{\lambda \leq (n)} . \]
Fact 1.3. If \( x = (x_n : n \in \omega) \) is a sequence in a c.B.a. \( B \) and \( f \in \omega^{\omega} \), then \( y = x \circ f \) is a subsequence of \( x \) and for the \( \mathbb{B} \)-names \( \tau_x \) and \( \tau_y \) we have

(a) \( 1 \Vdash \tau_y = f^{-1}[\tau_x] \);
(b) \( \limsup y = \| f[\omega]^\omega \cap \tau_x \| = \check{\omega} \| \);
(c) \( \liminf y = \| f[\omega]^\omega \subseteq \ast \tau_x \| \);
(d) \( \liminf x \leq \liminf y \leq \limsup y \leq \limsup x \).

2. The convergence \( \lambda_1 \)

First, choosing a convenient notation, we present this research in the context of some previous results. Let \( B \) be a complete Boolean algebra and let the convergences \( \lambda_i : B^\omega \to P(B) \), for \( i \in \{0, 1, 2, 3, 4\} \), be defined by

\[
\lambda_i(x) = \begin{cases} 
\{b_i(x)\} & \text{if } b_i(x) = b_4(x), \\
\emptyset & \text{if } b_i(x) < b_4(x),
\end{cases}
\]

where

\[
b_0(x) = \| \tau_x \text{ is cofinite} \| = \liminf x,
b_1(x) = \| \tau_x \text{ is old infinite} \|,
b_2(x) = \| \tau_x \text{ is supported} \|,
b_3(x) = \| \tau_x \text{ is infinite dependent} \|,
b_4(x) = \| \tau_x \text{ is infinite} \| = \limsup x.
\]

Then \( \lambda_0 \) is the well known algebraic convergence \[11\] generating the sequential topology \( O_{\lambda_0} \) on \( B \) \[1\] related to the von-Neumann and the Maharam problem, \( \lambda_1 \) will be considered in this paper and the convergences \( \lambda_2, \lambda_3 \) and \( \lambda_4 \) were investigated in \[8\] and \[9\] and are related in the following way (see \[8\]).

Fact 2.1. Let \( B \) be a complete Boolean algebra. Then

(a) \( \lambda_2 \leq \lambda_3 \leq \lambda_4 \);
(b) \( \lambda_2, \lambda_3 \) and \( \lambda_4 \) satisfy condition (L1), but do not satisfy (L2);
(c) \( \lambda_3 = \lambda_4 \) iff \( \lambda_2 = \lambda_4 \) iff the algebra \( B \) is \( (\omega, 2) \)-distributive;
(d) \( \lambda_3 = \lambda_4 \) iff forcing by \( B \) does not produce splitting reals.

Thus \( \lambda_4(x) = \{ \limsup x \} \), if \( \| \tau_x \text{ is old infinite} \| = \| \tau_x \text{ is infinite} \| \) and \( \lambda_4(x) = \emptyset \), otherwise. Preliminarily we have
The convergence of the sequences coding the ground model reals

**Theorem 2.2.** Let \( \mathcal{B} \) be a complete Boolean algebra. Then

(a) For each sequence \( x \) in \( \mathcal{B} \) we have

\[
\lambda_1(x) = \begin{cases} 
\{\limsup x\} & \text{if } 1 \Vdash \tau_x \text{ is old,} \\
\emptyset & \text{otherwise;}
\end{cases}
\]

(b) \( \lambda_0 \leq \lambda_1 \leq \lambda_2 \);

(c) The convergence \( \lambda_1 \) satisfies condition (L1), but does not satisfy (L2);

(d) \( \lambda_1 = \lambda_2 \) iff the algebra \( \mathcal{B} \) is \((\omega, 2)\)-distributive;

(e) \( \lambda_1 = \lambda_4 \) iff the algebra \( \mathcal{B} \) is \((\omega, 2)\)-distributive.

**Proof.** (a)

\[
\lambda_1(x) \neq \emptyset \iff \|\tau_x\text{ is infinite}\| \land \|\tau_x\text{ is old}\| = \|\tau_x\text{ is infinite}\|
\]

\[
\iff \|\tau_x\text{ is infinite}\| \leq \|\tau_x\text{ is old}\|
\]

\[
\iff 1 \Vdash \tau_x \text{ is infinite} \Rightarrow \tau_x \text{ is old}
\]

\[
\iff 1 \Vdash \tau_x \text{ is finite} \lor \tau_x \text{ is old}
\]

\[
\iff 1 \Vdash \tau_x \text{ is old.}
\]

(b) follows from Fact 1.2(g).

(c) For a constant sequence \( x = \langle a \rangle \) we have \( a \Vdash \tau_x = \bar{\omega} \) and \( a' \Vdash \tau_x = \emptyset \), which implies \( 1 \Vdash "\tau_x \text{ is old}" \). Since \( \limsup x = a \), by (a) we have \( a \in \lambda_1(\langle a \rangle) \) and (L1) holds. For the sequence \( x = \langle 1, 0, 1, 0, \ldots \rangle \) we have \( 1 \Vdash \tau_x = \{0, 2, 4, \ldots \} \in V \) and, by (a), \( \lambda_1(x) = \{\limsup x\} = \{1\} \). But \( y = \langle 0, 0, 0, \ldots \rangle \prec x \) and \( 1 \Vdash \tau_y = \emptyset \), which, by (a), implies \( \lambda_1(y) = \{0\} \not\subseteq \lambda_1(x) \).

(d) By Theorem 7.5 of [8] for each sequence \( x \) in \( \mathcal{B} \) we have

\[
\lambda_2(x) = \begin{cases} 
\{\limsup x\} & \text{if } 1 \Vdash \tau_x \text{ is finite or supported,} \\
\emptyset & \text{otherwise.}
\end{cases}
\]

(\( \Leftarrow \)) If \( \mathcal{B} \) is \((\omega, 2)\)-distributive, it does not produce new reals and, hence, for each sequence \( x \) in \( \mathcal{B} \) we have \( 1 \Vdash "\tau_x \text{ is old}" \) and, clearly, \( 1 \Vdash "\tau_x \text{ is finite or supported}" \). So, by (a) and (2), \( \lambda_1(x) = \{\limsup x\} = \lambda_2(x) \).

(\( \Rightarrow \)) Suppose that the algebra \( \mathcal{B} \) is not \((\omega, 2)\)-distributive. Then there is an extension \( V[G] \) containing a new set \( X \subseteq \omega \). Let \( \sigma \) be a \( \mathcal{B} \)-name such that \( X = \sigma_G \) and \( 1 \Vdash \sigma \subset \bar{\omega} \) and let \( b \in G \), where

\[
b \Vdash \sigma \text{ is new.}
\]
If \( y = \langle y_n : n \in \omega \rangle \), where \( y_n = \| \hat{n} \in \sigma \| \), \( n \in \omega \), for \( \tau_y = \{ \langle \hat{n}, y_n \rangle : n \in \omega \} \) we have
\[
1 \models \sigma = \tau_y.
\]
For \( x = \langle y_0, 1, y_1, 1, y_2, 1, \ldots \rangle \) we have \( 1 \models \{ 1, 3, 5, \ldots \} \subset \tau_x \) and, hence, \( 1 \models \tau_x \) is supported, which, by (2) implies \( \lambda_2(x) \neq \emptyset \).

On the other hand \( y = x \circ f \), where \( f : \omega \to \omega \) is defined by \( f(k) = 2k \), so, by Fact 1.3(a), \( 1 \models \tau_y = f^{-1}[\tau_x] \), which, together with (3) and (4), implies \( b \models \text{”} f^{-1}[\tau_x] \text{ is new”} \). Now, since \( f \in V \), we have \( b \models \text{”} \tau_x \text{ is new”} \) and, by (a), \( \lambda_1(x) = \emptyset \). So \( \lambda_1 \neq \lambda_2 \).

(e) follows from (d) and Fact 2.1(c).

Remark 2.3. Imitating the proof of the part (a) of the previous theorem one can easily show that, if \( B \) is a complete Boolean algebra, \( x \) a sequence in \( B \) and \( \tau_x \) the corresponding name for a real, then the real determined by \( \tau_x \) is
- always old iff \( \lambda_1(x) \neq \emptyset \);
- sometimes new, but always supported iff \( \lambda_1(x) = \emptyset \) and \( \lambda_2(x) \neq \emptyset \);
- sometimes unsupported, but always unsplitting iff \( \lambda_2(x) = \emptyset \) and \( \lambda_3(x) \neq \emptyset \);
- sometimes splitting iff \( \lambda_3 = \emptyset \) and \( \lambda_4(x) \neq \emptyset \).

(Here “always” means in each and “sometimes” in some generic extension.)

By Fact 2.1(a) and Theorem 2.2(b) we have \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \); by Fact 2.1(c), \( \lambda_3 < \lambda_4 \) is impossible and, by Fact 2.1(c) and Theorem 2.2(d), \( \lambda_1 = \lambda_2 \Leftrightarrow \lambda_2 = \lambda_3 \). Now, using Fact 2.1(c), (d) and Theorem 2.2(d), we show that, up to these restrictions, everything is possible.

Example 2.4. \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 \) holds in each \((\omega, 2)\)-distributive and, in particular, each atomic complete Boolean algebra.
\( \lambda_1 < \lambda_2 < \lambda_3 = \lambda_4 \) holds in each complete Boolean algebra which produces new reals, but does not produce splitting reals, for example in \( \text{r.o.}(P) \), where \( P \) is the Sacks or the Miller forcing.
\( \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 \) holds in each complete Boolean algebra which produces splitting reals, for example in \( \text{r.o.}(\mathbb{P}) \), where \( \mathbb{P} \) is the Cohen or the random forcing.

3. The closure of \( \lambda_1 \) under (L2)

By Theorem 2.2(c), the convergence \( \lambda_1 \) does not satisfy (L2) and, hence, it is never a topological convergence. The minimal closures of a convergence under (L2) and (L3) are described in the following general fact (see [8]).
Fact 3.1. Let \( \lambda : X^\omega \to P(X) \) be a convergence satisfying condition (L1). Then

(a) The mapping \( \tilde{\lambda} : X^\omega \to P(X) \) defined by \( \tilde{\lambda}(y) = \bigcup_{x \in X^\omega, f \in \omega^\omega, y = x \circ f} \lambda(x) \) is the minimal convergence bigger than \( \lambda \) and satisfying (L1) and (L2);

(b) \( \check{\lambda}^* : X^\omega \to P(X) \) defined by \( \check{\lambda}^*(y) = \bigcap_{f \in \omega^\omega} \bigcup_{g \in \omega^\omega} \check{\lambda}(y \circ f \circ g) \) is the minimal convergence bigger than \( \check{\lambda} \) and satisfying (L1)–(L3);

(c) \( \lambda \leq \check{\lambda} \leq \check{\lambda}^* \leq \lim \mathcal{O}_\lambda \);

(d) \( \mathcal{O}_\lambda = \mathcal{O}_{\check{\lambda}} = \mathcal{O}_{\check{\lambda}^*} \).

For a subset \( A \) of a complete Boolean algebra \( \mathbb{B} \) let \( A \uparrow = \{ b \in \mathbb{B} : \exists a \in A \ a \leq b \} \). The (L2)-closures of the convergences \( \lambda_2 \), \( \lambda_3 \) and \( \lambda_4 \) are described in the following fact (see [8] and [9]).

Fact 3.2. Let \( \mathbb{B} \) be a complete Boolean algebra. Then

(a) \( \check{\lambda}_4(y) = \{ \limsup y \} \uparrow \), for each sequence \( y \) in \( \mathbb{B} \);

(b) \( \check{\lambda}_2 = \check{\lambda}_3 = \check{\lambda}_4 \);

(c) The convergence \( \check{\lambda}_4 \) generalizes the convergence on the Aleksandrov cube.

Now, concerning the convergence \( \lambda_1 \) we have

Theorem 3.3. Let \( \mathbb{B} \) be a complete Boolean algebra. Then

(a) The closure of \( \lambda_1 \) under (L2) is given by

\[
\check{\lambda}_1(y) = \begin{cases} 
\{ \limsup y \} \uparrow & \text{if } 1 \forces \tau_y \text{ is old}, \\
\emptyset & \text{otherwise};
\end{cases}
\]

(b) \( \check{\lambda}_1 = \check{\lambda}_4 \) iff the algebra \( \mathbb{B} \) is \((\omega, 2)\)-distributive.

Proof. (a) Claim 1. \( \check{\lambda}_1(y) = \{ \limsup y \} \uparrow \) if and only if \( 1 \forces \tau_y \) is old.

Proof of Claim 1. (\( \Rightarrow \)) Let \( \check{\lambda}_1(y) = \{ \limsup y \} \uparrow \). Then, by Fact 3.1(a) the set \( \lambda_1(y) = \bigcup_{x \in \mathbb{B}^\omega, f \in \omega^\omega, y = x \circ f} \lambda_1(x) \) is nonempty and, hence there are \( x \in \mathbb{B}^\omega \) and \( f \in \omega^\omega \) such that \( y = x \circ f \) and \( \lambda_1(x) \neq \emptyset \). By Theorem 2.2(a), \( 1 \forces \tau_x \) is old and by Fact 1.3(a), \( 1 \forces \tau_y = f^{-1}[^1[\tau_x]], \) which implies \( 1 \forces \tau_y \) is old".

(\( \Leftarrow \)) Let \( 1 \forces \tau_y \) is old. According to Fact 3.1(a) we show that

\[
\bigcup_{x \in \mathbb{B}^\omega, f \in \omega^\omega, y = x \circ f} \lambda_1(x) = \{ \limsup y \} \uparrow .
\]

(\( \subseteq \)) Suppose that \( x \in \mathbb{B}^\omega, f \in \omega^\omega, y = x \circ f \) and \( b \in \lambda_1(x) \). Then \( b = \limsup x \) and, since \( y \prec x \), by Fact 1.3(d) we have \( \limsup y \leq \limsup x = b \).
(>) Let \( b \geq \limsup y \). Let \( x = \langle y_0, b, y_1, b, y_2, \ldots \rangle \) and \( f, g \in \omega^\omega \), where \( f(k) = 2k \) and \( g(k) = 2k + 1 \). Then \( y = x \circ f \) and, if \( z = x \circ g \), using Facts 1.2(f) and 1.3(b) we have

\[
\limsup x = ||\tau_x| = \hat{\omega}|| = ||\tau_x \cap f[\omega]| = \hat{\omega}|| \lor ||\tau_x \cap g[\omega]| = \hat{\omega}||
\]

\[
= ||\tau_y| = \hat{\omega}|| \lor ||\tau_z| = \hat{\omega}|| = ||\tau_y| = \hat{\omega}|| \lor b = b.
\]

So, by Theorem 2.2(a), for a proof that \( b \in \lambda_1(x) \) it remains to be shown that \( 1 \vdash \text{“} \tau_x \text{ is old} \)”, which follows from \( 1 \vdash \text{“} \tau_y \text{ is old} \) and the following subclaim.

Subclaim 1. (i) \( b' \vdash \tau_x = \bar{f}[\tau_y] \); (ii) \( b \vdash \tau_y = \bar{f}[\tau_y] \cup \{1, 3, 5, \ldots \} \).

Proof of Subclaim 1. By Fact 1.3(a) we have \( 1 \vdash \tau_y = \bar{f}^{-1}[\tau_x] \) and, hence,

\[
1 \vdash \bar{f}[\tau_y] \subset \tau_x.
\] (6)

Let \( G \) be a \( \mathcal{B} \)-generic filter over \( V \).

(i) If \( b' \in G \), then for \( n \in \langle \tau_x \rangle_G \) we have \( x_n \in G \) and, since \( b \notin G \), there is \( k \in \omega \) such that \( x_n = x_{2k} = y_k \). Hence \( k \in (\tau_y)_G \) and \( n = f(k) \in f(\tau_y)_G \). So \( b' \vdash \tau_x \subset \bar{f}[\tau_y] \) and, by (6), \( b' \vdash \tau_x = \bar{f}[\tau_y] \).

(ii) Clearly, \( b \vdash \{1, 3, 5, \ldots \} \subset \tau_x \) and, by (6), \( b \vdash \bar{f}[\tau_y] \subset \tau_x \). On the other hand, let \( b \in G \) and \( n \in (\tau_x)_G \), that is \( x_n \in G \). If \( n \) is odd, we are done. Otherwise, as in (a) we show that \( n \in f(\tau_y)_G \). Claim 1 is proved.

Claim 2. \( \lambda_1(y) \neq \emptyset \iff 1 \vdash \tau_y \text{ is old} \).

Proof of Claim 2. (\( \Rightarrow \)) Suppose that \( a \in \lambda_1(y) \). Then, by Fact 3.1(a), there are \( x \in \mathcal{B}^\omega \) and \( f \in \omega^\omega \) such that \( y = x \circ f \) and \( a \in \lambda_1(x) \), which, by Theorem 2.2(a), implies \( 1 \vdash \text{“} \tau_x \text{ is old} \)”. By Fact 1.3(a) we have \( 1 \vdash \tau_y = f^{-1}[\tau_x] \) and, consequently, \( 1 \vdash \text{“} \tau_y \text{ is old} \)”. (\( \Leftarrow \)) If \( \lambda_1(y) = \emptyset \), then, since \( \lambda_1 \leq \lambda_1 \), we have \( \lambda_1(y) = \emptyset \) and, by Theorem 2.2(a), \( \neg 1 \vdash \tau_y \text{ is old} \).

(b) It is well known [4] that \( \mathcal{B} \) is \( (\omega, 2) \)-distributive iff forcing by \( \mathcal{B} \) does not produce new reals, that is \( 1 \vdash \tau_y \text{ is old} \), for each sequence \( y \) in \( \mathcal{B} \). So we apply (a) and Fact 3.2(a).

4. The topology generated by \( \lambda_1 \)

By Theorem 3.3(b) and Fact 3.1(d), if \( \mathcal{B} \) is an \( (\omega, 2) \)-distributive algebra, then \( \mathcal{O}_{\lambda_1} = \mathcal{O}_{\lambda_1} \). In this section we show more, that on each complete Boolean algebra the convergences \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \) generate the same topology, investigated in [9]. Concerning the convergences \( \lambda_0, \lambda_2, \lambda_3 \) and \( \lambda_4 \) we have (see [8] and [9])
Let \( \mathcal{O}_2 = \mathcal{O}_3 = \mathcal{O}_4 \); 

(b) \( \mathcal{O}_4 \) is a sequential \( T_0 \) connected compact topology on \( \mathbb{B} \); 

(c) \( \mathcal{O}_4 \) and its dual generate the sequential topology, \( \mathcal{O}_{\lambda_0} \), when \( \mathbb{B} \) is a Maharam algebra. 

We will use the following general fact (see [8]). 

**Fact 4.2.** Let \( \lambda : X^\omega \to P(X) \) be a convergence satisfying (L1) and (L2) and let the mappings \( u_\alpha^n : P(X) \to P(X), \alpha \leq \omega_1 \), be defined by recursion in the following way: for \( A \subset X \) 

\[
\begin{align*}
    u_0^n(A) & = A, \\
    u_{\alpha+1}^n(A) & = u_\alpha(u_\alpha^n(A)) \text{ and} \\
    u_\gamma^n(A) & = \bigcup_{n<\gamma} u_\alpha^n(A), \text{ for a limit } \gamma \leq \omega_1. 
\end{align*}
\]

Then \( u_\alpha^n \) is the closure operator in the space \( \langle X, \mathcal{O}_\lambda \rangle \). 

We will say that a subset \( A \) of a c.B.a. \( \mathbb{B} \) is **upward closed** iff \( A = A^\uparrow \). A sequence \( x \) in \( \mathbb{B} \) will be called **decreasing** if \( x_0 \geq x_1 \geq x_2 \geq \ldots \).

**Lemma 4.3.** Let \( \mathbb{B} \) be a complete Boolean algebra. Then 

(a) The set \( \tilde{\lambda}_1(x) \) is upward closed, for each sequence \( x \) in \( \mathbb{B} \); 

(b) If \( x \) is a decreasing sequence in \( \mathbb{B} \), then \( \tilde{\lambda}_1(x) = \{ \bigwedge_{n \in \omega} x_n \} \); 

(c) If \( A \subset \mathbb{B} \) is an upward closed set, then \( u_{\tilde{\lambda}}(A) = u_{\tilde{\lambda}}(A) \); 

(d) The set \( u_{\tilde{\lambda}}(A) \) is upward closed, for each \( A \subset \mathbb{B} \). 

**Proof.** (a) follows from Theorem 3.3. 

(b) If \( x = \langle x_n : n \in \omega \rangle \) is decreasing, then \( \limsup x = \bigwedge_{n \in \omega} \bigvee_{k \geq n} x_k = \bigwedge_{n \in \omega} x_n \) and, by Theorem 2.2(a), it remains to be shown that \( 1 \vdash \tau \) is old. If \( G \) is a \( \mathbb{B} \)-generic filter over \( V \), then \( (\tau_G) = \{ n : x_n \in G \} \), so if \( m < n \in (\tau_G)_G \), then \( x_m \geq x_n \in G \), which implies \( x_m \in G \) and, hence, \( m \in (\tau_G)_G \). Thus \( (\tau_G)_G \) is either a finite set or equal to \( \omega \) and, consequently, belongs to \( V \). 

(c) Let \( A \subset \mathbb{B} \) be an upward closed set. 

(\( \supset \)) Since \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \), by the minimality of \( \lambda_1 \) (see Fact 3.1(a)) we have \( \lambda_1 \leq \lambda_2 \) and, hence, \( u_{\lambda_1}(A) = \bigcup_{x \in A^\omega} \lambda_1(x) \subseteq \bigcup_{x \in A^\omega} \lambda_3(x) = u_{\lambda_2}(A) \). 

(\( \subset \)) By Fact 3.2 we have \( \lambda_3(x) = \{ \limsup x \}^\uparrow \). So, for \( x \in A^\omega \) we show that \( \{ \limsup x \}^\uparrow \subset u_{\lambda_1}(A) \). Let \( \limsup x = b \). Then the sequence \( t = \langle t_n : n \in \omega \rangle \) defined by 

\[
t_n = b \lor \bigvee_{k \geq n} x_k
\]
is decreasing and, since $t_n \geq x_n \in A$, we have $t \in A^\omega$. Since
\[
\bigwedge_{n \in \omega} t_n = b \lor \bigvee_{n \in \omega} \bigvee_{k \geq n} x_k = b \lor \limsup x = b
\]
by (b) we have $b \in \lambda_1(t) \subset \lambda_1(t)$ and, by (a), \{b\} $\uparrow \lambda_1(t) \uparrow= \lambda_1(t) \subset u\lambda_1(A)$.

(d) We prove that $u\lambda_1(A) \subset u\lambda_1(A)$. If $b \geq a \in u\lambda_1(A)$, then there is $x \in A^\omega$ such that $a \in \lambda_1(x)$. By (a) we have $b \in \lambda_1(x)$, which implies $b \in u\lambda_1(A)$. $\square$

**Theorem 4.4.** Let $\mathbb{B}$ be a complete Boolean algebra. Then
\begin{enumerate}[(a)]
  \item $u^{\omega_1}_A(u^{\omega_1}_A(A))$, for each $A \subset \mathbb{B}$;
  \item $O\lambda_1 = O\lambda_1 = O\lambda_2 = O\lambda_3$.
\end{enumerate}

**Proof.** (a) (\subset) Since $\lambda_1 \leq \lambda_2$, we have $u^{\omega_1}_A(A) \subset u^{\omega_1}_2(A)$, for each $A \subset \mathbb{B}$.

(\supset) First, for $A \subset \mathbb{B}$ using induction we show that for each $\alpha \leq \omega_1$\begin{equation}
\text{(7)} \quad u_{\lambda_1}^{\omega_1}(u_{\lambda_1}(A)) = u_{\lambda_1}^{\omega_1}(u_{\lambda_1}(A)) \text{ and this set is upward closed.}
\end{equation}
By Lemma 4.3(d), (7) is true for $\alpha = 0$.

Let $\beta \leq \omega_1$ and suppose that (7) holds for each $\alpha < \beta$.

If $\beta$ is a limit ordinal, then, by the induction hypothesis, we have
\[
u_{\lambda_2}^{\beta}(u_{\lambda_1}(A)) = \bigcup_{\alpha < \beta} u_{\lambda_2}^{\alpha}(u_{\lambda_1}(A)) = \bigcup_{\alpha \leq \beta} u_{\lambda_1}^{\omega_1}(u_{\lambda_1}(A)) = u_{\lambda_1}^{\omega_1}(u_{\lambda_1}(A))
\]
and, since the union of upward closed sets is upward closed, (7) is true for $\beta$.

If $\beta = \alpha + 1$, then, by the induction hypothesis we have
\begin{equation}
\text{(8)} \quad u_{\lambda_2}^{\alpha+1}(u_{\lambda_1}(A)) = u_{\lambda_2}(u_{\lambda_1}(A)) = u_{\lambda_2}(u_{\lambda_1}^{\omega_1}(u_{\lambda_1}(A))).
\end{equation}
By the hypothesis the set $u_{\lambda_1}^{\omega_1}(u_{\lambda_1}(A))$ is upward closed and, by Lemma 4.3(c),
\begin{equation}
\text{(9)} \quad u_{\lambda_2}(u_{\lambda_1}^{\omega_1}(u_{\lambda_1}(A))) = u_{\lambda_1}(u_{\lambda_1}^{\omega_1}(u_{\lambda_1}(A))) = u_{\lambda_1}^{\omega_1+1}(u_{\lambda_1}(A))
\end{equation}
and $u_{\lambda_2}^{\beta}(u_{\lambda_1}(A)) = u_{\lambda_1}^{\omega_1}(u_{\lambda_1}(A))$ follows from (8) and (9). By Lemma 4.3(d) and (9) this set is upward closed and the proof of (7) is over.

Since $A \subset u_{\lambda_1}(A) \subset u_{\lambda_1}^{\omega_1}(A)$, by Fact 4.2 we have $u_{\lambda_1}(u_{\lambda_1}(A)) = u_{\lambda_1}^{\omega_1}(A)$. Using (7) we obtain $u_{\lambda_1}^{\omega_1}(A) \subset u_{\lambda_1}^{\omega_1}(u_{\lambda_1}(A)) = u_{\lambda_1}^{\omega_1}(u_{\lambda_1}(A)) = u_{\lambda_1}^{\omega_1}(A)$.

(b) By (a) and Fact 4.2 we have $O_{\lambda_2} = O_{\lambda_2}$ and, by Fact 3.1(d), $O_{\lambda_1} = O_{\lambda_2}$.

By Fact 4.1(a), the other two equalities hold as well. $\square$

Thus the topology $O_{\lambda_1}$, generated by the convergence $\lambda_1$, has the properties given in Fact 4.1(b) and (c).
5. Topological and weakly topological convergences

In this section we investigate the classes of complete Boolean algebras on which the convergence $\bar{\lambda}_1$ (satisfying conditions (L1) and (L2)) is topological or weakly topological. According to [8], a convergence $\lambda : X^\omega \to P(X)$ will be called weakly topological iff it satisfies conditions (L1) and (L2) and its (L3)-closure, $\lambda^*$, is a topological convergence. The following general fact can be found in [8].

**Fact 5.1.** A convergence $\lambda : X^\omega \to P(X)$ satisfying (L1) and (L2) is weakly topological iff $\lambda^* = \lim O\lambda$, that is for each $x \in X^\omega$ and $a \in X$

$$a \in \lim O\lambda(x) \iff \forall y \prec x \exists z \prec y a \in \lambda(z).$$

By [9], for the convergence $\bar{\lambda}_4$ we have

**Fact 5.2.** Let $B$ be a complete Boolean algebra. Then

(a) $\bar{\lambda}_4$ is a topological convergence iff the algebra $B$ is $(\omega, 2)$-distributive;

(b) If the algebra $B$ satisfies (h), then $\bar{\lambda}_4$ is a weakly topological convergence.

We note that, according to [7], a complete Boolean algebra satisfies condition (h) iff $\forall x \in B^\omega \exists y \prec x \forall z \prec y \lim sup z = \lim sup y$. More about condition (h) (implied by the ccc) can be found in [10].

For the convergence $\bar{\lambda}_1$ we have the following analogue of Fact 5.2(a).

**Theorem 5.3.** $\bar{\lambda}_1$ is a topological convergence iff the algebra $B$ is $(\omega, 2)$-distributive.

**Proof.** ($\Rightarrow$) Let $\bar{\lambda}_1$ be a topological convergence. Then, by Fact 1.1(h), $\lambda_1 = \lim O\lambda_1$. By Fact 3.1(d) and Theorem 4.4(b) we have $O\lambda_1 = O\lambda_1 = O\lambda_2$ thus $\lambda_1 = \lim O\lambda_1 \geq \lambda_2$. Since $\lambda_1 \leq \lambda_2 \leq \lambda_1$, by Fact 3.1(a) we have $\lambda_1 = \lambda_2$ and, by Fact 3.2(b) and Theorem 3.3(b) the algebra $B$ is $(\omega, 2)$-distributive.

($\Leftarrow$) follows from Theorem 3.3(b) and Fact 5.2(a).

Now we deal with the question on which algebras the convergence $\bar{\lambda}_1$ is weakly topological. First we describe its (L3)-closure, $\bar{\lambda}_1^*$, in terms of forcing.

**Theorem 5.4.** Let $B$ be a complete Boolean algebra. Then for $y \in B^\omega$ we have

(a) $\bar{\lambda}_1^*(y) = \cap_{A \in [\omega]^{\omega}} B \in [\omega]^{\omega} \cap \| y \cap B \| \text{ is old } = 1 \| y \cap B \| = \omega \uparrow;$

(b) $\bar{\lambda}_1^*(y) \neq \emptyset$ iff $D_y = \{ B \in [\omega]^{\omega} : \| y \cap B \| \text{ is old } = 1 \}$ is a dense set in the poset $(\langle [\omega]^{\omega}, \subset \rangle).$
(a) By Fact 3.1(b), for \( g \in B^2 \) we prove that
\[
\bigcap_{f \in \omega^1} \bigcup_{g \in \omega^1} \lambda_f(y \circ f \circ g) = \bigcap_{A \in \omega^1} \bigcup_{B \in [A]^\omega \wedge \|\tau_g \cap B\| = \omega^\uparrow} \|\tau_y \cap \bar{B}\| = \omega^\uparrow
\]

(\( \subseteq \)) Suppose that for each \( f \in \omega^1 \) \( f \neq \omega^1 \) there is \( g \in \omega^1 \) such that \( a \in \lambda_f(y \circ f \circ g) \), which, by Theorem 3.3 and Fact 1.3, means that
\[
\|\tau_g \circ f \circ g \| \text{ is old} \quad \text{and} \quad a \in \|\tau_y \cap (f \circ g)[\omega^1]\| \text{ is old}. \quad (10)
\]
Let \( A \in [\omega]^\omega \) and let \( f \in \omega^2 \), where \( A = f[\omega] \). By the assumption, there is \( g \in \omega^1 \) such that (10) holds. Then \( B = f[g[\omega]] \subset A \) and, since \( f \) and \( g \) are injections, \( B \in [A]^\omega \). By (10), \( a \in \|\tau_y \cap \bar{B}\| = \omega^\uparrow \). By Fact 1.3, in each generic extension \( V_\kappa[G] \) we have \( (\tau_{g \circ f \circ g})_G = (f \circ g)^{-1}[\tau_g]_G = (f \circ g)^{-1}[\tau_y]_G \cap B \) and, hence, \( (\tau_y)_G \cap B = f[g[\tau_{g \circ f \circ g}]_G] \). Thus
\[
\|\tau_{g \circ f \circ g} \| \text{ is old} \quad \Rightarrow \quad \tau_y \cap \bar{B} \text{ is old} \| = 1,
\]
which together with (10) implies \( \|\tau_y \cap \bar{B}\| = 1 \).

(\( \supseteq \)) Suppose that for each \( A \in [\omega]^\omega \) there is \( B \in [A]^\omega \) such that
\[
\|\tau_y \cap \bar{B}\| = 1 \quad \text{and} \quad a \in \|\tau_y \cap (f \circ g)[\omega^1]\| = \omega^\uparrow. \quad (12)
\]
Let \( f \in \omega^1 \) and \( A = f[\omega] \). By the assumption, there is \( B \in [A]^\omega \) such that (12) holds. If \( g \in \omega^1 \) where \( g[\omega] = f^{-1}[B] \), then \( B = (f \circ g)[\omega] \) and, by (12), we have
\[
a \in \|\tau_y \cap (f \circ g)[\omega^1]\| = \omega^\uparrow. \quad \text{By (11), } \|\tau_{g \circ f \circ g} \| = 1 \quad \text{and, thus } a \in \lambda_f(y \circ f \circ g).
\]
(b) \( \Rightarrow \) Let \( a \in \lambda_f(y) \) and \( A \in [\omega]^\omega \). By (a) there is \( B \in [A]^\omega \) such that \( \|\tau_y \cap \bar{B}\| = 1 \) and \( a \geq \|\tau_y \cap \bar{B}\| = \omega^\uparrow \). Thus \( B \subset A \) and \( B \in D_y \).

(\( \Leftarrow \)) Let \( D_y \) be a dense set in \( [\omega]^\omega \) and \( a = \|\tau_y\| = \omega^\uparrow \). Since for each \( A \in [\omega]^\omega \) there is \( B \in [A]^\omega \) such that \( \tau_y \cap \bar{B} \) is old \( \| = 1 \) and, clearly, \( a \geq \|\tau_y \cap \bar{B}\| = \omega^\uparrow \), by (a) we have \( a \in \lambda_f(y) \).

\[\Box\]

**Theorem 5.5.** If there is a sequence \( y \in B \) such that \( \|\tau_y\| \text{ is splitting} | > 0 \), then \( \lambda_f(y) = \emptyset \) and the convergence \( \lambda_1 \) is not weakly topological.

**Proof.** Let \( \|\tau_y\| \text{ is splitting} \| = b > 0 \) and suppose that \( \lambda_f(y) \neq \emptyset \). Then, by (b), there is \( B \in [\omega]^\omega \) such that \( 1 \cap \tau_y \cap \bar{B} \) is old. But then \( b \Rightarrow \text{"} \tau_y \cap \bar{B} \text{ is old and } \tau_y \text{ is splitting"} \), which is impossible. Thus \( \lambda_f(y) = \emptyset \). By Theorem 4.4, \( \lim_{\lambda_1(y)} = \lim_{\lambda_1(y)} \cup \lambda_4(y) \supset \lim_{\sup} y \) and, hence, \( \lambda_f(y) \neq \lim_{\lambda_1(y)} \) so \( \lambda_1 \) is not a weakly topological convergence. \[\Box\]
Concerning the previous theorem we remark that it is possible that the convergence \( \bar{\lambda}_1 \) is not weakly topological, although forcing by \( \mathbb{B} \) does not produce splitting reals (see Example 5.8). In contrast to Fact 5.2(b) we have

**Example 5.6.** The ccc (and, consequently, condition \((h)\)) does not imply that the convergence \( \bar{\lambda}_1 \) is weakly topological. The Cohen algebra is ccc, produces splitting reals and, by Theorem 5.5, \( \bar{\lambda}_1 \) is not a weakly topological convergence.

Now, inside a wide class of complete Boolean algebras, we characterize the algebras on which the convergence \( \bar{\lambda}_1 \) is weakly topological.

**Theorem 5.7.** Let \( \mathbb{B} \) be a Boolean algebra such that \( hcc(\mathbb{B}) > \epsilon \) (i.e. below each \( b \in \mathbb{B}^+ \) there is an antichain of size \( \epsilon \)). Then

\[
\bar{\lambda}_1 \text{ is a weakly topological convergence } \iff \mathbb{B} \text{ is } (\omega, 2)-\text{distribution}. \tag{13}
\]

**Proof.** (\( \Rightarrow \)) This implication follows from Theorem 5.3.

(\( \Leftarrow \)) If \( \mathbb{B} \) is not \((\omega, 2)\)-distributive, then \( b = \| \exists r \subset \bar{\omega} (r \text{ is new}) \| > 0 \) and, by the Maximum Principle, there is a name \( \pi \) such that

\[
b \models \pi \subset \bar{\omega} \land \pi \text{ is new.} \tag{14}
\]

We choose an enumeration \( [\omega]^\omega = \{ S_\alpha : \alpha < \epsilon \} \), bijections \( f_\alpha : S_\alpha \to \omega \), \( \alpha < \epsilon \), and a maximal antichain under \( b \), \( \{ b_\alpha : \alpha < \epsilon \} \). Now, for the \( \mathbb{B} \)-name \( \sigma \) defined by \( \sigma = \{ (\bar{n}, \bigvee_{\alpha < \epsilon} (b_\alpha \land \| f_\alpha(n)^- \in \pi \|)) : n \in \omega \} \) it is easy to prove that \( b_\alpha \models \sigma = f^{-1}_\alpha[\pi] \), for \( \alpha < \epsilon \), (see \([7, \text{Th. 4, Cl. 1}]\)) and, clearly, \( 1 \models \sigma = \tau_x \), where \( x = (x_n : n \in \omega) \) and \( x_n = \bigvee_{\alpha < \epsilon} b_\alpha \land \| f_\alpha(n)^- \in \pi \|, n \in \omega \). Thus

\[
b_\alpha \models \tau_x = f^{-1}_\alpha[\pi]. \tag{15}
\]

Let us prove

\[
\forall B \in [\omega]^\omega \| \tau_x \cap \bar{B} \text{ is new} \| > 0. \tag{16}
\]

Let \( B \in [\omega]^\omega \) and \( \alpha < \epsilon \), where \( B = S_\alpha \). Let \( G \) be a \( \mathbb{B} \)-generic filter over \( V \) containing \( b_\alpha \). Since \( b_\alpha < b \) we have \( b \in G \) and, by (14), \( \pi_G \notin V \). By (15), \( (\tau_x)_G = f^{-1}_\alpha[\pi_G] \subset B \). Now, \( f^{-1}_\alpha[\pi_G] \in V \) would imply \( f_\alpha[f^{-1}_\alpha[\pi_G]] = \pi_G \in V \), which is false. Thus \( f^{-1}_\alpha[\pi_G] = (\tau_x)_G = (\tau_x)_G \land \bar{B} \notin V \) and (16) is proved.

By (16) we have \( D_\alpha = \{ B \in [\omega]^\omega : 1 \models \tau_x \cap \bar{B} \text{ is old} \} = \emptyset \) so, by Theorem 5.4(b), \( \bar{\lambda}_1(x) = \emptyset \). But, by Theorem 4.4, \( \limsup x \in \lim_{\alpha, \omega} (x) = \lim_{\alpha, \omega} (x) \) and, by Fact 5.1, \( \bar{\lambda}_1 \) is not a weakly topological convergence.

**Example 5.8.** \( \bar{\lambda}_1 \) is not a weakly topological convergence on the Sacks algebra.

Namely, if \( \mathbb{B} \) is the Boolean completion of the Sacks forcing, \( \mathbb{B} \) is homogeneous, has antichains of size \( \epsilon \), adds new reals and we apply Theorem 5.7.
Is the equivalence (13) a theorem of ZFC? In the following theorem, using a result of Veličković [12], we show that under the CH, a possible counterexample can not be nicely definable.

**Theorem 5.9.** (CH) If \( B = r.o.(P) \), where \( P \) is a Suslin forcing notion, then

\[ \bar{\lambda}_1 \text{ is a weakly topological convergence } \Leftrightarrow B \text{ is } (\omega, 2)\text{-distributive.} \]

**Proof.** (\( \Leftarrow \)) This implication follows from Theorem 5.3.

(\( \Rightarrow \)) If the algebra \( B \) is not \( (\omega, 2)\)-distributive, then \( b = \| \exists r \subseteq \check{\omega} \text{ (} r \text{ is new)} \| > 0 \). If there exists an uncountable antichain below \( b \), then, as in the proof of Theorem 5.7, we show that \( \bar{\lambda}_1 \) is not a weakly topological convergence. Otherwise, \( B|b \) is a non-atomic ccc forcing, clearly, \( P \cap b \downarrow \) is a non-atomic ccc Suslin forcing and, by a result of Veličković [12], produces splitting reals. Now, by Theorem 5.5, \( \bar{\lambda}_1 \) is not a weakly topological convergence again.

\[ \square \]

6. A diagram

Here we describe the relations between the convergence structures considered in this paper.

**Theorem 6.1.** Let \( B \) be a complete Boolean algebra. Then

(a) If \( A \subset [\omega]^{\omega} \) is a mad family, \( y \) a sequence in \( B \) and \( 1 \Vdash \tau_y \text{ kills } \check{A} \), then \( \bar{\lambda}_1^*(y) = B \) and \( \bar{\lambda}_4(y) = \{1\} \);

(b) If forcing by \( B \) produces a splitting real in each extension, then the convergences \( \bar{\lambda}_1^* \) and \( \bar{\lambda}_4 \) are not comparable;

**Proof.** (a) Suppose that \( 1 \Vdash |\tau_y| = \check{\omega} \wedge \forall A \in \check{A} \ |\tau_y \cap A| < \check{\omega} \). Then \( \| \tau_y \text{ is infinite} \| = 1 \) and, by Facts 1.2 and 3.2(a), we have \( \lambda_4(y) = \{1\} \uparrow = \{1\} \).

Using Theorem 5.4(a) we prove that \( 0 \in \bar{\lambda}_1^*(y) \) (which implies \( \bar{\lambda}_1^*(y) = B \)). For \( A \in [\omega]^{\omega} \), by the maximality of \( A \), there is \( A_1 \in A \) such that \( B = A \cap A_1 \in [A]^{\omega} \). Since \( 1 \Vdash |\tau_y \cap A_1| < \check{\omega} \), we have \( \| |\tau_y \cap B| < \check{\omega} \| = 1 \) which implies \( \| \tau_y \cap B \text{ is old} \| = 1 \) and \( \| |\tau_y \cap B| = \check{\omega} \| = 0 \).

(b) \( \bar{\lambda}_4 \not\leq \bar{\lambda}_1^* \). By the assumption, there is \( y \in B^{\omega} \) such that \( \| \tau_y \text{ is splitting} \| > 0 \) so, by Theorem 5.5, \( \bar{\lambda}_1^*(y) = \emptyset \) and \( \bar{\lambda}_4(y) \neq \emptyset \).

(\( \bar{\lambda}_1^* \not\leq \lambda_4 \)). It is known (see [7, Lemma 1]) that there is a mad family \( A \subset [\omega]^{\omega} \) which is killed in each generic extension of the ground model containing new reals.

By the assumption, forcing by \( B \) produces new reals in each extension and, hence,
The convergence of the sequences coding the ground model reals

we have $1 \vDash \exists x \subset \omega \ (x \text{ kills } \dot{A})$ and, by the Maximum Principle, there is a $\mathbb{B}$-name $\sigma$ such that $1 \vDash \sigma \subset \omega$ and $1 \vDash \sigma \text{ kills } \dot{A}$. If $y_n = \|\bar{n} \in \sigma\|$, $n \in \omega$, then $1 \vDash \tau_y = \sigma$ and $1 \vDash \tau_y \text{ kills } \dot{A}$. By (a) we have $\bar{\lambda}_1^*(y) = \mathbb{B}$ and $\bar{\lambda}_4(y) = \{1\}$.

In the following diagram $\lambda' \leq \lambda''$ denotes that for each c.B.a. $\mathbb{B}$ and each sequence $x$ in $\mathbb{B}$, $\lambda'(x) \subset \lambda''(x)$.

In the sequel we show that the diagram is correct. By Fact 2.1(a) and Theorem 2.2(b) we have $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ and, by Example 2.4, all the inequalities can be strict. By Fact 3.2(b) we have $\bar{\lambda}_2 = \bar{\lambda}_3 = \bar{\lambda}_4$, which implies $\bar{\lambda}_2^* = \bar{\lambda}_3^* = \bar{\lambda}_4^*$ and $\lim_{\mathcal{O}_{\lambda_2}} = \lim_{\mathcal{O}_{\lambda_3}} = \lim_{\mathcal{O}_{\lambda_4}}$. By Theorem 4.4(b) we have $\lim_{\mathcal{O}_{\lambda_2}} = \lim_{\mathcal{O}_{\lambda_3}} = \lim_{\mathcal{O}_{\lambda_4}}$. By Theorem 5.5. If $y \in \mathbb{B}^{\omega}$, where $\|\tau_y\|$ is splitting $> 0$, then $\bar{\lambda}_1^*(y) = \emptyset$. For the sequence $z = \langle y_0, 1, y_1, 1, \ldots \rangle$ we have $y \prec z$ and, since $\bar{\lambda}_1^*(y) = \emptyset$. But $1 \vDash \{1, 3, 5, \ldots \} \subset \tau_z$, thus $1 \vDash \text{“} \tau_z \text{ is supported”}$ which, by (2), implies $\lambda_2(z) \neq \emptyset$.

The convergence $\bar{\lambda}_1$ is not comparable with $\lambda_2, \lambda_3, \lambda_4$ and $\bar{\lambda}_4$. The relation $\bar{\lambda}_1^* \not\geq \bar{\lambda}_4$ is proved in (b) of Theorem 6.1. For a proof that $\bar{\lambda}_1^* \not\geq \lambda_2$ we follow Theorem 5.5. If $y \in \mathbb{B}^{\omega}$, where $\|\tau_y\|$ is splitting $> 0$, then $\bar{\lambda}_1^*(y) = \emptyset$. For the sequence $z = \langle y_0, 1, y_1, 1, \ldots \rangle$ we have $y \prec z$ and, since $\bar{\lambda}_1^*(y) = \emptyset$. But $1 \vDash \{1, 3, 5, \ldots \} \subset \tau_z$, thus $1 \vDash \text{“} \tau_z \text{ is supported”}$ which, by (2), implies $\lambda_2(z) \neq \emptyset$.

The convergence $\bar{\lambda}_1$ is not comparable with $\lambda_2, \lambda_3$, and $\lambda_4$. Namely, $\lambda_1(\langle 0 \rangle) = \{0\} \uparrow$ and $\lambda_4(\langle 0 \rangle) = \{0\}$ implies $\bar{\lambda}_1 \not\geq \lambda_4$. The relation $\bar{\lambda}_1 \not\geq \lambda_2$ follows from $\bar{\lambda}_1^* \not\geq \lambda_2$, proved above. □
Remark 6.2. For the $\omega_1$-distributive algebras the diagram collapses to the diagram containing two elements, e.g. $\lambda_1$ and $\bar{\lambda}_1$ (see Example 2.4 and Fact 5.2(a)).

References