On the powers of integers and conductors of quadratic fields

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Abstract. We consider non-zero integers of the maximal order $O = O_F$ of the quadratic field $F = \mathbb{Q}(\sqrt{d})$ where $d \in \mathbb{Z}$ is square-free. Let $p$ be an odd prime and $0 \neq \alpha \in O_F$. Using the embedding into $\text{GL}(2, \mathbb{R})$ we obtain bounds for the first $\nu \in \mathbb{N}$ such that $\alpha^\nu \equiv 1 \mod p$. For a conductor $f$, we then study the smallest positive integer $n = n(f)$ such that $\alpha^n \in O_f$. We obtain bounds for $n(f)$ and for $n(fp^k)$. The most interesting case is where $\alpha$ is the fundamental unit of $\mathbb{Q}(\sqrt{d})$.

1. Introduction

We consider quadratic fields $F = \mathbb{Q}(\sqrt{d})$ where $d \in \mathbb{Z}$ is square-free. We write $d = 4q + r$ with $r \in \{1, 2, 3\}$. The algebraic integers $\alpha$ of $\mathbb{Q}(\sqrt{d})$ are given by

$$
\alpha = \begin{cases}
    a + b\sqrt{d}, & a, b \in \mathbb{Z} \\
    \frac{1}{2}(a + b\sqrt{d}), & a, b \in \mathbb{Z}, \ a + b \in 2\mathbb{Z}
\end{cases}
$$

if $r = 2, 3$ (1.1)

Throughout the paper $\alpha$ denotes a non-zero integer of $F$. Let $p$ be an odd prime. First we study the problem to find small exponents $n$ such that $\alpha^n \equiv 1 \mod p$. We will extensively use Legendre symbols.

We adapt the classical Chebyshev polynomials $T_n$ and $U_n$ (for detailed information see [9] Section 5.7, [1] Chapter 22) by defining

$$
t_n(x) = t_n(x; s) = 2s^{n/2}T_n\left(\frac{x}{2\sqrt{s}}\right), \quad (1.2)
$$

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for \( n \in \mathbb{N}_0 \) where \( s \) is the norm of a non-zero integer in the quadratic field \( F \). These are unimodular polynomials with integer coefficients. For technical reasons we use this modification of Chebyshev polynomials for treating the cases \( d \equiv 1 \mod 4 \) and \( d \equiv 2, 3 \mod 4 \) simultaneously. In Section 6 we present all properties of these adapted polynomials which we use for proving our results. Then we specialize the results of the paper [2] about \( \text{GL}(2, \mathbb{Z}) \) to quadratic fields. For previous works on this subject see e.g. [4], [5], [6].

In Section 2, we consider \( 2 \times 2 \) matrices over the rational integers and show how the integers of any quadratic field \( F = \mathbb{Q}(\sqrt{d}) \) can be embedded into \( \text{GL}(2, \mathbb{R}) \). We also prove that \( \alpha^n \equiv 1 \mod p \) holds if and only if \( A^n \equiv I \mod p \) where the matrix \( A \) is the image of \( \alpha \). In the next sections we consider non-zero integers \( \alpha \) of \( F \) and especially units \( \alpha \). In these sections we apply the results of [2] to the case of quadratic fields. Let \( f \) denote a conductor for \( F \). In Section 5, we give upper estimates for \( n(f) := \min\{\nu \in \mathbb{N} : \alpha^\nu \in \mathcal{O}_f\} \) and also for \( n(fp^k) \) where \( k \in \mathbb{N} \) and \( p \) is an odd prime.

2. The embedding of algebraic integers of \( \mathbb{Q}(\sqrt{d}) \) into \( \text{GL}(2, \mathbb{R}) \)

Let \( A \in \text{GL}(2, \mathbb{C}) \), that is
\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0.
\]
(2.1)

We always write
\[
x := \text{tr} A = a + d, \quad s := \det A = ad - bc.
\]
(2.2)

Proposition 2.1. For \( n \in \mathbb{N} \) we have
\[
A^n = u_{n-1}(x)A - su_{n-2}(x)I,
\]
(2.3)
\[
A^n = \frac{1}{2} t_n(x)I + u_{n-1}(x)(A - \frac{1}{2} xI).
\]
(2.4)

This proposition is known in various forms. For instance, (2.3) with \( s = 1 \) is Lemma 3.1.3 in [8] where \( p_n = u_{n-1} \) and \( q_n = u_{n-2} \). The last matrix in (2.4) has zero trace and it follows that
\[
\text{tr} A^n = t_n(x).
\]
(2.5)
With the notation (2.1) we can write (2.4) as

\[
A^n = \begin{pmatrix}
\frac{1}{2} t_n(x) + \frac{1}{2} (a-d)u_{n-1}(x) & bu_{n-1}(x) \\
\frac{1}{2} t_n(x) - \frac{1}{2} (a-d)u_{n-1}(x) & 1 - \frac{1}{2} t_n(x)
\end{pmatrix}.
\]  (2.6)

Now, we consider algebraic integers \( \alpha \) of \( \mathbb{Q}(\sqrt{d}) \) in the notation (1.1). We define a homomorphism \( \varphi \) of the multiplicative semigroup of non-zero integers \( \alpha \) into \( \text{GL}(2, \mathbb{R}) \). For \( r = 2, 3 \) we set (see e.g. [3, p. 38])

\[
\varphi(\alpha) := A = \begin{pmatrix} a & b \\ bd & a \end{pmatrix},
\]  (2.7)

whereas for \( r = 1 \) we set

\[
\varphi(\alpha) := A = \begin{pmatrix} \frac{1}{2} (a+b) & b \\ qb & \frac{1}{2} (a-b) \end{pmatrix}.
\]  (2.8)

It can be checked that this indeed defines an injective homomorphism. We have

\[
s = \det A = \text{Norm}(\alpha) = \begin{cases} 
a^2 - b^2d & \text{if } r = 2, 3 \\
\frac{1}{4} (a^2 - b^2d) & \text{if } r = 1,
\end{cases}
\]  (2.9)

\[
x = \text{tr } A = \begin{cases} 
2a & \text{if } r = 2, 3 \\
a & \text{if } r = 1.
\end{cases}
\]  (2.10)

Since \( A^n = \varphi(\alpha^n) \) and \( \varphi \) is injective, it follows from (2.6) that

\[
\alpha^n = \begin{pmatrix} \frac{1}{2} t_n(2a) + u_{n-1}(2a)b\sqrt{d} \\ \frac{1}{2} t_n(a) + \frac{1}{2} u_{n-1}(a)b\sqrt{d} \end{pmatrix} \text{ if } r = 2, 3
\]  (2.11)

\textbf{Proposition 2.2.} If \( p \) is an odd prime and \( \alpha_k, \alpha_m \) are integers of \( \mathbb{Q}(\sqrt{d}) \) then \( \alpha_k \equiv \alpha_m \text{ mod } p \) if and only if \( \varphi(\alpha_k) \equiv \varphi(\alpha_m) \text{ mod } p \).

\textbf{Proof.} We prove only the more complicated case \( r = 1 \) (see (1.1)). The statement can be proved in a similar way for \( r = 2, 3 \).
First we assume \( \alpha_k \equiv \alpha_m \mod p \) and we prove \( \varphi(\alpha_k) \equiv \varphi(\alpha_m) \mod p \). For \( \alpha_k \equiv \alpha_m \mod p \) with
\[
\alpha_k = \frac{1}{2} \left( a_k + b_k \sqrt{d} \right), \quad \alpha_m = \frac{1}{2} \left( a_m + b_m \sqrt{d} \right).
\]
we have \( a_k \equiv a_m \mod p \) and \( b_k \equiv b_m \mod p \). This implies \( a_k + b_k \equiv a_m + b_m \mod p \) and \( a_k - b_k \equiv a_m - b_m \mod p \). Since \( p \) is odd we obtain
\[
\frac{1}{2} (a_k + b_k) \equiv \frac{1}{2} (a_m + b_m) \mod p, \quad \frac{1}{2} (a_k - b_k) \equiv \frac{1}{2} (a_m - b_m) \mod p.
\]
Then (2.8) yields \( \varphi(\alpha_k) \equiv \varphi(\alpha_m) \mod p \).

Now we assume \( \varphi(\alpha_k) \equiv \varphi(\alpha_m) \mod p \) and prove \( \alpha_k \equiv \alpha_m \mod p \). Using the definition in (2.8) we can write
\[
\varphi(\alpha_j) = \begin{pmatrix} \frac{1}{2}(a_j + b_j) & b_j \\ qb_j & \frac{1}{2}(a_j - b_j) \end{pmatrix}
\]
for \( j = k, m \). We immediately see that \( b_k \equiv b_m \mod p \), \( \frac{1}{2}(a_k + b_k) \equiv \frac{1}{2}(a_m + b_m) \mod p \) and \( \frac{1}{2}(a_k - b_k) \equiv \frac{1}{2}(a_m - b_m) \mod p \) and obtain \( \alpha_k \equiv \alpha_m \mod p \), hence \( \alpha_k \equiv \alpha_m \mod p \). \( \square \)

**Proposition 2.3.** If \( p \nmid b \), \( p \nmid d \) then \( \alpha^n \equiv 1 \mod p \) if and only if \( A^n \equiv I \mod p \).

**Proof.** (a) First, we assume \( \alpha^n \equiv 1 \mod p \). For \( r = 2, 3 \),
\[
\alpha^n = \frac{1}{2} t_n(x) + u_{n-1}(x)b\sqrt{d} \equiv 1 \mod p
\]
with \( p \nmid b \), \( p \nmid d \) and \( x \) was defined in (2.10). Since \( u_{n-1}(x) \equiv 0 \mod p \) by (2.11) we get \( \frac{1}{2} t_n(x) \equiv 1 \mod p \). Hence, \( A^n = \frac{1}{2} t_n(x)I + u_{n-1}(x)(A - \frac{1}{2} x I) \equiv I \mod p \).

For \( r = 1 \), namely, \( \alpha^n = \frac{1}{2} t_n(x) + \frac{1}{2} u_{n-1}(x)b\sqrt{d} \), the proof is similar.

(b) We assume \( A^n \equiv I \mod p \). Then
\[
A^n = \frac{1}{2} t_n(x)I + u_{n-1}(x) \left( A - \frac{1}{2} x I \right) \equiv I \mod p
\]
and we want to prove \( \alpha^n = \frac{1}{2} t_n(x) + u_{n-1}(x)b\sqrt{d} \equiv 1 \mod p \) for \( r = 2, 3 \). By (2.6) we have \( bu_{n-1}(x) \equiv 0 \mod p \). Because of \( b \nmid 0 \mod p \) we get \( u_{n-1}(x)(A - \frac{1}{2} x I)v \equiv 0 \mod p \) and \( \text{tr}(A - \frac{1}{2} x I) \equiv 0 \mod p \), hence
\[
u_{n-1}(x) \begin{pmatrix} * & b \\ bd & * \end{pmatrix} \equiv 0 \mod p.
\]
This implies \( u_{n-1}(x)b \equiv 0 \mod p \). From (2.6) we obtain \( \frac{1}{2} t_n(x) \equiv 1 \mod p \) for the cases \( r = 2, 3 \) and \( r = 1 \), hence \( \alpha^n \equiv 1 \mod p \). \( \square \)
In this section, we specialize the results of [2] to the case of quadratic fields using the embedding introduced in Section 2. We note that we allow $d$ to be negative. Again we write $d = 4q + r$ and $s = \text{Norm}(\alpha)$ for non-zero integers $\alpha$ of $F = \mathbb{Q}(\sqrt{d})$ as in (1.1).

Let $p$ be an odd prime. We assume that $p \nmid d$, $p \nmid b$ and that
\[a^2 - 4s \not\equiv 0 \mod p\] for $r = 2, 3$, \[a^2 - s \not\equiv 0 \mod p\] for $r = 1$. \hfill (3.1)

Throughout the rest of the paper let $x$ be the trace and $s$ be the norm of $\alpha$ as defined in (2.10) and (2.9). Since $t_n$ and $u_n$ are polynomials with integer coefficients the identities in Section 6 can be transferred into congruences. We let $\ell$ be the Legendre symbol
\[\ell := \left(\frac{x^2 - 4s}{p}\right).\] \hfill (3.2)

Then $p - \ell$ becomes $= p \mp 1$ for $\ell = \pm 1$.

**Theorem 3.1.** Let $p$ be an odd prime with $p \nmid d$, $p \nmid b$ and $s = N(\alpha) \neq 0$. Let $\ell$ be the Legendre symbol defined above. We set $\sigma = 1$ for $\ell = +1$ and $\sigma = s$ for $\ell = -1$. Then
\[t_{p-\ell}(x) \equiv 2\sigma \mod p, \quad u_{p-\ell-1}(x) \equiv 0 \mod p.\]

We sum up the further results in the following table.

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$r = 2, 3$</th>
<th>$r = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\left(\frac{\alpha}{p}\right) = +1$</td>
<td>$t_{\frac{r-\ell}{2}}(2a)^2 \equiv 4\sigma \mod p$,</td>
<td>$t_{\frac{r-\ell}{2}}(a)^2 \equiv 4\sigma \mod p,$</td>
</tr>
<tr>
<td></td>
<td>$u_{\frac{r-\ell-1}{2}}(2a) \equiv 0 \mod p$</td>
<td>$u_{\frac{r-\ell-1}{2}}(a) \equiv 0 \mod p$</td>
</tr>
<tr>
<td>$\left(\frac{\alpha}{p}\right) = -1$</td>
<td>$t_{\frac{r-\ell}{2}}(2a) \equiv 0 \mod p$,</td>
<td>$t_{\frac{r-\ell}{2}}(a) \equiv 0 \mod p,$</td>
</tr>
<tr>
<td></td>
<td>$(a^2 - s)u_{\frac{r-\ell-1}{2}}(2a)^2 \equiv \sigma \mod p$</td>
<td>$(a^2 - 4s)u_{\frac{r-\ell-1}{2}}(a)^2 \equiv 4\sigma \mod p$.</td>
</tr>
</tbody>
</table>

This is [2, Theorem 4.1] specialized to our present situation.

The proof in [2] uses Chebyshev polynomials. In the present context of quadratic fields, many of the previous formulas can be proved by other methods, see for instance [3], [7, Theorem 1.7].
4. Units of $F$

First we consider the case $s = \text{Norm}(\alpha) = +1$. Again we let $\ell$ be the Legendre symbol defined in (3.2), and $x$ is defined in (2.10).

The following results are obtained by specializing the results in Sections 5 and 6 of [2]. The Legendre polynomials $t_n$ and $u_{n-1}$ depend only on $x$ and $s$ as defined in (2.9) and (2.10); the specific form (1.1) of $\alpha$ is not important.

**Proposition 4.1.** Let $k \in \mathbb{N}$ divide $p - \ell$ and we assume that $\ell = \left(\frac{x^2 - 4s}{p}\right) \neq 0$. If $x \equiv t_k(y) \mod p$ for some $y \in \mathbb{Z}$ then, with $n = \frac{p-\ell}{k}$,

$$t_n(x) \equiv 2 \mod p, \quad u_{n-1}(x) \equiv 0 \mod p, \quad \alpha^n \equiv 1 \mod p.$$  \hspace{1cm} (4.1)

For a proof compare [2, Theorem 5.1].

For the special case that $k = 2^j$ we can say much more. We construct $x_0, \ldots, x_m$ recursively by the following rule. Let $x_0 = x$. For $\left(\frac{x+2}{p}\right) = -1$ we set $m = 0$ and stop. Now let $\left(\frac{x+2}{p}\right) = +1$ and suppose that $x_0, \ldots, x_k$ have already been constructed such that $2^k \mid (p - \ell)$ and

$$x_{\nu-1} \equiv t_2(x_{\nu}) \mod p, \quad ((x_{\nu}^2 - 4)/p) = \ell \quad \text{for } 1 \leq \nu \leq k.$$  \hspace{1cm} (4.2)

For $2^{k+1} \nmid (p - \ell)$ or $\left(\frac{x+2}{p}\right) = -1$ we set $m = k$ and stop. Otherwise we have $2^{k+1} \mid (p - \ell)$ and $\left(\frac{x+2}{p}\right) = +1$. Then there exists $x_{k+1}$ subject to $x_k + 2 \equiv x_{k+1}^2 \mod p$ and thus $x_k = t_2(x_{k+1})$. It follows from (4.2) that

$$((x_k - 2)/p) = ((x_k + 2)/p)((x_k - 2)/p) = ((x_k^2 - 4)/p) = \ell$$

and therefore $((x_{k+1}^2 - 4)/p) = ((x_k - 2)/p) = \ell$. This completes our construction. We note that $2^n \mid (p - \ell)$.

**Theorem 4.2.** Let $N(\alpha) = 1$, $\ell = \left(\frac{x^2 - 4s}{p}\right) \neq 0$ and $x_0, \ldots, x_m$ be constructed as above. Then

$$t_{(p-\ell)/2^m}(x) \equiv 2 \mod p \quad \text{for } k = 0, \ldots, m,$$  \hspace{1cm} (4.3)

$$t_{(p-\ell)/2^{m+1}}(x) \equiv -2 \mod p \quad \text{or } 2^{m+1} \nmid (p - \ell).$$  \hspace{1cm} (4.4)

The proof is analogous to that of [2, Theorem 5.4].

**Corollary 4.3.** Let $s = N(\alpha) = 1$, $\ell = \left(\frac{x^2 - 4}{p}\right) \neq 0$ and let $x_0, \ldots, x_m$ be constructed as above. Setting $n = (p - \ell)/2^m$ we have

$$u_{n-1}(x) \equiv 0 \mod p, \quad \alpha^n \equiv 1 \mod p.$$  \hspace{1cm} (4.5)

For $2^{m+1} \mid (p - \ell)$ we additionally get

$$u_{\frac{x}{2}-1}(x) \equiv 0 \mod p, \quad \alpha^{n/2} \equiv -1 \mod p.$$  \hspace{1cm} (4.6)

These bounds are best possible: $2^{m+2} \mid (p - \ell)$ implies $u_{\frac{x}{2}-1}(x) \neq 0 \mod p$. 
Because of Let construction stops if \((p - \ell)\) the congruences (4.6) follow from (4.4) analogously. Finally, we let \(s\) that \(u\) step of our construction can always be carried out resulting in recursion formula for \(t_n(x)\) which is similar to that for \(u_n(x)\) in Section 6. Hence, \(u_{2^{-1}}(x) \neq 0 \mod p\).

Now we consider the more complicated case of units with norm \(-1\), i.e.\( t_n(x) = t_n(x; -1)\). As before we set \(\ell := (\frac{x^2 - 4}{p})\) and assume that (3.1) with \(s = -1\) holds. We set \(n = \frac{p - \ell}{2}\). Because of \((-1/p) = (-1)(p^{-1}/2)\) Theorem 3.1 (with \(\sigma = \ell\)) yields

\[
\begin{align*}
t_{2n}(x) &\equiv 2\ell \mod p, \quad t_n(x)^2 \equiv 4\ell \mod p, \quad u_{n-1}(x) \equiv 0 \mod p \\
&\text{for } p \equiv 1 \mod 4, \quad (4.7) \\
t_{2n}(x) &\equiv 2\ell \mod p, \quad t_n(x) \equiv 0 \mod p, \quad u_{n-1}(x) \not\equiv 0 \mod p \\
&\text{for } p \equiv 3 \mod 4. \quad (4.8)
\end{align*}
\]

Then (6.3) implies that

\[
t_{2(p-\ell)}(x) \equiv 2 \mod p. \quad (4.9)
\]

Hence, \(t_n(x) \equiv \pm 2 \mod p\) if and only if \(p \equiv 1 \mod 4\) and \(\ell = +1\). Assuming the latter we obtain from (6.7) with \(t_2(x; -1) = x^2 + 2\) that

\[
t_{2n}(x; -1) = t_n(x^2 + 2; 1) \quad \text{for } n \in \mathbb{N}. \quad (4.10)
\]

Because of \((-1/7) = +1\) there exists \(j \in \mathbb{Z}\) with \(j^2 \equiv -1 \mod p\). We now assume that \(x \not\equiv 0 \mod p\) and \(x \not\equiv \pm 2j \mod p\). This implies

\[
(x^2 + 2)^2 - 4 = x^2(x^2 + 4) \not\equiv 0 \mod p. \quad (4.11)
\]

Similar to Section 4, we construct numbers \(y_0, \ldots, y_m\) subject to the initial condition \(y_0 = x^2 + 2\) instead of \(x_0 = x\). It follows from (4.11) that also \((y_0^n - 4)/p = \ell\). We have \(y_0 + 2 = x^2 + 4\) and therefore \((y_0 + 2)/p = \ell = +1\). Hence, the first step of our construction can always be carried out resulting in \(m \geq 1\). The construction stops if \((y_m + 2)/p = -1\) or \(2^{m+1} \not\mid (p - 1)\).

**Theorem 4.4.** Let \(N(\alpha) = -1, p \equiv 1 \mod 4, a^2 + 4 \not\equiv 0 \mod p, \ell = +1\) and let \(y_0, \ldots, y_m\) be constructed as above. Then \(m \geq 1\) and

\[
t_{(p-1)/2k}(x) \equiv 2 \mod p \quad \text{for } k = 0, \ldots, m - 1, \quad (4.12)
\]
\[
t_{(p-1)/2m}(x) \equiv \begin{cases} 
-2 \mod p & \text{for } 2^{m+1} \mid (p-\ell), \\
0 \mod p & \text{for } 2^{m+1} \nmid (p-\ell).
\end{cases}
\tag{4.13}
\]

See [2, Theorem 6.1] for the proof. The next result is not a surprise because of \(N(\alpha^2) = 1\). The proof is similar to that of Corollary 4.3, so we omit it.

**Corollary 4.5.** Under the assumptions of Theorem 4.4, we now write \(n = (p-\ell)/2^{m-1}\). Then (4.5) holds, and in case \(2^{m+1} \mid (p-\ell)\) then (4.6) is also fulfilled.

**Theorem 4.6.** Let \(N(\alpha) = -1\) and \(k\) be odd with \(k \mid (p-\ell)\). We put \(n = (p-\ell)/k\). If \(x^2 + 4 \nmid 0 \mod p\) and \(x \equiv t_k(y; -1) \mod p\) for some \(y \in \mathbb{Z}\) then

\[
t_{2n}(x) \equiv 2 \mod p, \quad t_n(x) \equiv 2\ell \mod p, \quad \alpha^n \equiv \ell \mod p.
\tag{4.14}
\]

**Proof.** This was shown more generally in [2]. \qed

5. Estimates for conductors

We continue to study the quadratic field \(F = \mathbb{Q}(\sqrt{d})\) with \(d > 0\) and \(r \in \{1, 2, 3\}\). The order with conductor \(f \in \mathbb{N}\) is

\[
\mathcal{O}_f = \begin{cases} 
\{a'+b'f\sqrt{d} : a', b' \in \mathbb{Z}\} & \text{if } r = 2, 3, \\
\left\{ \frac{1}{2}(a'+(f-1)b') + \frac{1}{2}b'f\sqrt{d} : a', b' \in \mathbb{Z}, \ 2 \mid a'+b' \right\} & \text{if } r = 1.
\end{cases}
\tag{5.1}
\]

We fix an integer \(\alpha\) of \(\mathbb{Q}(\sqrt{d})\) with \(s = N(\alpha) \neq 0\). Let \(x\) be given by (2.10). Again we use the notation in (1.1). The most interesting case is that \(\alpha\) is the fundamental unit of \(\mathbb{Q}(\sqrt{d})\). Following Halter–Koch we define

\[
n(f) = n(f, \alpha) := \min\{\nu \in \mathbb{N} : \alpha^\nu \in \mathcal{O}_f\}.
\tag{5.2}
\]

**Lemma 5.1.** Let \(b \neq 0\) be given by (1.1) and \(s, x\) by (2.9). We write

\[
c := \gcd(b, f), b_0 := b/c, f_0 = f/c.
\tag{5.3}
\]

Then we have

\[
n(f) = n(f_0) = \min\{\nu \in \mathbb{N} : u_{\nu-1}(x; s) \equiv 0 \mod f_0\}.
\tag{5.4}
\]
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Proof. By (2.9) and (2.10) we have

$$\alpha^\nu \in \mathcal{O}_f \iff bu_{\nu-1}(x) \equiv 0 \mod f.$$ 

Since $$\gcd(b_0, f_0) = 1$$ it follows by (5.3) that

$$\alpha^\nu \in \mathcal{O}_f \iff b_0 u_{\nu-1}(x) \equiv 0 \mod f_0 \iff u_{\nu-1}(x) \equiv 0 \mod f_0.$$ 

We note that $$b$$ has not been replaced by $$b_0$$. Therefore we still have

$$u_{\nu-1}(x) = u_{\nu-1}(x; s)$$ with $$x$$ and $$s$$ unchanged. \(\square\)

Let $$g \in \mathbb{N}$$ and $$\gcd(b, g) = \gcd(f, g) = 1$$. Then it follows from (5.4) and (6.5) that $$u_{n(f)n(g)-1}(x; s) \equiv 0 \mod \operatorname{lcm}(f, g)f$$. Hence, we get

$$n(fg) \leq n(f)n(g) \quad \text{for } \gcd(f, g) = 1.$$ 

(5.5)

For an odd prime $$p$$ we define

$$q(p) = q(p; \alpha) := \min\{\nu \in \mathbb{N} : u_{\nu-1}(x; s) \equiv 0 \mod p\}.$$ 

(5.6)

The results of Sections 3 and 4 provide upper estimates for $$q(p)$$. These results depend explicitly on $$x$$ and $$s$$, and implicitly on $$a, b$$ and $$d$$ in (1.1).

First let $$\ell = \left(\frac{x^2 - 4s}{p}\right) \neq 0$$. For $$s = 1$$ it follows from Corollary 4.3 that

$$q(p) \leq \frac{p - \ell}{2m}, \quad \text{and} \quad q(p) \leq \frac{p - \ell}{2m+1} \quad \text{for } 2^{m+1} \mid (p - \ell).$$

If $$s = -1$$, $$p \equiv 1 \mod 4$$ and $$\ell = +1$$ then it follows from Corollary 4.5 that

$$q(p) \leq \frac{p - \ell}{2m} \quad \text{and} \quad q(p) \leq \frac{p - \ell}{2m} \quad \text{for } 2^m \mid (p - \ell).$$

Now let $$x^2 - 4s \equiv 0 \mod p$$. Then for all $$\nu \in \mathbb{N}$$ it follows from (6.1) that

$$2^{\nu-1}u_{\nu-1}(x; s) \equiv \nu x^{\nu-1} \mod p.$$ 

We conclude that $$q(p) = p$$ for $$p \mid s$$ and $$q(p) = 2$$ for $$p \nmid s$$.

Theorem 5.2. For $$\gcd(f, b) = 1$$ and $$p \nmid f$$ we have

$$n(p^k f) \leq q(p)p^{k-1}n(f) \quad \text{for all } k \geq 1.$$ 

(5.7)
Proof. We use induction on \(k\). By (5.4) and (6.5) we have
\[ u_{q(p)n(f)-1}(x;s) \equiv 0 \mod f. \]
By (5.6) and (6.5) this congruence also holds modulo \(p\). Since \(\gcd(f,p) = 1\) it follows that the congruence is true also modulo \(pf\). Hence (5.7) holds for \(k = 1\) in view of (5.4).

Now let (5.7) hold for \(k\). We write
\[ \nu = q(p)^{p-1}(x;s) \]
and have, by (5.7),
\[ u_{\nu-1}(x;s) \equiv 0 \mod p. \]
We apply (6.1) with \(n = p\) and with \(s'\) instead of \(s\). The binomial coefficients in the sum are divisible by the prime \(p\). Because of \(2^{p-1} \equiv 1 \mod p\) we get for \(z \in \mathbb{Z}\)
\[ u_{p-1}(z; s') \equiv (z^2 - 4s')^{(p-1)/2} \mod p. \]
For \(z = t_s(x;s)\) we obtain by (6.2) that
\[ u_{p-1}(t_s(x;s); s') \equiv [(x^2 - 4s)u_{\nu-1}(x;s)]^{\frac{p-1}{2}} \equiv 0 \mod p. \]
Here we used (5.8) for \(k \geq 1\). Now we apply (6.4) with \(m = p\) and \(n = \nu\). By (5.8) and (5.9) we obtain
\[ u_{q(p)p^{k-1}}(x;s) = u_{p^{k-1}}(x;s) \equiv 0 \mod p^{k+1}. \]
Hence, it follows from (5.4) that \(n(p^{k+1}f) \leq q(p)p^k\). \(\square\)

**Theorem 5.3.** Let \(f \in \mathbb{N}\) be odd and let \(f_0\) be defined as in (5.3). We write
\[ f_0 = \prod_{\nu=1}^{\mu} p_{\nu}^{k_{\nu}} \quad (k_{\nu} \in \mathbb{N}) \]
with different primes \(p_{\nu}\). Then
\[ n(f) \leq \prod_{\nu=1}^{\mu} (q(p_{\nu})p_{\nu}^{k_{\nu}-1}). \]

Proof. Let \(g_0 = 1\) and for \(1 \leq \lambda \leq \mu\)
\[ g_\lambda = \prod_{\nu=1}^{\lambda} p_{\nu}^{k_{\nu}} \quad (1 \leq \lambda \leq \mu). \]
Then \(g_\lambda = p^{k_s}g_{\lambda-1}\) and \(p_{\lambda} \nmid g_{\lambda-1}\). Hence we obtain from Theorem 5.2 applied to \(f_0\) that
\[ n(f_\lambda) \leq q(p_\lambda)p^{k_s-1}n(f_{\lambda-1}). \]
Hence, (5.11) with \(f\) replaced by \(f_0\) follows by induction. Finally, we use that Lemma 5.1 implies \(n(f) = n(f_0)\). \(\square\)
6. Addendum: useful formulas for Chebyshev polynomials

We present several formulas which we need in proving our results. We put our emphasis on the polynomials \( u_n \) defined in (1.3) (see [9, Section 5.7] and [2]).

For odd \( n \) and \( x, s \in \mathbb{C} \), we have

\[
 u_{n-1}(x; s) = \frac{1}{2^{n-1}} \sum_{k=0}^{(n-3)/2} \binom{n}{2k+1} x^{n-2k-1}(x^2 - 4s)^k + \frac{1}{2^{n-1}} (x^2 - 4s)^{n-1}. \tag{6.1}
\]

The recursion formula \( u_{n+1}(x) = xu_n(x) - su_{n-1}(x) \) shows that

\[
 u_0(x) = 1, \quad u_1(x) = x, \quad u_2(x) = x^2 - s, \quad u_3(x) = x^3 - 2sx,
 u_4(x) = x^4 - 3sx^2 + s^2, \quad u_5(x) = x^5 - 4sx^3 + 2s^2x.
\]

Furthermore, \( t_n(x; s) \) and \( u_n(x; s) \) are polynomials in \( \mathbb{Z}[x, s] \). For \( n \in \mathbb{N} \) we have

\[
 (x^2 - 4s)u_{n-1}(x; s)^2 \equiv t_n(x; s)^2 - 4s^n \tag{6.2}
\]

\[
 t_n(x; s)^2 = t_{2n}(x; s) + 2s^n. \tag{6.3}
\]

We need a relation for products which involves different parameters.

\[
 u_{mn-1}(x; s) = u_{m-1}(t_n(x; s); s^n) u_{n-1}(x; s) \quad (m, n \in \mathbb{N}). \tag{6.4}
\]

It follows that for \( \mu \in \mathbb{N} \) and \( x, s \in \mathbb{Z} \)

\[
 u_{n-1}(x; s) \equiv 0 \mod \mu \Rightarrow u_{mn-1}(x; s) \equiv 0 \mod \mu. \tag{6.5}
\]

To prove (6.4) it is sufficient to consider \( \frac{x}{2\sqrt{s}} = \cos \theta \) with real \( \theta \). Then it follows from (1.2), (1.3) and the properties [9, p. 257] of the \( T_n \) and \( U_n \) that

\[
 t_n(x; s) = 2s \frac{x}{\sqrt{s}} \cos(n\theta), \quad u_{m-1}(x; s) = s^{m-1} \frac{\sin(m\theta)}{\sin \theta}. \tag{6.6}
\]

By (1.3) and (1.2) we therefore have

\[
 u_{m-1}(t_n(x; s); s^n) = s^n \frac{m-1}{2} U_{m-1} \left( \frac{1}{2s^{n/2}t_n(x; s)} \right)
 = s^n \frac{m-1}{2} U_{m-1}(\cos(n\theta)) = s^{m-1} \frac{\sin(mn\theta)}{\sin n\theta}. \]
Now we multiply by $u_{n-1}(x; s)$. Using (6.6) we obtain
\[ u_{m-1}(t_n(x; s); s^n)u_{n-1}(x; s) = s \frac{m-1}{n-1} \frac{\sin(mn\theta)}{\sin n\theta} = u_{mn-1}(x; s) \]
using (6.6) again.

In Section 4 we use the following relation between the polynomials $t_n(x; s)$ with different parameters $s$. If $s \neq 0$ and $m, n \in \mathbb{N}$ then
\[ t_{mn}(x; s) = t_n(t_m(x; s); s^m). \quad (6.7) \]
Indeed, (1.2) and the composition property $T_{mn} = T_n \circ T_m$ imply that
\[ t_{mn}(x; s) = 2(s^m)^{n/2}T_n \left( T_m \left( \frac{x}{2\sqrt{s}} \right) \right) = t_n \left( \frac{1}{2^{n/2}}T_m \left( \frac{x}{2^{n/2}} \right); s^m \right) \]
from which (6.7) follows using (1.2).

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References

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