Distribution functions of ratio sequences, III

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Dedicated to Professor Kálmán Győry on the occasion of his 70th birthday

Abstract. In this paper we study the distribution functions \( g(x) \) of the sequence of blocks \( X_n = (\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_n}{x_n}) \), \( n = 1, 2, \ldots \), where \( x_n \) is an increasing sequence of positive integers. Assuming that the lower asymptotic density \( d \) of \( x_n \) is positive, we find the optimal lower and upper bounds of \( g(x) \). As an application, we also get the optimal bounds of limit points of \( \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n}, n = 1, 2, \ldots \).

1. Introduction

Let \( x_n, n = 1, 2, \ldots \), be an increasing sequence of positive integers (by “increasing” we mean strictly increasing). The double sequence \( x_m/x_n, m, n = 1, 2, \ldots \) is called the ratio sequence of \( x_n \), which has been introduced by T. Šalát [Sa]. He studied its everywhere density. For further study of ratio sequences, O. Strauch and J. T. Tóth [ST] introduced the sequence \( X_n \) of blocks

\[
X_n = \left( \frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_n}{x_n} \right), \quad n = 1, 2, \ldots
\]

and they studied the uniform distribution of \( X_n \) in the sense of the monographs [KN] and [DT]. The authors in [ST] further studied the distribution functions

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The motivation for this is that the existence of strictly increasing \( g(x) \) implies everywhere density of \( x_n/x_n \), which is the primary problem of Šalát in [Sa].

In what follows, we will use the following definitions, and basic properties, see [SP, p. 1–28, 1.8.23].

- Denote by \( F(X_n, x) \) the step distribution function

\[
F(X_n, x) = \frac{\# \{ i \leq n; \frac{x_n}{x_n} < x \}}{n},
\]

for \( x \in [0,1) \), and \( F(X_n, 1) = 1 \). Directly from definition we have

\[
F(X_m, x) = \frac{n}{m} F\left( X_n, \frac{x_m}{x_n} \right)
\]

for all pairs of integers \( m < n \), and every \( x \in [0,1) \).

- For any increasing sequence of positive integers \( x_n, n = 1, 2, \ldots \), we define a counting function \( A(t) \) as

\[
A(t) = \# \{ n \in \mathbb{N}; x_n < t \}.
\]

Then for every \( x \in (0,1] \) we have the equality

\[
\frac{nF(X_n, x)}{xx_n} = \frac{A(xx_n)}{xx_n},
\]

which we shall use to compute the asymptotic density of \( x_n \). Here, the lower asymptotic density \( \underline{d} \) and the upper asymptotic density \( \overline{d} \) of \( x_n, n = 1, 2, \ldots \) are defined as

\[
\underline{d} = \lim \inf_{t \to \infty} \frac{A(t)}{t} = \lim \inf_{n \to \infty} \frac{n}{x_n}, \quad \overline{d} = \lim \sup_{t \to \infty} \frac{A(t)}{t} = \lim \sup_{n \to \infty} \frac{n}{x_n}.
\]

- A non-decreasing function \( g : [0,1] \to [0,1] \), \( g(0) = 0 \), \( g(1) = 1 \) is called a distribution function. We shall identify any two distribution functions coinciding at common points of continuity.

- A distribution function \( g(x) \) is a distribution function of the sequence of blocks \( X_n, n = 1, 2, \ldots \), if there exists an increasing sequence \( n_1, n_2, \ldots \) of positive integers such that

\[
\lim_{k \to \infty} F(X_{n_k}, x) = g(x)
\]

almost everywhere in \([0,1] \). This is equivalent to weak convergence; it means that the preceding limit holds for every point \( x \in [0,1] \) of continuity of \( g(x) \).
In this paper we frequently use the following two theorems of Helly (see First and Second Helly theorem [SP, Theorem 4.1.0.10 and Theorem 4.1.0.11, p. 4–5]).

- **Helly’s selection principle**: For any sequence \( g_n(x), n = 1, 2, \ldots \), of distribution functions in \([0, 1]\) there exists a subsequence \( g_{n_k}(x), k = 1, 2, \ldots \), and a distribution function \( g(x) \) such that \( \lim_{n \to \infty} g_{n_k}(x) = g(x) \) almost everywhere.

- **Second Helly theorem**: If we have \( \lim_{n \to \infty} g_n(x) = g(x) \) almost everywhere in \([0, 1]\), then for every continuous function \( f: [0, 1] \to \mathbb{R} \) we have \( \lim_{n \to \infty} \int_0^1 f(x) dg_n(x) = \int_0^1 f(x) dg(x) \).

- Note that applying Helly’s selection principle, from the sequence \( F(X_n, x), n = 1, 2, \ldots \), one can select a subsequence \( F(X_{n_k}, x), k = 1, 2, \ldots \), such that \( \lim_{k \to \infty} F(X_{n_k}, x) = g(x) \) holds not only for the continuity points \( x \) of \( g(x) \), but also for all \( x \in [0, 1] \).

- Denote by \( G(X_n) \) the set of all distribution functions of \( X_n, n = 1, 2, \ldots \). For a singleton \( G(X_n) = \{g(x)\} \), the distribution function \( g(x) \) is also called asymptotic distribution function of \( X_n \).

- We will use the one-step distribution function \( c_\alpha(x) \) with the step 1 at \( \alpha \) defined on \([0, 1]\) via

\[
c_\alpha(x) = \begin{cases} 
0, & \text{if } x \leq \alpha; \\
1, & \text{if } x > \alpha,
\end{cases}
\]

while always \( c_\alpha(0) = 0 \) and \( c_\alpha(1) = 1 \).

- The lower distribution function \( g(x) \), and the upper distribution function \( \overline{g}(x) \) of a sequence \( x_n, n = 1, 2, \ldots \) are defined as

\[
g(x) = \inf_{g \in G(x_n)} g(x), \quad \overline{g}(x) = \sup_{g \in G(x_n)} g(x).
\]

In Section 2 of this paper we recall some known theorems, which we shall use and extend. In Section 3 (Theorem 5) we solve Open problem no. 7 from [SN, 1.9. Block sequence] stating that every sequence of blocks \( X_n \) has a distribution function \( g(x) \) such that \( g(x) \geq x \) for all \( x \in [0, 1] \). Then, assuming \( d > 0 \), we find (Theorem 6) boundaries \( h_1(x) \leq g(x) \leq h_2(x) \), which hold for every distribution function \( g(x) \in G(X_n) \), and which are, in a sense, optimal.

As a consequence, we produce boundaries (Theorem 7) for \( \frac{1}{n} \sum_{i=1}^{n} x_i \). In the last Section 4 (Example 3), we find the exact values of the lim inf and lim sup of \( \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n} \) for integers \( x_n \) from the intervals \((\gamma a^k, \delta a^k), k = 1, 2, \ldots \).
2. Basic known results

For an increasing sequence \( x_n, n = 1, 2, \ldots \) of positive integers the following theorems are known.

**Theorem 1** ([ST, Theorem 7.1]). For every sequence of positive integers \( x_n \) there exits \( g(x) \in G(X_n) \) such that
\[
\int_0^1 g(x) \, dx \geq \frac{1}{2}.
\] (4)

**Theorem 2** ([ST, Theorem 6.2 (ii),(iii)]). If \( d > 0 \), then there exits \( g \in G(X_n) \) such that \( g(x) \geq x \) for every \( x \in [0,1] \). Furthermore, for every \( g(x) \in G(X_n) \), and \( x \in [0,1] \) we have
\[
x \frac{d}{d} \leq g(x) \leq x \frac{d}{d}.
\]

**Theorem 3** ([ST, Propozicion 6.1]). Assume for a sequence \( n_k, k = 1, 2, \ldots \) that
(i) \( \lim_{k \to \infty} F(X_{n_k}, x) = g(x) \),
(ii) \( \lim_{k \to \infty} \frac{n_k}{x_{n_k}} = d_g \).

Then there exists
(iii) \( \lim_{k \to \infty} \frac{A(xx_{n_k})}{xx_{n_k}} = d_g(x) \) and
\[
\frac{g(x)}{x} d_g = d_g(x).
\] (5)

Here the limits (i), and (iii) can be considered for all \( x \in (0,1] \), or all continuity points \( x \in (0,1] \) of \( g(x) \).

**Theorem 4** ([ST, Theorem 4.1, Theorem 6.2]). Assume that every distribution function in \( G(X_n) \) is continuous at 1. Then each distribution function in \( G(X_n) \) is continuous in \( (0,1] \), i.e. the only point of discontinuity is possibly 0. Furthermore, if \( d > 0 \), then all distribution functions in \( G(X_n) \) are continuous in \([0,1]\).

3. Main results

We start with an extension of Theorem 1, and the first part of Theorem 2.

**Theorem 5.** For every increasing sequence of positive integers \( x_n, n = 1, 2, \ldots \), there exists \( g(x) \in G(X_n) \) such that \( g(x) \geq x \) for all \( x \in [0,1] \).
Proof. If \( d > 0 \), select \( n_k \) so that \( \frac{n_k}{x_{n_k}} \to d > 0 \), and \( F(X_{n_k}, x) \to g(x) \). For such \( g(x) \), (5) implies

\[
\frac{g(x)}{x} d \ge d.
\]

Now, let \( d = 0 \). Select \( n_k \) such that

\[
\frac{n_k}{x_{n_k}} = \min_{i \le n_k} \frac{i}{x_i},
\]

and \( F(X_{n_k}, x) \to g(x) \). Then for every \( x \in (0, 1] \),

\[
\frac{A(x x_{n_k})}{x x_{n_k}} \ge \frac{n_k - 1}{x_{n_k}}.
\]

Applying (2) yields

\[
\frac{F(X_{n_k}, x) n_k}{x x_{n_k}} \ge \frac{n_k - 1}{x_{n_k}},
\]

and taking the limit, as \( k \to \infty \), we obtain \( g(x) \ge x \) for all \( x \in [0, 1] \). \( \square \)

Now we are going to study in more detail the second part of Theorem 2.

Theorem 6. Let \( x_1 < x_2 < \ldots \) be a sequence of positive integers with positive lower asymptotic density \( d > 0 \) and upper asymptotic density \( \overline{d} \). Then all distribution functions \( g(x) \in G(X_n) \) are continuous, non-singular and bounded by \( h_1(x) \le g(x) \le h_2(x) \), where

\[
h_1(x) = \begin{cases} \frac{x \overline{d}}{d} & \text{if } x \in \left[0, \frac{1 - \overline{d}}{1 - d}\right] ; \\ \frac{d}{x - \left(1 - \overline{d}\right)} & \text{otherwise,} \end{cases}
\]

\[
h_2(x) = \min \left( \frac{x \overline{d}}{d}, 1 \right).
\]

Moreover, \( h_1(x) \) and \( h_2(x) \) are the best possible in the following sense: for given \( 0 < \overline{d} \le \overline{d} \), there exists \( x_1 < x_2 < \ldots \) with lower and upper asymptotic density \( \overline{d}, \overline{d} \), such that \( g(x) = h_1(x) \) for \( x \in \left[\frac{1 - \overline{d}}{1 - d}, 1\right] \); also, there exists \( x_1 < x_2 < \ldots \) with given \( 0 < \overline{d} \le \overline{d} \) such that \( \overline{g}(x) = h_2(x) \in G(X_n) \).

Proof. For \( g(x) \in G(X_n) \), let \( n_k, k = 1, 2, \ldots, \) be an increasing sequence of indices such that \( F(X_{n_k}, x) \to g(x) \). From \( n_k \) we can select a subsequence (for simplicity written as the original \( n_k \)) \(^1\) such that

\[
\frac{n_k}{x_{n_k}} \to d_g > 0.
\]

\(^1\)We call \( d_g \) a local asymptotic density related to \( g(x) \).
Then, by (5), we have
\[ g(x) = x \frac{d_g(x)}{d_g}, \quad \text{where} \quad \frac{A(xx_{n_k})}{xx_{n_k}} \to d_g(x) \]  
for arbitrary \( x \in (0, 1] \).

We will continue in six steps 1\(^0\)–6\(^0\).

1\(^0\). We prove the continuity of \( g(x) \) at \( x = 1 \) (improving (iv) in [ST, Theorem 6.2]) for each \( g(x) \in G(X_n) \).

In view of the definition of the counting function \( A(t) \)
\[ 0 \leq A(x_{n_k}) - A(xx_{n_k}) \leq x_{n_k} - xx_{n_k}; \]
thus,
\[ 0 \leq \frac{A(x_{n_k})}{x_{n_k}} - \frac{A(xx_{n_k})}{xx_{n_k}} = \frac{n_k - 1}{x_{n_k}} - \frac{A(xx_{n_k})}{xx_{n_k}} \leq 1 - x, \]
and, as \( k \to \infty \), we have \( 0 \leq d_g - d_g(x)x \leq 1 - x \), which implies
\[ 0 \leq d_g - d_g(x) + d_g(x)(1 - x) \leq 1 - x. \]
Consequently, \( \lim_{x \to 1} d_g(x) = d_g \), and so \( \lim_{x \to 1} g(x) = \lim_{x \to 1} x \cdot \frac{d_g(x)}{d_g} = 1 \). Since \( g(x) \in G(X_n) \) is arbitrary, Theorem 4 gives continuity of \( g(x) \) in the whole unit interval \([0, 1]\).

2\(^0\). We prove that \( g(x) \) has a bounded right derivative for every \( x \in (0, 1) \), and for each \( g(x) \in G(X_n) \).

For \( 0 < x < y < 1 \) again
\[ 0 \leq A(yx_{n_k}) - A(xx_{n_k}) \leq (y - x)x_{n_k}, \]
which implies
\[ 0 \leq \frac{A(yx_{n_k})}{yx_{n_k}} y - \frac{A(xx_{n_k})}{xx_{n_k}} x \leq y - x. \]
Letting \( k \to \infty \), we get \( 0 \leq d_g(y)y - d_g(x)x \leq y - x \), hence
\[ 0 \leq g(y) - g(x) = \frac{d_g(y)y - d_g(x)x}{d_g} \leq \frac{y - x}{d_g}. \]
Consequently,
\[ 0 \leq \frac{g(y) - g(x)}{y - x} \leq \frac{1}{d_g} \] (10)
for all $x, y \in (0, 1)$, $x < y$, which gives the upper bound of the right derivatives of $g(x)$ for every $x \in (0, 1)$. Note that a singular distribution function (continuous, strictly increasing, having zero derivative almost everywhere) has infinite right Dini derivatives in a dense subset of $(0, 1)$.

30. We prove a local form of Theorem 5.

As $\underline{d} \leq d_g \leq \overline{d}$, (9) implies

$$x \frac{d}{d_g} \leq g(x) \leq x \frac{\overline{d}}{d_g}$$

(11)

for every $x \in [0, 1]$. It follows from (10), that there exists an extreme point $A_g = (x_g, y_g)$ on the line $y = x \frac{d}{d_g}$ such that $g(x)$ has no common point with this line for $x > x_g$. This point $A_g$ is the intersection of the lines

$$y = x \frac{d}{d_g} \text{ and } \frac{1}{1 - \frac{d}{d_g}}$$

therefore,

$$A_g = (x_g, y_g) = \left( \frac{1 - d_g}{1 - \frac{d}{d_g}}, \frac{d}{1 - \frac{d}{d_g}} \right).$$

(13)

It means that for a given $g(x) \in G(X_n)$, $h_{1,g}(x) \leq g(x) \leq h_{2,g}(x)$, where

$$h_{1,g}(x) = \begin{cases} x \frac{d}{d_g} & \text{if } x < y_0 = \frac{1 - d_g}{1 - \frac{d}{d_g}}, \\ \frac{d}{d_g} + 1 - \frac{1}{d_g} & \text{if } y_0 \leq x \leq 1, \end{cases}$$

(14)

$$h_{2,g}(x) = \min \left( x \frac{d}{d_g}, 1 \right).$$

(15)

40. Now we find $h_1(x)$, and $h_2(x)$ such that

$$h_1(x) \leq h_{1,g}(x) \leq h_{2,g}(x) \leq h_2(x)$$

for every $g \in G(X_n)$.

In the parametric expression (13) of $A_g$, the local asymptotic density $d_g$ defined by (8) belongs to the interval $[\underline{d}, \overline{d}]$. The well-known Darboux property of the asymptotic density implies that for an arbitrary $d \in [\underline{d}, \overline{d}]$ there exists an increasing $n_k$, $k = 1, 2, \ldots$, such that $\frac{n_k}{x_{n_k}} \to d$ $^2$, and then the Helly selection

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A simple proof follows from the fact that for every $d \in (\underline{d}, \overline{d})$ there exist infinitely many $n \in \mathbb{N}$ such that $A(n)/n \leq d \leq A(n + 1)/(n + 1)$. These $n$ we denote as $n_k$. 

principle implies the existence of a subsequence of \( n_k \) such that \( F(X_{n_k}, x) \to g(x) \) for some \( g(x) \in G(X_n) \). Thus, if \( g(x) \) runs over \( G(X_n) \), then \( d_g \) runs over the entire interval \([d, d]\). Substituting \( d_g = 1 - x_g(1 - d) \) in \( A_g = (x_g, y_g) \) we get

\[
y_g = y_g(x_g) = \frac{d}{x_g - (1 - d)},
\]

where \( x_g = \frac{1 - d}{1 - d} \) runs through the interval \( I = \left[\frac{1 - d}{1 - d}, 1\right] \) for \( d_g \in [d, d] \). By putting \( x_g = x \), and \( y_g = h_1 \) we find a part of \( h_1(x) \) for \( x \in I \) in (6). The remaining part of \( h_1(x) \), and also the whole \( h_2(x) \), follow from the basic inequality (11), see Figure 1.

![Figure 1. Boundaries of \( g(x) \in G(X_n) \)](image)

50. The optimality of \( h_1(x) \) follows from the following example.

The increasing sequence \( x_n \) of the integers lying in the intervals

\[
(\gamma, \delta), (\gamma a, \delta a), \ldots, (\gamma a^k, \delta a^k), \ldots,
\]

where \( 1 \leq \gamma < \delta \leq a \), has been used in [ST, pp. 774–777, Example 11.2]. For its lower, and upper asymptotic densities \( d \) and \( d \), it has been shown that

\[
d = \frac{(\delta - \gamma)}{\gamma(a - 1)}, \quad d = \frac{(\delta - \gamma)a}{\delta(a - 1)},
\]

(16)
and that the graph of every $g \in G(X_n)$ lies in the intervals

$$[1/a, 1] \times [1/a, 1] \cup [1/a^2, 1/a] \times [1/a^2, 1/a] \cup \ldots$$

Moreover, the part of the graph in $[1/a^k, 1/a^{k-1}] \times [1/a^k, 1/a^{k-1}]$ is similar to the part of the graph in $[1/a^{k+1}, 1/a^k] \times [1/a^{k+1}, 1/a^k]$ with the scale $a$. It is also proved in [ST], that the graph of $g(x)$ in $[1/a, 1] \times [1/a, 1]$ has the form $g(x) = (1 + \frac{x}{a} (\frac{1}{a} - 1))^{-1}$ for $x \in [\frac{1}{a}, 1]$, and it coincides with the graph of $h(x)$ in the interval $I = [\frac{1}{a^2}, 1]$, since $\frac{1}{a^2} = \frac{1}{a}$.

$6^0$. Finally, we prove the optimality of $h_2(x)$. Before proving it in several substeps, note that in $5^0$ the graph of the upper distribution function $\overline{g}(x)$ in $[1/a, 1] \times [1/a, 1]$ is a straight line which intersects the line $y = 1$ at $x = \frac{d}{m_k} = \frac{1}{a}$. Thus, $\overline{g}(\frac{1}{a}) = h_2(\frac{1}{a}) = 1$ proving that the point $(\frac{1}{a}, 1)$ is optimal.

To complete the proof of $6^0$, in the following steps a)- f) we shall construct a sequence of positive integers $x_1 < x_2 < \ldots$ with $0 < d < \overline{d}$ such that $h_2(x) \in G(X_n)$. This implies $h_2(x) = \overline{g}(x)$.

a) The condition $h_2(x) \in G(X_n)$ for a sequence of positive integers $x_1 < x_2 < \ldots$ is equivalent to the existence of an increasing sequence of indices $n_k$ such that $F(X_{n_k}, x) \rightarrow h_2(x)$ for $x \in [0, 1]$, and $\frac{m_k}{n_{k+1}} \rightarrow 0$. An application of (1) yields that this is equivalent (see Fig. 2) to the existence of $m'_k < m_k < n_k$ such that the values $x_{m'_k} < x_{m_k} < x_{n_k}$ satisfy

(i) $\frac{x_{m_k}}{x_{n_k}} \rightarrow \frac{d}{\overline{d}}$,

(ii) $\frac{m_k}{n_k} \rightarrow 1$,

(iii) $\frac{x_{m'_k}}{x_{n_k}} \rightarrow 0$,

(iv) $\frac{m'_k}{n_k} \rightarrow 0$.

Moreover, because the sequence of positive integers $x_n$ increases, we have (see Figure 3)

(v) $x_{m_k} - x_{m'_k} \geq m_k - m'_k$,

(vi) $x_{n_k} - x_{m_k} \geq n_k - m_k$,

(vii) $x_{m'_{k+1}} - x_{n_k} \geq m'_{k+1} - n_k$,

(viii) $n_k < m'_{k+1}$,

(ix) $m'_1 \leq x_{m'_1}$.
b) Before solving (i)-(ix) we must capture a role of $d$ and $\overline{d}$. By (i) and (ii) we have the limit

$$\frac{n_k x_{m_k}}{m_k x_{n_k}} \to \frac{d}{\overline{d}}.$$ 

Selecting a subsequence of $(m_k, n_k), k = 1, 2, \ldots$, we can assume the existence of the limits $\frac{n_k}{m_k} \to d_{h_2}$, and $\frac{m_k}{x_{m_k}} \to d_g$ (for simplicity, also assume $F(X_{m_k}, x) \to g(x)$). Then

$$\frac{n_k x_{m_k}}{m_k x_{n_k}} = \frac{x_{m_k}}{x_{n_k}} \frac{m_k}{m_k} \frac{m_k}{x_{m_k}} \to \frac{1}{\frac{d_g}{d_h}} = \frac{d}{\overline{d}}.$$
and since
\[ \frac{d}{d} = \min_{d, d' \in [0, d]} \frac{d_1}{d_2}, \]
we have \( d_{h_2} = d \) and \( d_g = \overline{d} \). This yields the additional conditions

\[(x) \quad \frac{m_k}{x_{n_k}} \to d, \]
\[(xi) \quad \frac{m_k}{x_{m_k}} \to \overline{d}. \]

c) In what follows, we assume \( d < \overline{d} \), because from Theorem 2, by \( 0 < d = \overline{d} \), we get \( G(X_n) = \{x\} \), and also \( h_2(x) = x \).

d) To find a sequence \( x_n \) which satisfies (i)–(xi), we define \( x_{n_k}, x_{m_k}, m_k, x_{m'_k}, m'_k \) by using \( n_k \) (for a simplifying the definitions, the integer part will be omitted):

\[ x_{n_k} = \frac{n_k}{d}, \]
\[ x_{m_k} = x_{n_k} \frac{d}{d} = \frac{n_k}{\overline{d}}, \]
\[ m_k = x_{m_k} \overline{d} - o(n_k) = n_k - \sqrt{n_k}, \]
\[ x_{m'_k} = \sqrt{x_{m_k}} = \sqrt{\frac{n_k}{d}}, \]
\[ m'_k = d' x_{m'_k} = d' \sqrt{\frac{n_k}{d}}, \]

for some \( d' \in (\overline{d}, d) \).

These \( x_{n_k}, x_{m_k}, x_{m'_k}, m_k, m'_k \) satisfy (i)–(vii), (x), (xi). For (viii) we need

\[ n_{k+1} > \overline{d} \frac{1}{d^2} \frac{d}{d^2} n_k^2 \]

for \( k = 2, 3, \ldots \), and for (ix) the \( n_1 \) must be large.

e) For linearity of \( h_2(x) \) in \([0, \overline{d}]\), and to guarantee the asymptotic densities \( d, \overline{d} \), define

\[(xii) \quad x_n = x_a + (n - a) \frac{x_{b - a} - x_{a}}{b - a} \quad \text{for} \quad n \in (a, b), \quad \text{where} \quad (a, b) \quad \text{coincides successively with} \quad (m'_k, m_k), (m_k, n_k), \quad \text{or} \quad (n_k, m'_k+1). \]

Then

\[ \frac{n}{x_n} = \frac{a + (n - a) \frac{x_{b - a} - x_{a}}{b - a}}{x_a + (n - a) \frac{x_{b - a} - x_{a}}{b - a}} \]
and because the derivative \((Ax + B)' = \frac{AD - BC}{(Cx + D)}\), the minimum and maximum of \(\frac{n}{x_n}\) for \(n \in (a, b)\) are attained at the endpoints \(n = a, b\), i.e., for \(n = m_k, m_k, m_k\). Since the limits of \(\frac{n}{x_{n_k}}, \frac{m_k}{x_{n_k}}\) are from \([d, \overline{d}]\), and the boundary points are attained, \(\liminf_{n \to \infty} \frac{n}{x_n} = \overline{d}\) and \(\limsup_{n \to \infty} \frac{n}{x_n} = d\).

f) For such \(x_1 < x_2 < \ldots\) we have \(d, \overline{d}\), and \(F(X_{n_k}, x) \to h_2(x)\) for \(x \in [0, 1]\); hence, the proof of Theorem 6 is finished. \(\square\)

Remark 1. In a sharp contrast to \(h_2(x) \in G(X_n)\) in 60 we note that for every sequence of integers \(x_1 < x_2 < \ldots, 0 < \underline{d} < \overline{d}\), we have \(h_1(x) \notin G(X_n)\), because for every \(g(x) \in G(X_n), h_{1,g}(x) \leq g(x) \leq h_{2,g}(x)\), and \(h_{1,g}(x) \neq h_1(x)\).

Theorem 6 implies the following best possible boundaries of the sum

\[
\frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n}
\]

**Theorem 7.** For every increasing sequence \(x_1 < x_2 < \ldots\) of positive integers with \(0 < \underline{d} \leq \overline{d}\) we have

\[
\frac{1}{2} \frac{\underline{d}}{\overline{d}} \leq \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n},
\]

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n} \leq \frac{1}{2} + \frac{1}{2} \left(1 - \min\left(\sqrt{\underline{d}}, \overline{d}\right)\right) \left(1 - \frac{\underline{d}}{\min\left(\sqrt{\underline{d}}, \overline{d}\right)}\right).
\]

Here the equality in both (17) and (18)\(^3\) can be attained.

**Proof.** By the Helly theorem, \(F(X_{n_k}, x) \to g(x)\) forces

\[
\int_{0}^{1} x dF(X_{n_k}, x) = \frac{1}{n_k} \sum_{i=1}^{n_k} \frac{x_i}{x_{n_k}} \rightarrow \int_{0}^{1} x dg(x) = 1 - \int_{0}^{1} g(x)dx;
\]

thus,

\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n} = 1 - \max_{g \in G(X_n)} \int_{0}^{1} g(x)dx,
\]

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n} = 1 - \min_{g \in G(X_n)} \int_{0}^{1} g(x)dx.
\]

\(^3\)If \(\sqrt{\underline{d}} \leq \overline{d}\) then the right-hand side in (18) is \(\frac{1}{1 + \sqrt{\underline{d}}}\).
If $d > 0$, then by Theorem 6, $h_1(x) \leq g(x) \leq h_2(x)$, which implies

$$1 - \int_0^1 h_2(x) \, dx \leq \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n} \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n} \leq 1 - \int_0^1 h_1(x) \, dx. \quad (22)$$

For $x_1 < x_2 < \ldots$ in step 6, where $h_2(x) \in G(X_n)$, we have equality on the left hand side of (22). On the other hand, Remark 1 implies a sharp inequality on the right hand side, therefore,

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n} < 1 - \int_0^1 h_1(x) \, dx.$$  

(23)

holds for an arbitrary sequence of integers $x_1 < x_2 < \ldots$ with $0 < d < \overline{d}$.

Applying the inequality $h_1,g(x) \leq g(x) \leq h_2,g(x)$ for every $g \in G(X_n)$ from step 3 to (19), we obtain

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n} \leq \max_{g(x) \in G(X_n)} \left( 1 - \int_0^1 h_{1,g}(x) \, dx \right). \quad (24)$$

If the maximum in (24) is attained for $g_0(x) \in G(X_n)$, and $h_{1,g_0}(x) \in G(X_n)$, then $g_0(x) = h_{1,g_0}(x)$, and

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n} = 1 - \int_0^1 h_{1,g_0}(x) \, dx. \quad (25)$$

Using (14) we get

$$\int_0^1 h_{1,g}(x) \, dx = \frac{1}{2} \left( 1 + \frac{1 - d_g}{1 - d} \left( \frac{d}{d_g} - 1 \right) \right),$$

and taking derivative with respect to $d_g \in [d, \overline{d}]$

$$\left( \int_0^1 h_{1,g}(x) \, dx \right)' = \frac{1}{2(1 - d)} \left( 1 - \frac{d}{(d_g)^2} \right)$$

shows that $\min \int_0^1 h_{1,g}(x) \, dx$ is attained for $d_{g_0} = \min(\sqrt{\overline{d}}, \overline{d})$.

Now, to prove (25) we shall construct integers $x_1 < x_2 < \ldots$ with $0 < d \leq \overline{d}$ such that $h_{1,g_0}(x) \in G(X_n)$. We start with the sequence of indices $n_k$, and then
by (14) we shall find indices $m'_{k} < m_{k} < n_{k}$, and integers $x_{m'_{k}} < x_{m_{k}} < x_{n_{k}}$ such that

(i) $\frac{n_{k}}{x_{n_{k}}} \to d_{n_{k}}$,  
(ii) $\frac{m_{k}}{n_{k}} \to \frac{d_{k} \cdot (1 - d_{n_{k}})}{d_{n_{k}} \cdot (1 - d_{k})}$,  
(iii) $\frac{x_{m_{k}}}{x_{n_{k}}} \to \frac{1 - d}{d}$,  
(iv) $\frac{x_{m'_{k}}}{x_{n_{k}}} \to 0$,  
(v) $\frac{m_{k}}{n_{k}} \to 0$,  
(vi) $\frac{m'_{k}}{n_{k}} \to \frac{d}{d}$.

Then from (i), (ii), and (iii) it follows that $\frac{n_{k}}{x_{n_{k}}} \to d_{k}$. Furthermore, assume

(v) $x_{m_{k}} - x_{m'_{k}} \geq m_{k} - m'_{k}$,  
(vi) $x_{n_{k}} - x_{m_{k}} \geq n_{k} - m_{k}$,  
(vii) $x_{m_{k+1}} - x_{m_{k}} \geq m'_{k+1} - n_{k}$,  
(viii) $n_{k} < m'_{k+1}$,  
(ix) $m'_{1} < x_{m'_{1}}$.

For these (i)–(ix) a sequence of integers $x_{n}$ can be found similarly to 60d). The rest of the terms of $x_{n}$ define linearly as in e).

\section{Examples}

\textbf{Example 1.} a) If $0 < d = \frac{1}{2}$, then the bounds in both (17), and (18) equal to $\frac{1}{2}$, which implies

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_{i}}{x_{n}} = \frac{1}{2}.$$  

This also follows from the fact that $G(X_{n}) = \{x\}$, see Theorem 2.

b) If $d = \frac{1}{2}$, and $d = 1$, then by (23), $\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_{i}}{x_{n}} < 2 - \log 4 < 1$. Using (18) we have an even better estimate $\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_{i}}{x_{n}} \leq 2 - \sqrt{2}$.

\textbf{Example 2.} Omitting $d > 0$, we can find a sequence of positive integers $x_{1} < x_{2} < \ldots$ such that $c_{0}(x), c_{1}(x) \in G(X_{n})$, where $c_{0}(x), c_{1}(x)$ are one-steps.
distribution functions defined by (3) in the Introduction. In this case
\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} x_i x_n = 0 = 1 - \int_{0}^{1} c_0(x) \, dx,
\]
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} x_i x_n = 1 = 1 - \int_{0}^{1} c_1(x) \, dx.
\]
We shall construct such \(x_n\) by applying [GS, Theorem 5]. For the index sequences
\[m_k' < m_k < n_k' < n_k\]
we shall find sequences of positive integers
\[x_{m_k'} < x_{m_k} < x_{n_k'} < x_{n_k}\]
such that
(i) \(n_{k'} / n_k \to 0\),
(ii) \(m_k / m_k \to 1\),
(iii) \(x_{n_k'} / x_{n_k} \to 1\),
(iv) \(x_{m_k'} / x_{m_k} \to 0\).
Furthermore,
(v) \(x_{n_k} - x_{n_k'} \geq n_k - n_k'\),
(vi) \(x_{n_k'} - x_{m_k} \geq n_k' - m_k\),
(vii) \(x_{m_k} - x_{m_k'} \geq m_k - m_k'\),
(viii) \(m_{k+1} \geq n_k\),
(ix) \(m_{k+1} \geq m_k'\),
(x) \(x_{m_{k+1}} - x_{m_k} \geq m_{k+1} - n_k\).
Then (i)-(x) will be satisfied, if for a given \(n_k\) we put
\[x_{n_k} = n_k^2, \quad x_{n_k'} = n_k^2 - n_k, \quad x_{m_k} = n_k^2 - 2n_k, \quad x_{m_k'} = n_k, \quad n_k' = \sqrt{n_k} - \sqrt[n_k]{n_k}, \quad m_k = \sqrt{n_k} - \sqrt[n_k]{n_k}, \quad m_k' = \sqrt{n_k} - 2\sqrt[n_k]{n_k};\]
\[m_k = \sqrt{n_k} - \sqrt[n_k]{n_k}, \quad m_k' = \sqrt{n_k} - 2\sqrt[n_k]{n_k};\]
further, (vii) holds if \(n_{k+1} \geq n_k^4\). For the other \(n_k'\)'s in the intervals \((m_k', m_k)\), \((m_k, n_k')\), \((n_k', n_k)\), and \((n_k, m_{k+1}')\) define \(x_n\) linearly.

Now, by (i), and (iii) we have \(F(X_{n_k}, x) \to c_1(x)\), and (ii), (iv) imply \(F(X_{m_k}, x) \to c_0(x)\).

Example 3. In this example we extend a characterization of distribution functions of the sequence \(x_1 < x_2 < \ldots\) in [ST, Example 11.2]. This sequence was used in the proof of Theorem 6, part 5°.
Let \( x_n, n = 1, 2, \ldots \), be the increasing sequence of all integer points in the sequence of intervals \((\gamma a^k, \delta a^k)\) (in short \(a^k(\gamma, \delta)\)), \( k = 0, 1, 2, \ldots \), where \( 1 \leq \gamma < \delta \leq a \) are real numbers.

It is proved in [ST, Ex. 11.2] that

1\(^{\text{st}}\). The set of all distribution functions can be expressed in parametric form as \( G(X_n) = \{g_t(x); t \in [0, 1]\} \), where

\[
F(X_{n_k}, x) \to g_t(x) \quad \text{for } n_k \quad \text{such that } x_{n_k} = \lfloor a^k \gamma + t a^k (\delta - \gamma) \rfloor \tag{26}
\]

2\(^{\text{nd}}\). The function \( g_t(x) \) has constant values \( g_t(x) = \frac{1}{a^{(1+t)(a-1)}} \) for \( x \in \frac{(\delta - a \gamma)}{a^{i+1}(\gamma + t(a-1))}, \ i = 0, 1, 2, \ldots \), and in the component intervals it has a constant derivative

\[
\gamma_t(x) = \frac{(a-1)(\gamma + t(a-1))}{\gamma + t(a-1)(\gamma - \delta)} \quad \text{for } x \in \frac{(\gamma + t(a-1)(\gamma - \delta))}{a^{i+1}(\gamma + t(a-1))}, \ i = 0, 1, 2, \ldots \]

3\(^{\text{rd}}\). The graph of every \( g \in G(X_n) \) lies in the intervals

\([1/a, 1] \times [1/a, 1] \cup [1/a^2, 1/a] \times [1/a^2, 1/a] \cup \ldots \),

and the graph of \( g \) in \([1/a^k, 1/a^k-1] \times [1/a^k, 1/a^k-1] \) is similar to the graph of \( g \) in \([1/a^{k+1}, 1/a^{k+1}] \times [1/a^{k+1}, 1/a^{k+1}] \) with coefficient \( a \).

4\(^{\text{th}}\). We have \( g_0(x) = \overline{g}(x), \ g_1(x) \notin G(X_n) \), and the asymptotic densities \( d, \overline{d} \) are

\[
d = \frac{(\delta - \gamma)}{\gamma(a - 1)}, \quad \overline{d} = \frac{(\delta - \gamma)a}{\delta(a - 1)}.
\]

We can add the following new properties 5\(^{\text{th}}\)–8\(^{\text{th}}\):

5\(^{\text{th}}\). By definition (8) of the local asymptotic density \( d_{g_t} \), along with (26) for \( g(x) = g_t(x) \) we get

\[
d_{g_t} = \lim_{k \to \infty} \frac{n_k}{x_{n_k}} = \lim_{k \to \infty} \frac{\sum_{i=0}^{k-1} a^i (\delta - \gamma) + t a^k (\delta - \gamma)}{a^k \gamma + t a^k (\delta - \gamma)} = \frac{(\delta - \gamma)(1 + t(a - 1))}{(a - 1)(\gamma + t(\delta - \gamma))}, \tag{27}
\]

for \( t = 0, d_{g_0} = d, \) for \( t = 1, d_{g_1} = \overline{d}, \) and we have

\[
g_t'(x) = \frac{1}{d_{g_t}} \tag{28}
\]

for \( x \) in intervals where the derivative of \( g_t(x) \) is constant.

\[\text{Here, as above, we write } (x, y, z) = (x, y)z, \text{ and } (x/z, y/z) = (x, y)/z.\]
6°. For the function \( h_{1,g}(x) \) defined in (14), putting \( g(x) = g_t(x) \), we have

\[
\frac{d}{d_{g_t}} = \frac{\gamma + t(\delta - \gamma)}{\gamma(1 + t(a - 1))}, \quad \frac{1 - d_{g_t}}{1 - \delta} = \frac{\gamma}{\gamma + t(\delta - \gamma)}.
\]

Then

\[
h_{1,g_t}(x) = \begin{cases} 
  g_t(x) = x \frac{1}{d_{g_t}} + 1 - \frac{1}{d_{g_t}}, & \text{for } x \in \left( \frac{\gamma}{\gamma + t(\delta - \gamma)}, 1 \right); \\
  g_t(x) = \frac{1}{a^i(1 + t(a - 1))}, & \text{for } x = \frac{\gamma}{a^i(\gamma + t(\delta - \gamma))}, \quad i = 0, 1, 2, \ldots,
\end{cases}
\]

(29)

see Figure 4.

\[
\begin{tikzpicture}
  \node at (0,0) {\(g_t(x)\)};
  \node at (1,1) {\(h_{1,g_t}(x)\)};
  \node at (2,2) {\((1,1)\)};
  \node at (0,2) {\(g(x)\)};
  \node at (-1,0) {\((\frac{1}{a}, 0)\)};
  \node at (-1,1) {\((\frac{1}{a}, \frac{1}{a})\)};
  \node at (-1,2) {\((\frac{1}{a^2}, \frac{1}{a})\)};
  \node at (-1,3) {\((\frac{1}{a^3}, \frac{1}{a^2})\)};
\end{tikzpicture}
\]

\textbf{Figure 4:} \(g_t(x)\) and \(h_{1,g_t}(x)\).

7°. In the proof of the upper bound (18) we have proved that \(1 - \int_0^1 h_{1,g}(x)dx\) is maximal for \( d_{g} = \min(\sqrt{\frac{a}{d}}, \alpha) \). Let \( t_0 \in [0, 1] \) be such that \( d_{g_{t_0}} = \min(\sqrt{\frac{a}{d}}, \alpha) \). This \( t = t_0 \) we shall find from (27) as

\[
t = \frac{d_{g_t}(a - 1)\gamma - (\delta - \gamma)}{(\delta - \gamma)(a - 1)(1 - d_{g_t})}.
\]

(30)

8°. Let \( P(t) \) be the area in \([\frac{1}{a}, 1] \times [\frac{1}{a}, 1] \) bounded by the graph of \( g_t(x) \). Then

\[
\int_0^1 g_t(x)dx = P(t) \left( \frac{1}{1 - \frac{1}{a^i}} + \frac{1}{a + 1} = \frac{1}{2} + \frac{1}{2(a + 1)} \cdot \frac{1}{(1 + t(a - 1))(\gamma + t(\delta - \gamma))} \right)
\]
\[
\frac{1}{n+1} \sum_{i=1}^{n+1} \frac{x_i}{x_{n+1}} - \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n} = \frac{1}{n+1} - \left( \frac{1}{x_n + 1} + \frac{1}{n+1} \right) \left( \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n} \right) > 0,
\]

and, because \( c_1(x) \notin G(X_n) \), \( \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n} < 1 \). Denoting the index \( n_k \) for \( x_{n_k} = [a^k \delta] \), the limit sup of \( \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n} \) is attained over \( n = n_k, \ k = 0, 1, 2, \ldots \), and for such \( n_k \) (see (26)) we have \( F(X_{n_k}, x) \to g_1(x) \) for \( x \in [0,1] \).

It follows, by (20), and (21) that

\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n} = 1 - \int_0^1 g_0(x)dx = \frac{1}{2} - \frac{1}{2} (a+1) \left( \frac{\gamma a - \delta}{\gamma} \right),
\]

(32)

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n} = 1 - \int_0^1 g_1(x)dx = \frac{1}{2} + \frac{1}{2} (a+1) \left( \frac{\gamma a - \delta}{\delta} \right).
\]

(33)

The upper bound in (18) coincides with the maximal value of \( 1 - \int_0^1 h_{1,\delta}(x)dx \) attained for \( d_\delta = \min(\sqrt{\pi \delta}, \bar{d}) \). Since \( 1 - \int_0^1 g_1(x)dx \) is maximal for all \( 1 - \int_0^1 g_t(x)dx, \ t \in [0,1], \) and \( 1 - \int_0^1 g_1(x)dx \leq 1 - \int_0^1 h_{1,\delta_1}(x)dx \), the upper bound (33) satisfies (18).

Using the explicit formulas (16) for asymptotic densities, we see again that (32), and (33) satisfy (17), and (18), respectively, in Theorem 7.

References


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