On several classes of additively non-regular $C$-semirings

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Abstract. In this paper, the authors study several classes of additively non-regular $C$-semirings whose additive idempotents are central, including the generalized $C$-rpp semirings, $C$-rpp semirings, generalized $C$-abundant semirings and $C$-abundant semirings. After introducing the concept of generalized $C$-rpp semirings, the authors obtain their equivalent characterizations, and show that a semiring is a generalized $C$-rpp semiring if and only if it is a strong b-lattice of additively left cancellative halfrings, and if and only if it is a subdirect product of a b-lattice and an additively left cancellative halfring. Also, the authors give the constructions of $C$-rpp semirings, generalized $C$-abundant semirings and $C$-abundant semirings. Consequently, the corresponding results of Clifford semirings and generalized Clifford semirings in [7] and [29] are extended and generalized.

1. Introduction

A semiring is an algebra $(R, +, \cdot)$ with two binary operations $+$ and $\cdot$ such that both $(R, +)$ and $(R, \cdot)$ are semigroups and such that the distributive laws

$$x(y + z) \approx xy + xz \quad \text{and} \quad (x + y)z \approx xz + yz$$

are satisfied.

The additive identity (if it exists) of a semiring $R$ is called zero and denoted by 0. An additively commutative semiring $R$ with a zero satisfying $0x = x0 = 0$
for all $x \in R$, is called a hemiring. A halfring is a hemiring whose additive reduct $(R, +)$ is a cancellative monoid, i.e., for any $a, b, c \in R$, $a + b = a + c$ or $b + a = c + a$ implies $b = c$. A skew-ring $(R, +, \cdot)$ [29] is a semiring whose additive reduct $(R, +)$ is a group, not necessarily an abelian group. An additively cancellative skew-halfring (additively left cancellative skew-halfring, respectively) is a semiring whose additive reduct is an additively cancellative monoid (left cancellative monoid, respectively), not necessarily to be additively commutative.

Also, a semiring $(R, +, \cdot)$ is said to be a b-lattice [29] if its additive reduct $(R, +)$ is a semilattice and its multiplicative reduct $(R, \cdot)$ is a band.

The algebraic theory of semirings have some important applications in automation theory, optimization theory and models of discrete event networks etc. There are a series of papers in the literature considering semirings (for example, see [2], [7]–[10], [16]–[17], [20]–[21], [23]–[32]).

Since semirings are generalizations of distributive lattices, b-lattices, rings, skew-rings, skew-halfrings and left skew-halfrings, it is interesting to use those semirings to establish the constructions of some semirings. In [2], Bandelt and Petrich introduced Bandelt–Petrich Construction in semirings and described the semirings with regular addition which is a subdirect products of a distributive lattice and a ring. In [7], Ghosh established the constructions of strong distributive lattice of semirings which include the Bandelt–Petrich Construction, and characterized all semirings which are subdirect products of a distributive lattice and a ring. In particular, the authors introduced Clifford semirings, and showed that a semiring is a Clifford semiring if and only if it is a strong distributive lattice of rings, and if and only if it is an inverse subdirect product of a distributive lattice and a ring. Later, Sen, Maity and Shum in [29] defined the Clifford semiring which is a completely regular and an additively inverse semiring such that the set of its additive idempotents is a distributive sublattice as well as a k-ideal (without assuming that its additive reduct is commutative) and verified that a semiring is a Clifford semiring if and only if it is a strong distributive lattice of skew-rings. Meanwhile, they introduced generalized Clifford semirings which are completely regular and inverse semirings such that its additive idempotent set is a k-ideal, and obtained that a semiring is a generalized Clifford semiring if and only if it is a strong b-lattice of skew-rings, and if and only if it is an additively inverse semiring and is a subdirect product of a b-lattice and a skew-ring. It is not hard to see that all the semirings studied in [2], [7] and [29] are additively regular.

On the other hand, as we know, in order to generalize regular semigroups, new Green’s relations, namely, the Green’s $\ast$-relations on a semigroup have been
introduced as follows (see [3], [19], or [22]):

\[
\mathcal{L}^* = \{(a, b) \in S \times S : (\forall x, y \in S^1) ax = ay \iff bx = by\},
\]

\[
\mathcal{R}^* = \{(a, b) \in S \times S : (\forall x, y \in S^1) xa = ya \iff xb = yb\},
\]

\[
\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*.
\]

\[
\mathcal{D}^* = \mathcal{L}^* \lor \mathcal{R}^*.
\]

It is clear that \(\mathcal{L} \subseteq \mathcal{L}^*\), \(\mathcal{R} \subseteq \mathcal{R}^*\), \(\mathcal{H} \subseteq \mathcal{H}^*\), \(\mathcal{D} \subseteq \mathcal{D}^*\). A semigroup \(S\) is abundant [3] if its each \(\mathcal{L}^*\)-class and each \(\mathcal{R}^*\)-class contains an idempotent, a semigroup \(S\) is an rpp semigroup (a lpp semigroup, respectively) if its each \(\mathcal{L}^*\)-class (\(\mathcal{R}^*\)-class, respectively) contains an idempotent (see [5]). A semigroup \(S\) is a \(C\)-rpp semigroup ([5]) if its every \(\mathcal{L}^*\)-class contains an idempotent and \(E(S)\) is central. Dually, we will get the definition of \(C\)-lpp semigroups. A semigroup \(S\) is said to be a \(C\)-abundant semigroup if it is abundant and \(E(S)\) is central, i.e., it is both a \(C\)-lpp semigroup and a \(C\)-rpp semigroup. In general, abundant semigroups, \(C\)-rpp semigroups, \(C\)-lpp semigroups and \(C\)-abundant semigroups are not regular, so we will call them non-regular semigroups in the following. There are also a series of papers in the literature considering non-regular semigroups (for example, see [1], [3]–[5], [11]–[15], [18] etc.).

In this paper, we will study several classes of additively non-regular \(C\)-semirings whose additive idempotents are central, including the generalized \(C\)-rpp semirings, \(C\)-rpp semirings, generalized \(C\)-abundant semirings and \(C\)-abundant semirings. Our purpose is to extend the results of Clifford semirings and generalized Clifford semirings in [29] and the semirings which are subdirect products of a distributive lattice and a ring in [7] to the non-regular \(C\)-semirings. We will show that a semiring is a generalized \(C\)-rpp semiring (\(C\)-rpp semiring, generalized \(C\)-abundant semiring, \(C\)-abundant semiring, respectively) if and only if it is a strong \(b\)-lattice (strong distributive lattice, strong \(b\)-lattice, strong distributive lattice, respectively) of additively left cancellative (left cancellative, cancellative, respectively) halfrings, and if and only if it is a subdirect product of a \(b\)-lattice (distributive lattice, \(b\)-lattice, distributive lattice, respectively) and an additively left cancellative (left cancellative, cancellative, cancellative, respectively) halfring.

For notations and terminologies not mentioned in this paper, the readers are referred to [3], [8] or [29].
2. Generalized $C$-rpp semirings and $C$-rpp semirings

In this section, we will study the classes of generalized $C$-rpp semirings and $C$-rpp semirings, and show that a semiring is a generalized $C$-rpp semiring ($C$-rpp semiring, respectively) if and only if it is a strong b-lattice (strong distributive lattice, respectively) of additively left cancellative halfrings, and if and only if it is a subdirect product of a b-lattice (distributive lattice, respectively) and an additively left cancellative halfring. Also, we will give some other characterizations of such semirings.

Let $(R, +, \cdot)$ be a semiring. We denote the Green’s relations $\mathcal{L}, \mathcal{R}, \mathcal{H}$ on additive reduct $(R, +)$ by $\mathcal{L}^+, \mathcal{R}^+, \mathcal{H}^+$, respectively. These are also equivalence relations on semiring $(R, +, \cdot)$. Now, we introduce Green’s $\ast$-relations $\mathcal{L}^\ast, \mathcal{R}^\ast, \mathcal{H}^\ast$ on semiring $R$ which are given by

$\mathcal{L}^\ast = \{(a, b) \in R \times R : (\forall x, y \in R^+) a + x = a + y \iff b + x = b + y\}$,

$\mathcal{R}^\ast = \{(a, b) \in R \times R : (\forall x, y \in R^+) x + a = y + a \iff x + b = y + b\}$,

$\mathcal{H}^\ast = \mathcal{L}^\ast \cap \mathcal{R}^\ast$.

It is clear that $\mathcal{L}^\ast \subseteq \mathcal{L}^+, \mathcal{R}^\ast \subseteq \mathcal{R}^+, \mathcal{H}^\ast \subseteq \mathcal{H}^+$ on $(R, +, \cdot)$. In particular, if $R$ is an additively regular semiring, $\mathcal{L}^\ast = \mathcal{L}^+, \mathcal{R}^\ast = \mathcal{R}^+, \mathcal{H}^\ast = \mathcal{H}^+$ [4]. In general, Green’s equivalence relations $\mathcal{L}^\ast, \mathcal{R}^\ast$ and $\mathcal{H}^\ast$ are not congruences on $(R, +, \cdot)$.

For a semiring $R$, we denote by $E^+(R)$ the set of all additive idempotents of $R$. For any $e, f \in E^+(R)$, we write $e \leq_+ f$ if $e + f = f = f + e$. Remark that $\leq_+$ is a partial order which is compatible with the multiplication.

In the following, we will introduce the concepts of strong b-lattice and strong distributive lattice of semirings.

Definition 1 (Definition 2.3 in [29]). Let $T$ be a b-lattice and $\{R_\alpha : \alpha \in T\}$ be a family of pairwise disjoint semirings which are indexed by the elements of $T$. For each $\alpha \leq \beta$ in $T$, we now embed $R_\alpha$ in $R_\beta$ via a semiring monomorphism $\phi_{\alpha, \beta}$ satisfying the following conditions:

1. $\phi_{\alpha, \alpha} = I_{R_\alpha}$, the identity mapping on $R_\alpha$;
2. $\phi_{\alpha, \beta} \phi_{\beta, \gamma} = \phi_{\alpha, \gamma}$ if $\alpha \leq \beta \leq \gamma$;
3. $R_\alpha \phi_{\alpha, \beta} R_\beta \phi_{\beta, \gamma} \subseteq R_\alpha \beta \phi_{\alpha, \beta, \gamma}$ if $\alpha + \beta \leq \gamma$, i.e., $\alpha + \beta + \alpha \beta \leq \gamma$. 

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On $R = \cup_{\alpha \in \gamma} R_\alpha$, we define addition $+$ and multiplication $\cdot$ for $a \in R_\alpha$, $b \in R_\beta$, as follows:

(1.4) 

$$a + b = a_{\phi,\alpha,\beta} + b_{\phi,\beta,\alpha}$$

and such that

$$c_{\phi,\alpha,\beta} = a_{\phi,\alpha,\beta} + b_{\phi,\beta,\alpha}.$$ 

Same as the notation of strong semilattice of semigroups, we denote the above system by $R = \langle T, R_\alpha, \phi_{\alpha,\beta} \rangle$ and call it the strong b-lattice $T$ of the semirings $R_\alpha, \alpha \in T$.

In an obvious way, we may replace b-lattice $T$ in the above definition by distributive lattice $D$, $R = \langle D, R_\alpha, \phi_{\alpha,\beta} \rangle$ and call it strong distributive lattice $D$ of the semirings $R_\alpha, \alpha \in D$.

**Lemma 1** (Theorem 2.4 in [29]). The system $R = \langle T, R_\alpha, \phi_{\alpha,\beta} \rangle$ defined above is a semiring.

**Lemma 2** ([5]). A semigroup $S$ is a $C$-rpp semigroup if and only if it is a strong semilattice of left cancellative monoids.

By Lemma 2, we can dually obtain that a semigroup $S$ is a $C$-lpp semigroup if and only if it is a strong semilattice of right cancellative monoids.

From [5], it is also known that a semigroup $(S,+)$ is called a [right, left, respectively]adequate semigroup if its idempotents commute and every $L^*$-class and $R^*$-class $[L^*]$-class, $R^*$-class, respectively] contain a unique idempotent. For an element $a$ of such a semigroup, the unique idempotent in the $L^*$-class $[R^*$-class, respectively] containing $a$ is denoted by $a^*[a^+]$. A [right, left, respectively] adequate semigroup $S$ is called [right, left, respectively] type A if $[e + a = a + (e + a), a + e = (a + e)^* + a]$ and $a + e = (a + e)^* + a$ for $a \in S$ and $e \in E^+(S)$.

By the definition of $C$-rpp semigroups, it is not hard to see that a right type A semigroup is a $C$-rpp semigroup.

**Lemma 3** (Corollary 2.8 in [4]). Let $(S,+)$ be a right type A semigroup with semilattice of idempotents $E = E(S)$ and $\mu_L$ the largest congruence contained in $L^*$. Then the following conditions are equivalent:

(1) $S/\mu_L \cong E$;
Lemma 4 (Proposition 2.9 in [4]). Let \((S, +)\) be an adequate semigroup with semilattice of idempotents \(E = E(S)\) and \(\mu\) the largest congruence contained in \(\mathcal{H}^+\). Then the following conditions are equivalent:

1. \(S/\mu \cong E\);
2. \(\mu = \mathcal{H}^+\);
3. \(E\) is central in \(S\);
4. \(S\) is a strong semilattice of left cancellative monoids.

From Lemma 2, it is known that a semigroup \(S\) is a C-rpp semigroup if and only if it is a strong semilattice of left cancellative monoids. Now, we will similarly give the definition of generalized C-rpp semirings and then investigate some of their equivalent characterizations and constructions.

Definition 2. A semiring \(R\) is said to be a generalized C-rpp semiring if it is a strong b-lattice of additively left cancellative skew-half rings.

In the following, for any \(a \in R\), the unique idempotent in the \(\mathcal{H}^+\)-class containing \(a\) is denoted by \(a^0\).

Theorem 1. Assume that \(R\) is a generalized C-rpp semiring. Then the following conditions hold:

1. \((R, +)\) is a C-rpp semigroup;
2. \(E^+(R)\) is a b-lattice;
3. for any \(a, b \in S\), \((ab)^0 + a^0 b^0 = a^0 b^0\);
4. if \(a^0 = b^0\) and \(a + e = b + e\) for \(a, b \in R\) and some \(e \in E^+(R)\), then \(a = b\).

Proof. Assume that \(R\) is a generalized C-rpp semiring, then it is a strong b-lattice of additively left cancellative skew-half rings, say \(R = (T, R_\alpha, \phi_{\alpha, \beta})\), where \(R_\alpha\) are the additively left cancellative skew-half rings in which the zero of additive reduct is denoted by \(0_\alpha\) and \(T\) is a b-lattice.

i) Since \(R\) is a strong b-lattice of left additively cancellative skew-half rings \(R_\alpha, (R, +)\) is a strong semilattice of left cancellative monoids \((R_\alpha, +)\), by Lemma 2, \((R, +)\) is a C-rpp semigroup, and condition (GCR1) holds.
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ii) Notice that $E^+(R) = \{0_\alpha \mid \alpha \in T\} \cong T$, where $T$ is a b-lattice, then $(E^+(R), +, \cdot)$ is also a b-lattice, and condition (GCR2) holds.

iii) We will show that $a^0 = 0_\alpha$ for any $a \in R_\alpha$ at first. In fact, notice that $(R, +)$ is a strong semilattice of left cancellative monoids $(R_\alpha, +)$, by Lemma 3, we obtain that $a \overset{L^*}{\to} R_\alpha$ for any $a \in R_\alpha$. And then, $a^0 = 0_\alpha$. Thus, for any $a \in R_\alpha$, $b \in R_\beta$, we have $(ab)^0 = 0_{\alpha\beta} = 0_\alpha 0_\beta = a^0 b_\beta$, $(ab)^0 + a^0 b_\beta = a^0 b_\beta$. The condition (GCR3) holds.

iv) Assume that $a^0 = b^0$ and $a + e = b + e$ for $a, b \in R$ and some $e \in E^+(R)$, then there exist $\alpha, \beta \in T$, s.t., $a, b \in R_\alpha$ and $e \in E(R_\beta)$. Now let $f = a^0 + e$. Then

$$a + f = b + f,$$

i.e.,

$$a \varphi_{\alpha, \alpha + \beta} + f \varphi_{\alpha + \beta, \alpha + \beta} = b \varphi_{\alpha, \alpha + \beta} + f \varphi_{\alpha + \beta, \alpha + \beta}.$$ 

Since $\varphi_{\alpha + \beta, \alpha + \beta}$ is a monomorphism, we have

$$f \varphi_{\alpha + \beta, \alpha + \beta} = f,$$

where $f$ is the additive identity of $S_{\alpha + \beta}$. And then, we have

$$a \varphi_{\alpha, \alpha + \beta} = b \varphi_{\alpha, \alpha + \beta}.$$ 

Also, notice that $\varphi_{\alpha, \alpha + \beta}$ is a monomorphism, we immediately get $a = b$. The condition (GCR4) holds.

Actually, the converse of the above theorem also holds. To show this, we need the following proposition.

**Proposition 1.** If $R$ satisfies the conditions (GCR1)–(GCR3), then the following conclusions hold:

1. $L^*$ is a semiring congruence;
2. $R/L^* \cong E^+(R)$.

**Proof.** (1) Assume that $R$ satisfies the conditions (GCR1)–(GCR3), we will show that $L^*$ is a semiring congruence.

Firstly, since (GCR1) holds, by Lemma 3, $(R, +)$ is a strong semilattice $Y$ of left cancellative monoids $(R_\alpha, +)$, where $Y \cong (E^+(R), +)$. By Lemma 3 again, $(R, +)/L^* \cong (E^+(R), +)$, we obtain that $L^*$ is a semilattice congruence on $(R, +)$. To show that $L^*$ is a semiring congruence, we only need to prove that $L^*$ is a multiplicative congruence on $(R, \cdot)$, i.e., for any $a, b \in R$,

$$(ab)^0 = a^0 b = ab^0 = a^0 b^0.$$
In fact, for any \( a, b \in R \), since \( a^0b, ab^0, a^0b^0 \in E^+(R) \), by condition (GCR2), we have
\[
(ab)^0 = [(a + a^0)(b + b^0)]^0 = (ab + a^0b + ab^0 + a^0b^0)^0 = (ab)^0 + a^0b + ab^0 + a^0b^0.
\]
And then
\[
(ab)^0 + a^0b^0 = (ab)^0.
\]
Together with (GCR3), we have
\[
(ab)^0 = a^0b^0.
\]
Notice that \((ab)^0 + a^0b = (ab)^0\) and \((ab)^0 + ab^0 = (ab)^0\) also hold, we immediately get
\[
(ab)^0 = a^0b = ab^0 = a^0b^0.
\]

(2) Define a mapping
\[
\phi : S/L^* \to E^+(S), \quad aL^* \mapsto a^0.
\]
It is a routine way to check that \(\phi\) is bijective, and
\[
(aL^* + bL^*)\phi = [(a + b)L^*]\phi = (a + b)^0 = a^0 + b^0,
\]
\[
[(aL^*)(bL^*)]\phi = [(ab)L^*]\phi = (ab)^0 = a^0b^0.
\]
Thus, \(S/L^* \cong E^+(S)\). \(\square\)

Now, we have the following theorem.

**Theorem 2.** A semiring \( R \) is a generalized C-rpp semiring if and only if it satisfies the conditions (GCR1)–(GCR4).

**Proof.** We only need to show the sufficiency. By Proposition 1, it is known that if \( S \) satisfies the conditions (GCR1)–(GCR4), then \( L^* \) is a semiring congruence on \((R, +, \cdot)\), and \( R/L^* \cong E^+(R) \) is a b-lattice. Also, for any \( a \in R \), notice that \((L^*_a, +)\) is an additively left cancellative monoid, we obtain that \( R \) is a b-lattice of additively left cancellative skew-halfrings.

For any \( e, f \in E^+(R) \) with \( e \leq f \), define a mapping
\[
\phi_{e,f} : L^*_e \to L^*_f, \quad a \mapsto a + f.
\]
In the following, we will show that \( R = (E^+(R), L^*_e, \phi_{e,f}) \) is a strong b-lattice of the additively left cancellative skew-halfrings \( L^*_e, e \in E^+(R) \). That is, we will show that \(\phi_{e,f} \) satisfies the conditions of strong b-lattice.
For any \( a, b \in L^*_e \),
\[
(a + b) \phi_{e,f} = a + b + f = a + (f + b + f) = (a + f) + (b + f) = a \phi_{e,f} + b \phi_{e,f}.
\]
Also, since \( af \in L^*_e \cap E^+(R) \), we have \( af = ef \). And then,
\[
(ab) \phi_{e,f} = ab + f = ab + e + f = ab + (e + f)^2 = ab + e + ef + fe + f
= ab + af + fb + f = (a + f)(b + f) = a \phi_{e,f} b \phi_{e,f}.
\]
Hence, \( \phi_{e,f} \) is a semiring morphism.

For any \( a, b \in L^*_e \), if \( a \phi_{e,f} = b \phi_{e,f} \), we have \( a + f = b + f \). Notice that \((L^*_e, +)\) is an additively left cancellative monoid, we will get \( a^0 = b^0 = e \). It follows from (GCR4) that \( a = b \). Thus, \( \phi_{e,f} \) is a semiring monomorphism.

Moreover, we can check that the monomorphism \( \phi_{e,f} \) satisfies the conditions (1.1)–(1.4) of Definition 1.

(i) \( \phi_{e,e} \) is clearly an identity morphism.

(ii) For any \( e, f, g \in E^+(R) \) with \( e \leq f \leq g \), we have
\[
a \phi_{e,f} \phi_{f,g} = a + f + g = a + g = a \phi_{e,g}.
\]
Hence, \( \phi_{e,f} \phi_{f,g} = \phi_{e,g} \).

(iii) For any \( e, f, g \in E^+(R) \), if \( e + f \leq g \), then for any \( a \in L^*_e, b \in L^*_f \),
\[
a \phi_{e,g} b \phi_{f,g} = (a + g)(b + g) = ab + ag + gb + g = ab + eg + gf + g
= ab + g = (ab) \phi_{e,f,g}.
\]

(iv) For any \( e, f \in E^+(R), a \in L^*_e, b \in L^*_f \), we have
\[
a \phi_{e,e+f} + b \phi_{f,e+f} = (a + e + f) + (b + e + f) = a + f + b + e
= a + b + e + f = a + b;
\]
\[
a \phi_{e,e+f} b \phi_{f,e+f} = (a + e + f)(b + e + f) = ab + a(e + f) + (e + f)b + (e + f)
= ab + (e + f) = (ab) \phi_{e,f,e+f}.
\]

Thus, we have shown that \( R \) is a strong b-lattice of additively left cancellative skew-halfrings. And then it is a generalized C-rpp semiring. \( \square \)

Example 1. Let \((A, +)\) and \((B, +)\) be the infinite cyclic monoids generated by \( a \) and \( b \) respectively. Let \( M = A \cup B \cup \{0\} \) with additive identity \( 0 \) and addition \( + \) defined by
\[
ma + nb = (m + n)b, nb + ma = (n + m)a
\]
for any \( m, n \in \mathbb{N}^+ \). Also, we define the multiplication \( \cdot \) of \( M \) as follows: \( s_1 \cdot s_2 = 0 \) for any \( s_1, s_2 \in M \), it is a routine way to check that \( (M, +, \cdot) \) is an additively left cancellative skew-halfring.

On the other hand, let \( D = \{ e, f \} \) such that \( e + e = e \cdot e = e, f + f = e + f = f \cdot f = e \cdot f = f \cdot e = f \). Then \( (D, +, \cdot) \) is a b-lattice.

Now, construct the direct product of \( D \) and \( M \), and denote it by \( R \), i.e., \( R = D \times M \). Then, we can check that \( E^+(R) = D \times \{ e_M \} \), where \( e_M \) is the identity element of \( (M, +) \). It is also not hard to check that \( (R, +, \cdot) \) is a semiring which satisfies the following conditions:

(i) \( (R, +) \) is a \( C \)-rpp semigroup;
(ii) \( (E^+(R), +, \cdot) \) is a b-lattice;
(iii) for any \( a, b \in R \), \( (ab)^0 + a^0 b^0 = a^0 b^0 \);
(iv) if \( a^0 = b^0 \) and \( a + e = b + e \) for \( a, b \in R \) and some \( e \in E^+(R) \), then \( a = b \).

By Theorem 2, \( (R, +, \cdot) \) is just a generalized \( C \)-rpp semiring.

Next, we will give another construction of generalized \( C \)-rpp semirings. Recall that a subdirect product algebra \( T \) is a subalgebra of a direct product of algebras such that the projection mapping from the algebra \( T \) to each of its components is surjective.

**Theorem 3.** A semiring \( R \) is a generalized \( C \)-rpp semiring if and only if it is a subdirect product of a b-lattice and an additively left cancellative skew-halfring.

**Proof.** (\( \Rightarrow \)) Suppose that \( R \) is a subdirect product of a b-lattice \( T \) and an additively left cancellative skew-halfring \( M \). Consider \( R \subseteq T \times M \). For each \( \alpha \in T \), let \( R_\alpha = (\{ \alpha \} \times M) \cap R \). Then \( R_\alpha \) is an additively left cancellative skew-halfring for each \( \alpha \in T \) and \( R = \cup_{\alpha \in T} R_\alpha \). Now for each pair \( \alpha, \beta \in T \) with \( \alpha \leq_+ \beta \), define a mapping

\[
\phi_{\alpha, \beta} : R_\alpha \rightarrow R_\beta, (\alpha, r) \phi_{\alpha, \beta} = (\beta, r).
\]

Then \( \phi_{\alpha, \beta} \) is clearly a monomorphism satisfying the conditions \( \phi_{\alpha, \alpha} = I_{R_\alpha} \) and \( \phi_{\alpha, \beta} \phi_{\beta, \gamma} = \phi_{\alpha, \gamma} \) if \( \alpha \leq_+ \beta \leq_+ \gamma \) for \( \alpha, \beta, \gamma \in T \).

Let \( \alpha, \beta, \gamma \in T \) be such that \( \alpha + \beta \leq_+ \gamma \). Denote \( a = (\alpha, r) \in R_\alpha \), \( b = (\beta, r') \in R_\beta \). And then

\[
a + b = (\alpha, r) + (\beta, r') = (\alpha + \beta, r + r') \in R_{\alpha + \beta}
\]

and

\[
ab = (\alpha, r)(\beta, r') = (\alpha \beta, rr') \in R_{\alpha \beta}.
\]
Now, we have
\[(a\phi_{\alpha,\gamma})(b\phi_{\beta,\gamma}) = (\alpha\beta, rr')\phi_{\alpha,\gamma} = (ab)\phi_{\alpha,\gamma}.\]
Also, since
\[a + b = (\alpha, r) + (\beta, r') = (\alpha + \beta, r + r') = (\alpha + \beta, r) + (\alpha + \beta, r') = a\phi_{\alpha,\beta} + b\phi_{\beta,\alpha + \beta}\]
and
\[(a\phi_{\alpha,\alpha + \beta})(b\phi_{\beta,\alpha + \beta}) = (\alpha + \beta, r)(\alpha + \beta, r') = (\alpha + \beta, rr')\]
\[= (\alpha\beta, rr')\phi_{\alpha,\beta} \alpha + \beta = (ab)\phi_{\alpha,\beta} \alpha + \beta.\]

\(R\) is a strong b-lattice of additively left cancellative shew-halfrings. Hence, \(R\) is a generalized C-rpp semiring.

(\(\Rightarrow\)) Assume that \(R\) is a generalized C-rpp semiring. We will show that it is a subdirect product of a b-lattice and an additively left cancellative shew-halfring by the following steps.

Firstly, from Proposition 1, \(L^*\) is a semilattice congruence on \((R, +)\) and a semiring congruence on \(R\). Also, since \(R\) is a generalized C-rpp semiring, we have \(aa^0 = a\) for any \(a \in R\) and then \(a = aa^0 L^* a^2\). Hence, \(R/L^*\) is an idempotent semiring with the semilattice additive reduct and band multiplicative reduct, i.e., \(R/L^*\) is a b-lattice.

Secondly, define a binary relation
\[\theta = \{(a, b) \mid (\exists e \in E^+(R)) a + e = b + e\}.\]
It can be easily seen that \(\theta\) is an equivalence relation on \(R\). Moreover, \(\theta\) is the minimum additively left cancellative shew-halfring congruence on \((R, +, .)\). In fact, by the Proposition 1.7 in [18], \(\theta\) is a minimum left cancellative monoid congruence on the additive reduct \((R, +)\). Also, if \(a\theta b\) for some \(a, b \in R\), there exists \(e \in E^+(R)\) such that \(a + e = b + e\). Now, for any \(c \in R\), we have
\[ac + ec = bc + ec, ca + ec = cb + ec.\]
Notice that \(ce, ec \in E^+(R)\). We immediately obtain that
\[acebdc, caeacb.\]
Thus, we have shown that \(\theta\) is the minimum additively left cancellative shew-halfring congruence on \((R, +, .)\). This also shows that \(R/\theta\) is an additively left cancellative shew-halfring.
Finally, define a mapping
\[ \Phi : R \to R/\theta \times R/L^*, \quad a \mapsto (a\theta, aL^*) \]
It is a routine way to check that \( R \) can be embedded into \( R/\theta \times R/L^* \), and the projection mapping from \( R \) into each of its components is surjective. Consequently, \( R \) is a subdirect product of a b-lattice and an additively left cancellative skew-halfring.

So we have obtained some constructions and characterizations of generalized C-rpp semirings. In the following, we will investigate another class of additive non-regular C-semirings, called C-rpp semirings.

Definition 3. A semiring \( R \) is said to be a C-rpp semiring if it is a strong distributive lattice of additively left cancellative skew-halfrings.

Theorem 4. Assume that \( R \) is a C-rpp semiring. Then the following conditions hold:

\begin{enumerate}
  \item[(CR1)] \((R, +)\) is a C-rpp semigroup;
  \item[(CR2)] \((E^+(R), +, \cdot)\) is a distributive lattice;
  \item[(CR3)] for any \( a, b \in R \), \((ab)^0 + a^0b^0 = a^0b^0\);
  \item[(CR4)] if \( a^0 = b^0 \) and \( a + e = b + e \) for \( a, b \in R \) and some \( e \in E^+(R) \), then \( a = b \).
\end{enumerate}

Conversely, if a semiring \( R \) satisfies the conditions (CR1)–(CR4), then it is a C-rpp semiring.

Proof. \((\Rightarrow)\) From Definition 2 and Definition 3, it is known that a C-rpp semiring is a generalized C-rpp semiring. Thus, by Theorem 2, (CR1), (CR3), (CR4) hold. We only need to prove that (CR2) holds.

Assume that \( S \) is a C-rpp semiring. Then it is a strong distributive lattice of additively left cancellative skew-halfrings, say \( R = \langle D, R_\alpha, \phi_{\alpha, \beta} \rangle \), where each \( R_\alpha \) is an additively left cancellative skew-halfring in which the additive identity is denoted by \( 0_\alpha \) and \( T \) is a distributive lattice. Notice that \( E^+(R) = \{ 0_\alpha \mid \alpha \in T \} \cong T \), we immediately obtain that \((E^+(R), +, \cdot)\) is also a distributive lattice. Hence, (GC2) holds.

\((\Leftarrow)\) Assume that the semiring \( R \) satisfies the conditions (CR1)–(CR4), then by Theorem 2, it is clearly a generalized C-rpp semiring. Also, note that (CR2) holds. By analogy with the discussions of Theorem 1, \( R \) is a C-rpp semiring.

Example 2. Let \((M = A \cup B \cup \{ 0 \}, +, \cdot)\) be an additively left cancellative skew-halfring as defined in Example 1. Let \( D = \{ e, f \} \) be such that \( e + e = e \cdot e = \)}
On several classes of additively non-regular C-semirings

\[ e + f = f + e = e, f + f = f \cdot f = e \cdot f = f \cdot e = f. \]
Then \((D, +, \cdot)\) is a distributive lattice.

Now, construct the direct product of \(D\) and \(M\) and denote it by \(R\), i.e., \(R = D \times M\). Clearly, \(E^+(R) = D \times \{e_M\}\), where \(e_M\) is the identity element of \((M, +)\). It is also not hard to check that \((R, +, \cdot)\) is a semiring which satisfies the following conditions:

(i) \((R, +)\) is a C-rpp semigroup;
(ii) \((E^+(R), +, \cdot)\) is a distributive lattice;
(iii) for any \(a, b \in R\), \((ab)^0 + a^0b^0 = a^0b^0;
(iv) if \(a^0 = b^0\) and \(a + e = b + e\) for \(a, b \in R\) and some \(e \in E^+(R)\), then \(a = b\).

Thus, by Theorem 4, \((R, +, \cdot)\) is a C-rpp semiring.

Further, by analogy with the discussions of the subdirect decompositions of generalized C-rpp semirings, we have the following theorem.

**Theorem 5.** A semiring \(R\) is a C-rpp semiring if and only if it is a subdirect product of a distributive lattice and an additively left cancellative skew-halfring.

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3. Generalized C-abundant semirings and C-abundant semirings

In this section, we will study generalized C-abundant semirings and C-abundant semirings, and will show that a semiring is a generalized C-abundant semiring (C-abundant semiring, respectively) if and only if it is a strong b-lattice (strong distributive lattice, respectively) of additively cancellative skew-halfrings, and if and only if it is a subdirect product of a b-lattice (distributive lattice, respectively) and an additively cancellative skew-halfring. Also, we will give some characterizations of such semirings.

Firstly, by Lemma 2 and its dual, we immediately have

**Lemma 5.** A semigroup \(S\) is a C-a(or C-abundant) semigroup if and only if it is a strong semilattice of cancellative monoids.

**Definition 4.** A semiring \(R\) is said to be a generalized C-abundant semiring if it is a strong b-lattice of additively cancellative skew-halfrings.

**Theorem 6.** Assume that \(R\) is a generalized C-abundant semiring, then the following conditions hold:

(GCA1) \((R, +)\) is a C-abundant semigroup;
(GCA2) \((E^+(R), +, \cdot)\) is a b-lattice;
for any \( a, b \in R \), \((ab)^0 + a^0b^0 = a^0b^0;\)

(GCA4) if \( a^0 = b^0 \) and \( a + e = b + e \) for \( a, b \in S \) and some \( e \in E^+(R) \), then \( a = b \).

**Proof.** From Definition 2 and Definition 4, it is known that, a generalized \( C \)-abundant semiring is a generalized \( C \)-rpp semiring, then the conditions (GCA2)–(GCA4) hold. We only need to show that condition (GCA1) holds.

Actually, if \( R \) is a generalized \( C \)-abundant semiring, then it is a strong \( b \)-lattice of additively cancellative skew-halfrings, say \( R = \langle T, R_\alpha, \phi_\alpha, \beta \rangle \), where each \( R_\alpha \) is an additively cancellative skew-halfring and \( T \) is a \( b \)-lattice. It follows that \((R, +)\) is strong semilattice of cancellative monoids \((R_\alpha, +)\), i.e., \((R, +)\) is a \( C \)-abundant semigroup. Thus, the condition (GCA1) holds.

**Proposition 2.** Assume that a semiring \( R \) satisfies the conditions (GCA1)–(GCA3). Then the following conclusions hold:

(1) \( \mathcal{H}^* \) is a semiring congruence;

(2) \( R/\mathcal{H}^* \cong E^+(R) \).

**Proof.** (1) Assume that semiring \( R \) satisfies the conditions (GCA1)–(GCA3). We will show that \( \mathcal{H}^* \) is a semiring congruence.

Firstly, since condition (GCA1) holds, by Lemma 4 or Lemma 5, \((R, +)\) is a strong semilattice \( Y \) of cancellative monoids \( R_\alpha \), where \( Y \cong (E^+(R), +) \). By Lemma 4 again, we obtain that \( \mathcal{H}^* = \mathcal{L}^* = R^* \) is a semilattice congruence on \((R, +)\). And then, by analogy with with the discussions of Proposition 1, we can get \( \mathcal{H}^* = \mathcal{L}^* = R^* \) is a semiring congruence.

(2) Define a mapping

\[ \phi : R/\mathcal{H}^* \rightarrow E^+(R), a\mathcal{H}^* \mapsto a^0. \]

It is not hard to check that \( \phi \) is bijective, and

\[ (a\mathcal{H}^* + b\mathcal{H}^*)\phi = [(a + b)\mathcal{H}^*] \phi = (a + b)^0 = a^0 + b^0, \]

\[ [(a\mathcal{H}^*)(b\mathcal{H}^*)] \phi = [(ab)\mathcal{H}^*] \phi = (ab)^0 = a^0b^0. \]

Thus, \( R/\mathcal{H}^* \cong E^+(R) \).

**Theorem 7.** A semiring \( R \) is a generalized \( C \)-abundant semiring if and only if it satisfies the conditions (GCA1)–(GCA4).
Proof. We only need to show the sufficiency. By Proposition 2, it is known that if \( S \) satisfies the conditions (GCA1)–(GCA4), then \( \mathcal{H}^* \) is a semiring congruence on \( (R, +, \cdot) \), and \( R/\mathcal{H}^* \cong E^+(R) \) is a b-lattice. Also, notice that \( (\mathcal{H}^*_e, +) \) is an additively cancellative monoid, we obtain that \( R \) is a b-lattice of additively cancellative skew-halfrings.

For any \( e, f \in E^+(R) \) with \( e \leq_+ f \), define mapping
\[
\phi_{e,f} : H^*_e \to H^*_f, \quad a \mapsto a + f.
\]
In the following, we begin to show that \( R = \langle E^+(R), R_e, \phi_{e,f} \rangle \) is a strong b-lattice of the semirings \( R_e, e \in E^+(R) \).

For any \( a, b \in H^*_e \),
\[
(a + b)\phi_{e,f} = a + b + f = a + (f + b + f) = (a + f) + (b + f) = a\phi_{e,f} + b\phi_{e,f}.
\]
Also, since \( af \in H^*_e \cap E^+(R) \), we have \( af = ef \). And then,
\[
(ab)\phi_{e,f} = ab + f = ab + e + f = ab + (e + f) = ab + e + ef + fe + f
\]
\*
\[
= ab + af + fb + f = (a + f)(b + f) = a\phi_{e,f}b\phi_{e,f}.
\]
Hence, \( \phi_{e,f} \) is a semiring morphism.

For any \( a, b \in H^*_e \), if \( a\phi_{e,f} = b\phi_{e,f} \), we have \( a + f = b + f \). Notice that \( (\mathcal{H}^*_e, +) \) is an additively cancellative monoid, we have \( a^0 = b^0 = e \). By condition (GCA4), we have \( a = b \). Thus, \( \phi_{e,f} \) is a semiring monomorphism.

Moreover, we can check that the monomorphism \( \phi_{e,f} \) satisfies the conditions (1.1)–(1.4) of Definition 1.

(i) \( \phi_{e,e} \) is clearly an identity morphism.

(ii) For any \( e, f, g \in E^+(R) \) with \( e \leq_+ f \leq_+ g \), we have
\[
a\phi_{e,f} \phi_{f,g} = a + f + g = a + g = a\phi_{e,g}.
\]
Hence, \( \phi_{e,f} \phi_{f,g} = \phi_{e,g} \).

(iii) For any \( e, f, g \in E^+(S) \), if \( e + f \leq_+ g \), then for any \( a \in H^*_e, b \in H^*_f \),
\[
a\phi_{e,g} b\phi_{f,g} = (a + g)(b + g) = ab + ag + gb + g = ab + eg + gf + g = (ab)\phi_{e,f,g},
\]
\i.e.,
\[
\phi_{e,g} b\phi_{f,g} = \phi_{ef,g}.
\]
(iv) For any $e, f \in E^+(R)$, $a \in H^*_e$, $b \in H^*_f$, we have
\[
\begin{align*}
a \phi_{e,e+f} + b \phi_{f,e+f} &= (a + e + f) + (b + e + f) = a + f + b + e \\
&= a + b + (e + f) = a + b; \\
abla \phi_{e,e+f} b \phi_{f,e+f} &= (a + e + f)(b + e + f) = ab + a(e + f) + (e + f)b + (e + f) \\
&= ab + (e + f) = (ab) \phi_{e,e+f}. 
\end{align*}
\]
Thus, we have shown that $R$ is a strong b-lattice of additively cancellative skew-halfrings, and then it is a generalized $C$-abundant semiring.

Example 3. Let $T$ be a b-lattice and $M$ an additively cancellative skew-halfring. Construct the direct product of $T$ and $R$, and denote it by $M_T \times R$. Then, we can check that $E^+(R) = T \times \{e_M\}$, where $e_M$ is the identity element of $(M, +)$. We can also check that $(R, +, \cdot)$ is a semiring which satisfies the conditions (GCA1)-(GCA4). Thus, by Theorem 7, $(R, +, \cdot)$ is really a generalized $C$-abundant semiring.

Theorem 8. A semiring $R$ is a generalized $C$-abundant semiring if and only if it is a subdirect product of a b-lattice and an additively cancellative skew-halfring.

Proof. (⇒) By Theorem 3 and its dual, the sufficiency is clear.

(⇒) Assume that $R$ is a generalized $C$-rpp semiring, we will show that it is a subdirect product of a b-lattice and an additively cancellative skew-halfring by the following steps.

Firstly, from Proposition 2, $H^*$ is a semilattice congruence on $(R, +)$ and a semiring congruence on $R$. Also, since $R$ is a generalized $C$-a semiring, we have $a \theta a = a$ for any $a \in R$, and then $a = a \theta a$. Hence, $R/H^*$ is an idempotent semiring with the semilattice additive reduct and band multiplicative reduct, i.e., $R/H^*$ is a b-lattice.

Secondly, define a binary relation
\[
\theta = \{(a, b) \mid (\exists e \in E^+(R))a + e = b + e\}.
\]
It can be easily seen that $\theta$ is an equivalence relation on $(R, +, \cdot)$. Moreover, $\theta$ is the minimum additively cancellative skew-halfring congruence on $(R, +, \cdot)$. In fact, by the Proposition 1.7 in [18] and its dual, $\theta$ is a minimum cancellative monoid congruence on the additive reduct $(R, +)$. Also, if $a \theta b$ for some $a, b \in R$, there exists $e \in E^+(R)$ such that $a + e = b + e$. Now, for any $c \in R$, we have
\[
a c + e c = b c + e c, c a + e c = c b + e c.
\]
Notice that $ce, ec \in E^+(R)$, we immediately obtain that

$$ac\theta bc, ca\theta cb.$$ 

Thus, we have shown that $\theta$ is the minimum additively cancellative skew-halfring congruence on $(R, +, \cdot)$. This also shows that $R/\theta$ is an additively cancellative skew-halfring.

Finally, define a mapping

$$\Phi : R \rightarrow R/\theta \times R/\mathcal{H}^*, \quad a \mapsto (a\theta, a\mathcal{H}^*).$$

It is a routine way to check that $R$ can be embed into $R/\theta \times R/\mathcal{H}^*$, and the projection mapping from $R$ into each of its components is surjective. Consequently, $R$ is a subdirect product of a b-lattice and an additively cancellative skew-halfring. \(\square\)

**Remark 1.** From Theorem 8, we can see that the class of generalized $C$-abundant semirings is actually a general extension of the class of generalized Clifford semirings studied in [29].

At the end of this section, we will study $C$-abundant semirings.

**Definition 5.** A semiring $R$ is said to be a $C$-abundant semiring if it is a strong distributive lattice of additively cancellative skew-halfrings.

Some characterizations of such semirings are also given below.

**Theorem 9.** If $R$ is a $C$-abundant semiring, then the following conditions hold:

(CA1) $(R, +)$ is a $C$-abundant semigroup;
(CA2) $(E^+(R), +, \cdot)$ is a distributive lattice;
(CA3) for any $a, b \in R, (ab)^0 + a^0b^0 = a^0b^0$;
(CA4) if $a^0 = b^0$ and $a + e = b + e$ for $a, b \in S$ and some $e \in E^+(R)$, then $a = b$.

Conversely, if a semiring $R$ satisfies the conditions (CA1)–(CA4), then it is a $C$-abundant semiring.

**Proof.** ($\Rightarrow$) By Definition 4 and Definition 5, a $C$-a semiring is clearly a generalized $C$-abundant semiring. Thus, by Theorem 6, condition (CA1), (CA3), (CA4) hold. We only need to prove that condition (CA2) holds.

Assume that $R$ is a $C$-abundant semiring. Then it is a strong distributive lattice of additively cancellative skew-halfrings, say $R = \langle D, R_\alpha, \phi_{\alpha, \beta} \rangle$, where each $R_\alpha$ is an additively cancellative skew-halfring in which the additive identity is denoted by $0_\alpha$ and $D$ is a distributive lattice. Notice that $E^+(R) = \{0_\alpha \mid \alpha \in \ldots$
\[ D \cong D \], we immediately obtain that \( (E^+(R), +, \cdot) \) is also a distributive lattice. (CA2) holds.

(\( \Leftarrow \)) Assume that the semiring \( R \) satisfies the conditions (CA1)–(CA4), then by Theorem 7, it is clearly a generalized \( C \)-abundant semiring. Also, note that (CA2) holds, by analogy with the discussions of Theorem 1, together with Definition 5, \( R \) is a \( C \)-abundant semiring.

\[ \square \]

**Example 4.** Let \( D \) be a distributive lattice and \( M \) an additively cancellative skew-halfring. Construct the direct product of \( D \) and \( M \), and denote it by \( R \), i.e., \( R = D \times M \). Then, \( E^+(R) = D \times \{ e_M \} \), where \( e_M \) is the identity element of \( (M, +) \). We can also check that \( (R, +, \cdot) \) is a semiring which satisfies the conditions (CA1)–(CA2). Thus, by Theorem 9, \( (R, +, \cdot) \) is a \( C \)-abundant semiring.

By analogy with the discussions of the subdirect decompositions of the generalized \( C \)-a semirings, we will have the following theorem.

**Theorem 10.** A semiring \( R \) is a \( C \)-abundant semiring if and only if it is a subdirect product of a distributive lattice and an additively cancellative skew-halfring.

**Remark 2.** From Theorem 10, we can see that the class of \( C \)-abundant semirings is actually a general extension of the one of Clifford semirings studied in [7] and [29].

**References**

On several classes of additively non-regular $C$-semirings


