(Para)quaternionic geometry, harmonic forms, and stochastical relaxation

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This paper is dedicated to Professor Lajos Tamássy on the occasion of his ninetieth birthday

Abstract. Both quaternionic and para-quaternionic geometry are important when studying harmonic forms and stochastical relaxation with the help of Fokker–Planck-type or Oguchi-type parabolic equations. In a recent paper the first-named author and H. M. Polatoglou (2012) have shown that the five-dimensional case is the simplest case that the use of para-quaternions is more convenient than the use of quaternions. Now we discuss that case in some detail.

1. Introduction and preliminaries

Quaternionic geometry was studied e.g. in [1], [8], [18]–[22], including the twistor aspect; para-quaternionic geometry was investigated e.g. in [27]–[29], [7], [17]. The initial difference is due to the replacement of matrix units 1, iσ_1, iσ_2, iσ_3 of the usual quaternions, where

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

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are generators of the Pauli algebra, by the units
\[ 1, \tilde{i} = i\sigma_2, \tilde{j} = \sigma_1, \tilde{k} = \sigma_3 \]
of para-quaternions, so that our \( \tilde{i} \tilde{j}, \) and \( \tilde{k} \) mean \( j, (1/i)i, \) and \( (1/i)k \) in [8], respectively. This is due to our definition of the real Clifford algebra \( \tilde{H} \) of para-quaternions as generated by 1 and imaginary units \( \tilde{i}, \tilde{j}, \tilde{k} \) satisfying
\[ -\tilde{i}^2 = \tilde{j}^2 = \tilde{k}^2 = 1, \quad \tilde{i}\tilde{j} = -\tilde{j}\tilde{i} = \tilde{k}. \] (1)

For a para-quaternionic structure the left module structure is defined up to conjugation in \( \tilde{H} \).

In a more general setting, let \( V \) be a real vector space. A complex structure on \( V^{2n} \) is an endomorphism \( J \in \text{End}(V) \) such that \( J^2 = -\text{Id} \). A hypercomplex structure \( H \) on \( V^{4n} \) is a triple \( (J_a) = (J_1, J_2, J_3) \) of anticommuting complex structures on \( V \) satisfying \( J_1J_2 = J_3 \); it defines on \( V \) the structure of left vector space over quaternions \( H = \text{span}_R \{1, i, j, k\} \) such that multiplications by \( i, j \) and \( k \) are given by \( J_1, J_2 \) and \( J_3 \). A quaternionic structure on \( V^{4n} \) is the 3-dimensional subspace \( Q \subset \text{End}(V) \) spanned by a hypercomplex structure \( H \), i.e. \( Q = \text{span}_R \{J_1, J_2, J_3\} \).

A triple \( \tilde{J}_1, \tilde{J}_2, \tilde{J}_3 \) of anticommuting endomorphisms of \( V \) satisfying the relations
\[ -\tilde{J}_1^2 = \tilde{J}_2^2 = \tilde{J}_3^2 = \text{Id}, \quad \tilde{J}_1\tilde{J}_2 = \tilde{J}_3 \]
is called a para-hypercomplex structure on \( V \). Observe that \( (\tilde{J}_1 \text{ is a complex structure and) } \tilde{J}_2 \text{ and } \tilde{J}_3 \text{ are para-complex structures on } V \). A Lie subalgebra \( \tilde{Q} \subset \text{gl}(V) \) is called a para-quaternionic structure on \( V \) if there exists a basis \( \tilde{J}_1, \tilde{J}_2, \tilde{J}_3 \) satisfying the above relations. A para-hypercomplex structure \( (\tilde{J}_1, \tilde{J}_2, \tilde{J}_3) \) defines on \( V \) the structure of a left module over the Clifford algebra generated by unity 1 and generators \( \tilde{i}, \tilde{j}, \tilde{k} \) satisfying (1).

The Hurwitz twistors are deduced from quaternions and Clifford structures as follows. Let \( \mathbb{C}^4(I_2, I_2) \) be the 4-dimensional complex space with the indefinite hermitian metric
\[ \kappa = I_{2,2} = \text{diag}^*(I_2, -I_2) = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}, \quad I_2 = 1, \]
and \( \mathbb{R}^5(I_{2,3}) \) – the 5-dimensional real space with the indefinite symmetric metric \( I_{2,3} = \text{diag}(I_2, -I_3) \). Let \( (e_1, \ldots, e_4) \) and \( (e_1, \ldots, e_5) \) denote the corresponding
canonical bases, and \( \circ \) the multiplication acting from \( \mathbb{R}^3(I_{2,3}) \otimes \mathbb{C}^4(I_{2,2}) \) to \( \mathbb{C}^4(I_{2,2}) \). Let us set
\[
\epsilon_\alpha \circ \epsilon_k = C^{4}_{\alpha k} \epsilon_1 + \cdots + C^{4}_{\alpha k} \epsilon_4, \quad C_\alpha + (C^j_\alpha), \quad j = 1, \ldots, 5.
\]
Consider the algebra \( A_{2,3} \) generated by \( \{ C^\#_{\alpha \beta} : \alpha \leq \beta \} \) where \( C^\#_\alpha = \kappa C^*_\alpha K^{-1} \).

An element \( x \in A_{2,3} \) is called Hurwitz twistor \([12], [13]\) whenever \( x \) has the form
\[
x = \sum_{\alpha < \beta} \xi_{\alpha, \beta} C^\#_{\alpha \beta}, \quad \xi_{\alpha, \beta} \in \mathbb{C},
\]
and \( \text{im} \, x^2 = 0 \), where \( x \in A_{2,3} \) is defined in the following manner: \( x \in A_{2,3} \) can be written uniquely as
\[
x = \sum_{k=0}^{4} x_k, \quad x = \sum_{\alpha_1 < \beta_1 < \cdots < \alpha_k < \beta_k} \xi_{\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k} C^\#_{\alpha_1 \beta_1} \cdots C^\#_{\alpha_k \beta_k},
\]
with \( x_0 = \xi_0 I_4 \) for \( k = 0 \). We set \( \text{im} \, x := x - x_0 \) and denote the collection of Hurwitz twistors by \( H \):
\[
H = \left\{ x = \sum_{\alpha < \beta} \xi_{\alpha, \beta} C^\#_{\alpha \beta} : \text{im} \, x^2 = 0 \right\}.
\]
Traditionally, the 5-dimensional space-time is \( \mathbb{R}^5(I_{1,4}) \); when speaking on Hurwitz twistors, it seems convenient to associate them with \( \mathbb{R}^5(I_{3,2}) \) instead of \( \mathbb{R}^5(I_{2,3}) \).

It appears that the expression (2) is an element of \( H \), if and only if the following \( \binom{5}{4} \) equations hold:
\[
\begin{align*}
\xi_{12} \xi_{34} - \xi_{13} \xi_{24} + \xi_{14} \xi_{23} &= 0, \\
\xi_{12} \xi_{35} - \xi_{13} \xi_{25} + \xi_{14} \xi_{24} &= 0, \\
\xi_{12} \xi_{45} - \xi_{14} \xi_{25} + \xi_{15} \xi_{24} &= 0, \\
\xi_{13} \xi_{45} - \xi_{14} \xi_{35} + \xi_{15} \xi_{34} &= 0, \\
\xi_{23} \xi_{45} - \xi_{24} \xi_{35} + \xi_{25} \xi_{34} &= 0.
\end{align*}
\]

In analogous way the anti-objects, called anti-Hurwitz twistors, correspond to \( \mathbb{R}^5(1, 4) \) and are determined by \( \binom{5}{3} \) similar equations as well; we denote the collection of those anti-objects by \( aH \). Still in analogy we consider \( \mathbb{C}^{16}(I_{8,8}) \) and \( \mathbb{R}^8(8, 1) \) replacing it by \( \mathbb{R}^8(1, 8) \) which leads to pseudotwistors \([15]\):
\[
p = \left\{ x = \sum_{\alpha < \beta < \gamma} \xi_{\alpha, \beta} C^\#_{\alpha \beta} C_{\beta} : \text{im} \, x^2 = 0 \right\}.
\]
determined by \( \binom{13}{4} = 126 \) algebraic equations; we denote the collection of corresponding anti-objects by \( \mathcal{A} \). We may also consider \( \mathbb{C}^{64}(I_{32,32}) \) and \( \mathbb{R}^{13}(6,7) \) replaced by \( \mathbb{R}^{13}(7,6) \) which leads to bitwistors determined by \( \binom{13}{4} = 715 \) algebraic equations; we denote their collection by \( \mathcal{B} \) and the collection of their anti-objects – by \( \mathcal{AB} \). The above leads to the so-called Cartan-like triality [6].

Figure 1. Double Cartan-like triality of Hurwitz twistors, pseudotwistors, and bitwistors.

2. Some relationship with traditional harmonicity and holomorphy

Before we start to use quaternions or para-quaternions for investigating parabolic equations responsible for relaxation, we recall some known results on relations with traditional harmonicity and holomorphy.

2.1. Relationship with harmonic forms. Let \( Z_{A}^{(n)}(U) \) be the space of real-analytic solutions of the structure spinor equations (of spin \( \frac{1}{2} n \)) on an open set \( U \subset \mathbb{C}^{2k} \), \( k = 1, 2 \). Then [14] they can be written as harmonic forms, i.e., there exists a one-to-one correspondence between spinor solutions and harmonic forms.
with respect to:

the \((1,1)\)-metric

\[ ds^2 := dz^1d\bar{z}^1 - dz^2d\bar{z}^2 \text{ for } k = 1 \] (Hurwitz twistors);

the \((0,4)\)-metric

\[ ds^2 := -dz^1d\bar{z}^1 - dz^2d\bar{z}^2 - dz^3d\bar{z}^3 - dz^4d\bar{z}^4 \text{ for } k = 2 \] (pseudotwistors).

This correspondence can be expressed as:

\[ Z^{(n)}(U) \simeq H^1(U, \mathbb{C}^{2k-1(n-1)}) \text{ for } k = 1, 2, \]

where

\[ H^1(U, \mathbb{C}^{2k-1(n-1)}) = \{ \phi \in \Gamma^{1,0}(U, \mathbb{C}^{2k-1(n-1)}): \partial\phi = 0 \text{ and } \vartheta\phi = 0 \} \]

and \(\vartheta\) is the formally adjoint operator of \(\partial\) with respect to the indefinite fibre \((2^{k-1},0)\)-metric

\[ d\rho^2 := d\zeta^1d\bar{\zeta}^1 + d\zeta^2d\bar{\zeta}^2 + \ldots + d\zeta^{2k-1}d\bar{\zeta}^{2k-1}. \]

2.2. Relationship with the one-dimensional Dolbeault cohomology group. Set

\[ \mathcal{P}^1 := \{ L_1^1 : L_1^1 \subset \mathbb{C}^4, \text{ linear subspace, } \dim L_1^1 = 1 \} \simeq \mathbb{P}^3(\mathbb{C}), \]

\[ \mathcal{U}^1 := \{ L_2^1 : L_2^1 \subset \mathbb{C}^4, \text{ linear subspace, } \dim L_2^1 = 2 \} \simeq G(2,4), \]

\[ \mathcal{P}^2 := \{ L_1^2 : L_1^2 \subset \mathbb{C}^8, \text{ linear subspace, } \dim L_1^2 = 1 \} \simeq \mathbb{P}^7(\mathbb{C}), \]

\[ \mathcal{U}^2 := \{ L_2^2 : L_2^2 \subset \mathbb{C}^8, \text{ linear subspace, } \dim L_2^2 = 2 \} \simeq G(2,8), \]

where \(\mathbb{P}^3(\mathbb{C}), \mathbb{P}^7(\mathbb{C}), G(2,4), G(2,8)\) are the corresponding complex projective and Grassmannian spaces, respectively. Then we have the following correspondences:

\[
\begin{array}{cccc}
\mathbf{H} & \mathbf{P} & \mathbf{ab} \\
\mu_1 \swarrow & \triangledown \nu_1, & \mu_2 \swarrow & \triangledown \nu_2, & \mu_2 \swarrow & \triangledown \nu_2. \\
\mathcal{P}^1 & \mathcal{U}^1 & \mathcal{P}^2 & \mathcal{U}^2 & \mathcal{P}^2 & \mathcal{U}^2
\end{array}
\] (4)

Let \(Z^{(n)}_H(U_k)\) be the space of holomorphic solutions of the structure spinor equations (of spin \(\frac{1}{2}n\)) on an open set \(U_k\), whereas \(\mu_k\) and \(\nu_k\) be the related fibre bundles forming the diagrams (4). We set

\[ U'_k = \nu_k^{-1}(U_k) \text{ and } U''_k = \mu_k \circ \nu_k^{-1}(U_k) \text{ for } k = 1, 2. \]
Then, if every fibre of $\mu_k$ is connected, there exists a one-to-one correspondence [14]:

$$Z^n_k(U_k) \simeq H^1(U''_k, \mathcal{O}(-\alpha_k n - \beta_k)),$$

where $H^1$ denotes the one-dimensional Dolbeault cohomology group,

$$\mathcal{O}(-\alpha_k n - \beta_k) = \mathcal{O}([\epsilon]^{-\alpha_k n - \beta_k}),$$

$[\epsilon]$ being the canonical effective divisor of $\mathbb{P}^3(\mathbb{C})$, while $\alpha_k$ and $\beta_k$ are positive integers. Moreover,

$$\alpha_1 = 1, \beta_1 = 2; \quad \beta_2 \geq 2.$$

### 2.3. Relationship with traditional holomorphy.

Consider the holomorphic embeddings

$$\mathbb{C}^2 \cong \mathbb{R}^4 \xrightarrow{\iota} G(2,4), \quad \mathbb{R}^4 \ni x \mapsto \sum_{\alpha=1}^{3} x^{\alpha} S_{\alpha} + x^{4} I_{4}, \quad (5)$$

$$\mathbb{C}^4 \cong \mathbb{R}^8 \xrightarrow{\iota} G(8,16), \quad \mathbb{R}^8 \ni x \mapsto \sum_{\alpha=1}^{7} x^{\alpha} S_{\alpha} + x^{8} I_{8}, \quad (6)$$

where $G(\tau, \nu)$ stands for a $\tau$-dimensional Grassmannian submanifold, while $S_{\alpha}$ and $I_4$ or $I_8$ are generators of the corresponding algebra, proposed explicitly first in [13], so that they are real parts of holomorphic mappings in the classical sense. The result we are going to quote was first published without specification of the quaternionic or para-quaternionic dependence in [14], [15] and with specifying this dependence – in [7]. In the case $(\sigma - 1, \tau) = (0,4)$ resp. $(0,8)$ we are interested, it states that there exists a complex structure $I = I[\iota(\sigma - 1, \tau)]$ on the holomorphic embedding $(5)$ resp. $(6)$ with properties

$$\iota(0,4) = \iota(0,4)(\mathbb{H}), \text{ resp. } \iota(0,8) = \iota(0,8)(\mathbb{H}) \quad (7)$$

and the each embedding concerned is the real part of a holomorphic mapping in the classical sense.

We introduce seven $2 \times 2$-complex matrices which we call atoms:

$$A_0 = A_0(\mathbb{H}) = \begin{pmatrix} u & \hat{v} \\ -\hat{v} & u \end{pmatrix}, \quad A_1 = \begin{pmatrix} \hat{u} & \hat{v} \\ v & -u \end{pmatrix}, \quad A_2 = \begin{pmatrix} -u & -\hat{v} \\ -\hat{v} & \hat{u} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}, \quad A_4 = \begin{pmatrix} \hat{w} & 0 \\ 0 & \hat{w} \end{pmatrix}.$$
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\[ A_5 = A_5(\tilde{H}) \begin{pmatrix} t_{\tilde{H}} & 0 \\ 0 & t_{\tilde{H}} \end{pmatrix}, \quad A_6 = A_6(\tilde{H}) = \begin{pmatrix} t_{\tilde{H}} & 0 \\ 0 & t_{\tilde{H}} \end{pmatrix}, \]

where \( u \in A_5 \) and \( v_{\tilde{H}} \) are given in

\[ u = x_4 + ix_3 \in \mathbb{C}, \quad v_{\tilde{H}} = x_2 + ix_1 \in \mathbb{C}, \quad v_{\tilde{H}} = x_1 + ix_2 \in \mathbb{C}, \]

whereas \( u \in A_1, A_2; v, w, \) and \( t_{\tilde{H}} \) are given in

\[ u = x_3 + ix_8 \in \mathbb{C}, \quad v = x_1 + ix_2 \in \mathbb{C}, \quad w = x_4 + ix_5 \in \mathbb{C}, \quad t_{\tilde{H}} = x_7 + ix_6 \in \mathbb{C}, \quad t_{\tilde{H}} = x_6 + ix_7 \in \mathbb{C}. \]

Let \( \zeta_{\tilde{H}} = (u, v_{\tilde{H}}) \in \mathbb{C}^2 \) and \( z_{\tilde{H}} = (u, v, w, t_{\tilde{H}}) \in \mathbb{C}^4 \).

The atomization method allows us to find the following explicit formulae for the embedding in question:

\[ \iota(\zeta_{\tilde{H}}) = A_0 \text{ for } (0, 4), \quad \iota(z_{\tilde{H}}) = \begin{pmatrix} A_1 & A_3 & A_5 & 0 \\ A_4 & A_2 & 0 & A_5 \\ A_6 & 0 & A_2 & -A_3 \\ 0 & A_6 & -A_4 & A_5 \end{pmatrix} \text{ for } (0, 8). \]

2.4. Pseudotwistors of degree 3 vs. those of degree 1. Consider quaternion embeddings, like \( i_A = \text{diag}(A, A, A, A), A = A_3 \), acting from \( G(2, 4) \) to \( G(8, 16) \), instead of (2), pseudotwistors of degree \( k \) with \( x \) as in the second formula in (3), and collections \( J^{(k)} \) of all such \( x \) with \( \text{im } x^2 = 0 \). Consider the following analogues of (4):

\[ J^{(k)} \quad J^{(1)} \]

\[ \sqrt{k \in 1, 3}; \quad \sqrt{J^{(1)}_{\pm A}; \quad J^{(1)}_{\pm A}} \]

\[ J^{(3)}_{\pm A} \quad J^{(1)}_{\pm A}. \]

If \( k = 1 \), then for any quaternion embedding \( i_A \) of some \( V_A \) in \( G(2, 4) \) to \( G(8, 16) \) we have

\[ i_A^* \mathcal{J}^{(1)} = \mathcal{J}^{(1)}_A, \quad i_A^* \mathcal{J}^{(1)}_{\pm A} = \mathcal{J}^{(1)}_{\pm A}, \quad i_A^* \mathcal{J}^{(1)} = \mathcal{J}^{(1)}_A \]

and the diagram in (7) related with \( \mathcal{J}^{(1)}_A \). If \( k = 3 \), we have

\[ \mathcal{J}^{(3)}_{\pm A} \leq \mathcal{J}^{(1)}_{\pm A}, \quad \mathcal{J}^{(3)}_{\pm A} \leq \mathcal{J}^{(2)}_{\pm A}, \quad \mathcal{J}^{(3)}_{\pm A} \leq \mathcal{J}^{(2)}_{\pm A}, \]

where \( A \cup A^c = A^* \). The addends in (9) depend on the quaternionic or para-quaternionic structure according to dependence of \( A \) expressed in terms of \( A_1, \ldots, A_6 \).
3. Stochastical relaxation and the specific role of dimension 5

3.1. Setting of the problem. We consider a modified Oguchi equation [25], [4], [11]

\[
\frac{\partial}{\partial t} \langle s(t, \tilde{\tau}) \rangle = -\frac{1}{\tilde{\tau}} [\langle s(t, \tilde{\tau}) \rangle - \langle s(t, \tilde{\tau}) \rangle_{\text{eq}}]
\]

(10)

where \( \tilde{\tau} \) is the spin-lattice relaxation time related to a spin on R-site, \( R = (x_1, \ldots, x_\tau) \) in \( \mathbb{R}^r, r = 1, 2, \ldots \); \( \tilde{\tau} = x_{\tau+1} \) stands for the stochastic variable responsible for the stochastic behaviour of the lattice, describing thermal oscillations of spin, and \( \langle s(t, \tilde{\tau}) \rangle \) denotes the canonical average of spin; \( \langle s(t, \tilde{\tau}) \rangle_{\text{eq}} \) being its local equilibrium value. \( \langle s(t, \tau) \rangle \) does not depend on the positions in a fixed layer \( x_\tau = \hat{x}_\tau \). Set

\[
\Gamma = \frac{1}{\tilde{\tau}} \left[ 1 - \frac{1}{2} (1 - 4 \langle s \rangle^2) \frac{\hat{x}_\tau J}{k_B T} \right],
\]

(11)

\[
\Lambda = \frac{a^2}{\tilde{\tau}} \frac{1}{2} (1 - 4 \langle s \rangle^2) \frac{\hat{x}_\tau J}{k_B T},
\]

(12)

where \( J \) is the parameter of the theory responsible for the interaction between two neighbouring spins, and \( a \) is the lattice constant. The equation (10) can be transformed to

\[
\frac{\partial}{\partial t} \langle s(t, \tilde{\tau}) \rangle = -\Gamma \langle s(t, \tilde{\tau}) \rangle + \Lambda \left( \sum_{\nu=1}^\tau \frac{\partial^2}{\partial x^2_\nu} - \frac{\hat{\alpha}^2}{a^2} \frac{\partial^2}{\partial \tilde{\tau}^2} \right) \langle s(t, \tilde{\tau}) \rangle,
\]

where \( \Gamma \) and \( \Lambda \) are given by (11) and (12), respectively, while \( \hat{\alpha} \) is the amplitude of stochastic movement. Then the substitution \( \tilde{\tau} = (\hat{\alpha}/a) \tilde{\tau} \) brings the above equation to

\[
\frac{\partial}{\partial t} \langle s(t, \tilde{\tau}) \rangle = -\Gamma \langle s(t, \tilde{\tau}) \rangle + \Lambda \left( \sum_{\nu=1}^\tau \frac{\partial^2}{\partial x^2_\nu} - \frac{\hat{\alpha}^2}{a^2} \frac{\partial^2}{\partial \tilde{\tau}^2} \right) \langle s(t, \tilde{\tau}) \rangle.
\]

(13)

In [4], for solving (13), \( \tau = 2 \) and 3, the quaternionic approach was used systematically.

By (7), the 8- (resp. 4-) dimensional stochastical relaxation problem may be considered in relation with the pseudotwistors in \( \mathbb{P} \) (resp. anti-Hurwitz twistors in \( \mathbb{A} \)) in terms of para-quaternions (resp. quaternions) [11]. By restriction of solution of (13) an analogous conclusion holds for the 7-, 6-, and 5- (resp. 3-, 2-, and 1-) dimensional stochastical relaxation problems as well as for the 8-, 7-, 6-, and 5- (resp. 4-, 3-, 2-, and 1-) dimensional relaxation problems related with (13).
Figure 2. Applicability of para-quaternions (resp. quaternions) and pseudo-twistors (resp. anti-Hurwitz twistors) for 5-, 6-, 7-, 8- (resp. 1-, 2-, 3-, 4-)dimensional relaxation and stochastical relaxation problems.

for \( \langle s(t, \bar{\tau}) \rangle = \langle s(t) \rangle \), \( \langle s(t, \bar{\tau}) \rangle_{\text{e.}} = 0 \). The reasoning is illustrated by Figure 2; the family of solutions to (13) for \( \tau = 8 \) is represented by the point \((1,8)\) on the projection plane \( C_{\sigma,\tau} = \{ (\sigma, \tau) \} \).

It seems interesting to consider, with help of para-quaternions, the simplest proper case of equation (13), i.e. for \( \tau = 5 \). Let

\[
s(t, \bar{\tau}) = s(x, y, z; \xi, \eta, \bar{\tau}; t), \quad (x, y, z, \xi, \eta, \bar{\tau}) \in \mathbb{R}^6 \cong \mathbb{C}^3 \quad t \in \mathbb{R}^+.\]
Then the equation (13) reads
\[
\frac{\partial}{\partial t} \langle s(t, \tilde{\tau}) \rangle = -\Gamma \langle s(t, \tilde{\tau}) \rangle + \Lambda \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial \tilde{\tau}^2} \right) \langle s(t, \tilde{\tau}) \rangle.
\]
(14)

Mathematically, a specific position of this equation is connected with the fact that $\mathbb{C}^2 \simeq \mathbb{R}^4$ in (5) and $\mathbb{C}^4 \simeq \mathbb{R}^8$ in (6). We are going to discuss the equation (13) in detail.

3.2. Setting of a linearization procedure. As in [4], in relation with (13) we concentrate on the Fokker–Planck type [26] equation
\[
\frac{\partial}{\partial t} s(t) = -\Gamma s_*(t) + \Lambda \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial \tilde{\tau}^2} \right) s(t),
\]
\[ (x, y, z, \xi, \eta, \tilde{\tau}, t) \in \mathbb{R}^4, t \in \mathbb{R}^+, \]
(15)

where $s_*(t)$ is an arbitrary admissible function; in particular we may take [3]:
\[
s_* = s_0 \equiv -\int_0^t \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp \left[ \frac{(x-x')^2 + (y-y')^2 + (z-z')^2 + (\xi-\xi')^2 + (\eta-\eta')^2 - (\tilde{\tau}-\tilde{\tau}')^2}{4\Lambda(x', y', z', \xi', \eta', \tilde{\tau}', t')(t-t')} \right] \times \frac{(\Gamma s_0)(x-x', y-y', z-z', \xi-\xi', \eta-\eta', \tilde{\tau}-\tilde{\tau}', t-t')}{2\sqrt{\Lambda(x', y', z', \xi', \eta', \tilde{\tau}', t')(t-t')}} \times dx' dy' dz' d\xi' d\eta' d\tilde{\tau}' dt'.
\]
(16)

According to [23], [24] we need an 8-dimensional vector
\[
s = (s, s_0, s_1, \ldots, s_6) \in \mathbb{R}^8 \simeq \mathbb{C}^4
\]
(17)

and two bases:
\[
(\varepsilon, \varepsilon_0) = (\varepsilon, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_5)
\]
(18)
say, for the space $S$ of variables $x, y, z, \xi, \eta, \tilde{\tau}, t$, and
\[
(e, e_j) = (e, e_0, e_1, \ldots, e_6)
\]
(19)
for the space $V$ of solution (17). Hence, in our case, $(\varepsilon_\alpha)$ consists of complex $8 \times 8$-matrices. They have to satisfy the relations
\[
\varepsilon^2 = -\varepsilon_0, \quad \varepsilon_\alpha^2 = \varepsilon_0, \quad \alpha = 1, \ldots, 4; \quad \varepsilon_5^2 = 0,
\]
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\[ \varepsilon \varepsilon \alpha + \varepsilon_{\alpha} \varepsilon = 0, \quad \varepsilon_{\alpha} \varepsilon_{\beta} + \varepsilon_{\beta} \varepsilon_{\alpha} = 0, \quad \alpha, \beta = 1, \ldots, 5. \]  

(20)

The explicit formulae for (18) in terms of para-quaternions can be deduced from the corresponding formulae obtained for \( \tau = 1 \) and 2 in [4] (after converting quaternions to para-quaternions) with the use of interaction procedure of [18], formulae (1) and (18), expressed already in terms of para-quaternions. The explicit formulae will be published in a subsequent paper. The algebra determined by the basis (18) satisfying the conditions (20) is known as the Clifford-Grassmann algebra \( Cl_{1,4}^{\ast}(C) \).

Then we find analogues of the familiar operators \( \partial_\zeta \) and \( \partial_z \): \( \bar{\partial} \) and \( \partial \) (say):

\[ \partial_\zeta s = Ps - v, \quad \Lambda \partial(Ps) = (\partial/\partial t)s \quad \text{with} \quad \Lambda \partial v = -\Gamma s. \]  

(21)

Here

\[ v \in V, \quad \Lambda \partial(Ps) = \partial(Qs), \]  

(22)

\( Q \) being a polynomical of \( \varepsilon, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_4 \):

\[ Qs = s_1^1 \circ \varepsilon + s_2^1 \circ \varepsilon_0 + s_3^1 \circ \varepsilon_1 + \cdots + s_5^1 \circ \varepsilon_4 + s_4^1 \circ \varepsilon \varepsilon_0 + s_3^2 \circ \varepsilon \varepsilon_4 + s_5^2 \circ \varepsilon_0 \varepsilon_1 + \cdots + s_5^2 \circ \varepsilon_0 \varepsilon_3 + s_3^3 \circ \varepsilon_1 \varepsilon_4 + s_4^3 \circ \varepsilon_2 \varepsilon_3 + s_6^3 \circ \varepsilon_2 \varepsilon_4 + s_7^3 \circ \varepsilon_3 \varepsilon_4, \]  

(23)

where

\[ s_j^k, \quad j = 1, \ldots, 7; \quad k = 1, \ldots, 4, \]  

belong to \( V \) and are \( C^4 \)-valued, (24)

while \( \circ \) is the multiplication \( \circ : V \otimes S \rightarrow V \) in the algebra \( Cl_{1,4}^{\ast}(C) \). Indeed, from (18) and (19) we infer that

\[ \Lambda \partial \bar{\partial} s = \Lambda \partial(Ps) - \Lambda \partial v = \frac{\partial}{\partial t} s - \Gamma s \]

\[ = \Lambda \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial \bar{\tau}^2} \right) s, \]  

so

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial \bar{\tau}^2} \right) s = \partial \bar{\partial} s \]

with \( s \) being in \( V \) and \( C^4 \)-valued. Hence by (18)-(20) we arrive at the formulae

\[ s = s_1^1 \circ \varepsilon + s_2^1 \circ \varepsilon_0 + s_3^1 \circ \varepsilon_1 + \cdots + s_5^1 \circ \varepsilon_4 + s_4^2 \circ \varepsilon \varepsilon_0 + s_7^3 \circ \varepsilon \varepsilon_1 + \cdots \]
$s, s_0, s_1, \ldots, s_6$, being in $V$ and $C^4$-valued.  

3.3. The fundamental solution. In order to find $s$ effectively (in principle we do not need $s, s_0, s_1, \ldots, s_6$) we have to find the system of fundamental solutions of the equation (15) and to be able to compare on both sides of (25) the coordinates with respect to $e$. Here we have to remember that

$$s_j^k, j = 1, \ldots, 7; \quad k = 1, \ldots, 6,$$

are linear combinations of $e, e_0, e_1, \ldots, e_6$, so we need to determine the multiplication scheme for $e \circ e_\alpha, e_0 \circ e_\alpha, e_j \circ e_\alpha$ in the algebra $Cl_{1,4}^0$; the multiplication $\circ$ has to be compatible with the problem of solving the equation (15).

As far as the first question is concerned, we have

$$s = c_1^1 s^x + c_2^1 s^y + c_3^1 s^z + c_4^1 s^\xi + c_5^1 s^\eta + c_6^1 s^x y + c_3^2 s^x z + c_4^2 s^x \xi$$

$$+ c_5^2 s^x \eta + c_6^2 s^y z + c_2^3 s^y \xi + c_3^3 s^y \eta + c_4^3 s^z \xi + c_5^3 s^z \eta + c_6^3 s^z \xi,$$

$$+ c_4^4 s^x t + c_5^4 s^y t + c_6^4 s^z t + c_2^5 s^x \xi t + c_3^5 s^x \eta t + c_4^5 s^y \xi t + c_5^5 s^y \eta t + c_6^5 s^y \xi t,$$

$$+ c_4^6 s^z t + c_5^6 s^z \xi t + c_6^6 s^z \eta t + c_2^7 s^z \xi t + c_3^7 s^z \eta t + c_4^7 s^z \xi t,$$

where $c_j^k, j = 1, \ldots, 7; \quad k = 1, \ldots, 6,$

are complex constants to be determined from the initial conditions

$$s_{R=0}(0) = s_0, \quad s(t) \to \infty \quad \text{as} \quad (R, \tilde{R}, t) \to (\infty, \tilde{t}, 0),$$

$$s(t) \to 0 \quad \text{as} \quad (R, \tilde{R}, t) \to (R_0, \tilde{t}, 0) \quad \text{for some} \quad R_0, \tilde{t}, 0,$$

and the boundary conditions, and

$$s^x, s^y, s^z, s^\xi, s^\eta, s^x y, s^x z, s^x \xi, s^y \eta, s^y \xi, s^y \eta, s^z \xi, s^z \eta, s^z \xi, s^z \eta,$$

$$s^x^2, s^x^y, s^x^z, s^x^\xi, s^x^\eta, s^x^{y z}, s^x^{\xi \eta}, s^y^{x z}, s^y^{x \xi}, s^y^{x \eta}, s^z^{x \xi}, s^{x^2}, s^{x y}, s^{x z}, s^{x \xi}, s^{x \eta}, s^{y^2}, s^{y \xi}, s^{y \eta}, s^{z^2}, s^{z \xi}, s^{z \eta}.$$
are fundamental solutions of the equation (15). Explicitly, we get

\[ s^x = (\partial s_0) e, \quad s^{xy} = (\partial s_0) e e_0, \quad s^{xyz} = (\partial s_0) e e_0 e_2, \]
\[ s^y = (\partial s_0) e_0, \quad s^{yz} = (\partial s_0) e_1, \quad s^{y0} = (\partial s_0) e_0 e_3, \]
\[ s^z = (\partial s_0) e_1, \quad s^{zx} = (\partial s_0) e_2, \quad s^{z0} = (\partial s_0) e_0 e_4, \]
\[ s^\xi = (\partial s_0) e_2, \quad s^{zx^\xi} = (\partial s_0) e_3, \quad s^{z0^\xi} = (\partial s_0) e_1 e_2, \]
\[ s^\eta = (\partial s_0) e_3, \quad s^{z0^\eta} = (\partial s_0) e_1 e_4, \quad s^{z0^0} = (\partial s_0) e_1 e_4, \]
\[ s^{\xi\eta} = (\partial s_0) e_2 e_3, \quad s^{\xi\eta^\xi} = (\partial s_0) e_2 e_4 e_5, \quad s^{\xi\eta^\xi^\eta} = (\partial s_0) e_1 e_2 e_4 e_5, \]
\[ s^{\eta^\eta} = s_0 e_2 + (\partial s_0) e_2 e_5, \quad s^{\eta^\eta^\xi} = s_0 e_3 + (\partial s_0) e_3 e_5, \]
\[ s^{\eta^\eta^\eta} = s_0 e_4 + (\partial s_0) e_4 e_5, \quad s^{\eta^\eta^\eta^\eta} = s_0 e_1 e_2 + (\partial s_0) e_1 e_2 e_4 e_5. \]

(29) \hspace{2cm} (30) \hspace{2cm} (31)

where \( s_0 \) is determined by (16).

As far as second question is concerned, we have the multiplication rules

\[ e_j \odot e_\alpha = (e, e_j \odot e_\alpha) e + \sum_{k=0}^{6} (e_k, e_j \odot e_\alpha) e_k, \quad e_k = (e, e_k) e + \sum_{j=0}^{6} (e_j, e_k) e_j, \]
\[ [(e^j e^k)] = [(e_j, e_k)]^{-1}, \quad j, k = 0, \ldots, 6; \quad \alpha = 0, \ldots, 5, \]

and analogous rules for \((e, e), (e, e_j)\) and \((e_k, e)\).

Here \([(e_j, e_k)]\) denotes the matrix of all elements \((e_j, e_k)\) including \((e, e), (e_j, e)\) and \((e, e_k)\)-the scalar product of \(e_j\) and \(e_k\) etc., where we follow the convention (19).
3.4. Concluding the proof. Formulation of the result. Thanks to Sections 3.2 and 3.3 we have proved

**Theorem 1.** Let \( s = s(t) \) be a solution of the Fokker–Planck type equation (15), where \( \Gamma \) and \( \Lambda \) are \( C^1 \)-scalar functions of \( \mathbf{R} = (x, y, z, \xi, \eta) \in \mathbb{R}^5 \), \( \tau \in \mathbb{R} \), and \( t \in \mathbb{R}^+ \). Consider the space \( S \) of variables \( x, y, z, \xi, \eta, \tilde{\tau}, t \) with a basis (18) of complex \( 8 \times 8 \)-matrices specified the relations (20) and the space \( V \) of solutions (17) forming the algebra \( Cl_{1,4}^*(\mathbb{C}) \), with a basis (19) of the linearized problem

\[
\tilde{\partial} s = P s - v, \quad \Lambda \partial (P s) = (\partial/\partial t) s, \quad \Lambda \partial v = -\Gamma s, \tag{33}
\]

where \( Q \) being a polynomial of \( \varepsilon, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_4 \), where \( s \) is related to \( s \) by (17).

(a) Explicitly, a solution \( s \) of (33) can be expressed by (25) with coefficients as in (26) and (23), and the multiplication \( \circ : V \times S \to V \) in the algebra \( Cl_{1,4}^*(\mathbb{C}) \), which has to satisfy the rules (32). The polynomial \( Q \) is given by (23) and the matrices (18) can be satisfied according to (20) in the terms of para-quaternions.

(b) Then the general solution of the system (33) is a linear combination (27) with complex coefficients of 42 fundamental solutions which, in the case of (16), are explicitly given by the formulae (29)–(31).

Differently speaking, we have an equivalent

**Theorem 2.** (i) Let \( Q \) be a polynomial of \( \varepsilon, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_4 \), given by (23), where (18) are complex \( 8 \times 8 \)-matrices satisfying the relations (20) and forming a basis of the space \( V \) of variables \( x, y, z, \xi, \eta, \tilde{\tau}, t \) with \( P \) as in (34), where \( V \) is the space of solutions (17) forming the algebra \( Cl_{1,4}^*(\mathbb{C}) \) with a basis (19), of the linearized problem (33), corresponding to the Fokker–Planck equation (15), \( \Gamma \) and \( \Lambda \) are \( C^1 \)-scalar functions of \( \mathbf{R} = (x, y, z, \xi, \eta) \in \mathbb{R}^5 \), \( \tau \in \mathbb{R} \), and \( t \in \mathbb{R}^+ \), and a solution \( s = s(t) \) of (15) is related to \( s \) by (17). The multiplication \( \circ : V \times S \to V \) in algebra \( Cl_{1,4}^*(\mathbb{C}) \) has to satisfy the rules (32).

(ii) Then a solution \( s \) of (33) can be expressed as in (25) with coefficients as in (26) and (24), and the matrices (18) can be specified according to (20) in terms of para-quaternions. Moreover, the general solution of the system (33) is a linear combination (27) with complex coefficients of 42 fundamental solutions which, in the case of (16), are given explicitly by the formulae (29)–(31), where \( s_0 \) is determined by (16).

The results obtained have a clear physical significance [2], [5], [9], [11], [16], [17], [25], [26].
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