On the Diophantine equation \((x - 1)^k + x^k + (x + 1)^k = y^n\)

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Abstract. In this paper, we study the Diophantine equation \((x - 1)^k + x^k + (x + 1)^k = y^n\), \(n > 1\), and completely solve it for \(k = 2, 3, 4\).

1. Introduction

The Diophantine equation
\[1^k + 2^k + \cdots + x^k = y^n\]
was studied by LUCAS [9] for \((k, n) = (2, 2)\) and SCHÄFFER [12] for the general situation. There are many results on this equation, for example, see [1], [8], [11]. A generalization is to consider the equation
\[(x + 1)^k + (x + 2)^k + \cdots + (x + m)^k = y^n.
When \(m = 3\), redefining variables, we will consider the equation \((x - 1)^k + x^k + (x + 1)^k = y^n\). CASSELS [6] proved that \(x = 0, 1, 2, 24\) are the only integer solutions to this equation for \(k = 3, n = 2\).

Our result in this paper is the following.

Theorem 1.1. Let \(k = 2, 3, 4\), then the equation
\[(x - 1)^k + x^k + (x + 1)^k = y^n\]
has no integer solutions \((x, y)\) with \(n > 1\), unless \((x, y, k, n) = (1, \pm 3, 3, 2), (2, \pm 6, 3, 2), (24, \pm 204, 3, 2), (\pm 4, \pm 6, 3, 3)\) or \((x, y, k) = (0, 0, 3)\).

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2. Some preliminary results

In this section, we present some lemmas which will help us to prove Theorem 1.1. The first lemma is due to Nagell [10] and the second one is just Theorem 0.1 of [6].

**Lemma 2.1.** If $2 \nmid D$ and $D \geq 3$, then the equation

$$2 + Dx^2 = y^n, \ n > 2$$

has no integer solutions $(x, y, n)$ with $n \nmid h(-2D)$, where $h(-2D)$ is the class number of $\mathbb{Q}(\sqrt{-2D})$.

**Lemma 2.2.** The equation

$$3x(x^2 + 2) = y^2$$

has only the integer solutions $(x, y) = (0, 0), (1, \pm 3), (2, \pm 6), (24, \pm 204)$.

A special case of Theorem 1.5 in [4] which we need in this paper is the following result.

**Lemma 2.3.** Let $p \geq 5$ be a prime, $\alpha \geq 2$ be an integer, then the equation

$$x^p + 3^\alpha y^p = 2z^3$$

has no solutions in coprime integers with $|xy| > 1$.

In order to discuss the small exponents for $k = 4$ we also need the following result.

**Lemma 2.4.** Let $p \geq 3$ be a prime and $(x, y)$ be an integer solution to equation

$$3x^2 - 10 = y^p, \quad \text{(2)}$$

then

$$(\sqrt{3}x + \sqrt{10})^2 = (11 + 2\sqrt{30})^i(a + b\sqrt{30})^p$$

for some integers $a, b, i$ with $-\frac{p-1}{2} \leq i \leq \frac{p-1}{2}$.

**Proof.** From equation (2) one has $(3x^2 - 10)^2 = y^{2p}$, that is

$$(3x^2 + 10)^2 - 30(2x)^2 = (y^2)^p.$$
On the Diophantine equation \((x - 1)^k + x^k + (x + 1)^k = y^n\)

It is easy to see \(2 \nmid x, 5 \nmid x\), then \(\gcd(3x^2 + 10, 2x) = 1\), together with the fact that the class number of \(\mathbb{Q}(\sqrt{30})\) is 2 and \(11 + 2\sqrt{30}\) be the fundamental unit of this field, we have

\[
3x^2 + 10 + 2x\sqrt{30} = (11 + 2\sqrt{30})^i(a + b\sqrt{30})^p
\]

with \(-\frac{p-1}{2} \leq i \leq \frac{p-1}{2}\), that is

\[
(11 + 2\sqrt{30})^i(a + b\sqrt{30})^p = 3x^2 + 10 + 2x\sqrt{30}
\]

\[
= (\sqrt{3}x + \sqrt{10})^2, -\frac{p-1}{2} \leq i \leq \frac{p-1}{2}. \quad \Box
\]

3. The modular approach

Let \(E\) be an elliptic curve over \(\mathbb{Q}\) of conductor \(N\). For a prime of good reduction \(l\) we write \(#E(\mathbb{F}_l)\) for the number of points on \(E\) over the finite field \(\mathbb{F}_l\), and let \(a_l(E) = l + 1 - \#E(\mathbb{F}_l)\). By a newform \(f\), we will always mean a cuspidal newform of weight 2 with respect to \(\Gamma_0(N_0)\) for some positive integer \(N_0\), and \(N_0\) will be called the level of \(f\). Write \(f = q + \sum_{i \geq 2} c_i q^i\) the \(q\)-expansion of \(f\), then \(c_n\) will be called the Fourier coefficients of \(f\). Let \(\mathbb{K} = \mathbb{Q}(c_2, c_3, \ldots)\) be the field obtained by adjoining to \(\mathbb{Q}\) the Fourier coefficients of \(f\), then \(\mathbb{K}\) is a finite and totally real extension of \(\mathbb{Q}\) (see e.g. [7], Chapter 15).

We shall say that the curve \(E\) arises modulo \(p\) from the newform \(f\) (and write \(E \sim_p f\)) if there is a prime ideal \(p\) of \(\mathbb{K}\) above \(p\) such that for all but finitely many primes \(l\) we have \(a_l(E) \equiv c_l \pmod{p}\) (see [7], Definition 15.2.1).

We have the following result, which is just Lemma 2.1 of [5].

**Proposition 3.1.** Assume that \(E \sim_p f\). There exists a prime ideal \(p\) of \(\mathbb{K}\) above \(p\) such that for all primes \(l\),

(i) if \(l \nmid pN N_0\) then \(a_l(E) \equiv c_l \pmod{p}\),

(ii) if \(l \mid N\) but \(l \nmid p N_0\) then \(\pm(l + 1) \equiv c_l \pmod{p}\).

Moreover, if \(f\) is rational, then the above can be relaxed slightly as follows: for all primes \(l\),

(i) if \(l \nmid N N_0\) then \(a_l(E) \equiv c_l \pmod{p}\),

(ii) if \(l \mid N\) but \(l \nmid N_0\) then \(\pm(l + 1) \equiv c_l \pmod{p}\).
Proof of Theorem 1.1

Without loss of generality, we assume \( n = p \) and \( p \) is a prime. Expanding the left hand side of equation (1), one has

(i) \( 3x^2 + 2 = y^p \) when \( k = 2 \);

(ii) \( 3x(x^2 + 2) = y^p \) when \( k = 3 \);

(iii) \( 3x^4 + 12x^2 + 2 = y^p \) when \( k = 4 \).

We will discuss them separately.

(1) \( k = 2 \)

Applying Lemma 2.1, we conclude that there are no integer solutions for \( p \geq 3 \) since \( h(-6) = 2 \). When \( p = 2 \), the equation \( 3x^2 + 2 = y^2 \) modulo 3 yields a contradiction.

(2) \( k = 3 \)

From the result of Cassels, that is Lemma 2.2, one has \( (x, y) = (0, 0), (1, \pm 3), (2, \pm 6), (24, \pm 204) \) for \( p = 2 \).

When \( p \geq 3 \) we obtain \( (x, y) = (0, 0) \) if \( xy = 0 \). Now we assume \( xy \neq 0 \).

Since \( \gcd(x, x^2 + 2) = \gcd(x, 2) \in \{1, 2\} \), then equation \( 3x(x^2 + 2) = y^p \) (3) implies one of following cases:

(a) \( x = 3^{p-1}u^p, x^2 + 2 = v^p, 2 \nmid v; \)

(b) \( x = u^p, x^2 + 2 = 3^{p-1}v^p, 2 \nmid v; \)

(c) \( x = 2^{p-1} \times 3^{p-1}u^p, x^2 + 2 = 2v^p; \)

(d) \( x = 2^{p-1}u^p, x^2 + 2 = 2 \times 3^{p-1}v^p. \)

Firstly, in case (a), we can write equation (3) as \( 3^p(-3u^2)^p + v^p = 2 \) and find it has no integer solutions when \( p \geq 5 \) by Lemma 2.3. When \( p = 3 \), one has \( (3^2u^3)^2 = v^3 - 2 \), modulo 9 yields a contradiction.

In case (b), equation (3) turns into \( (-u^2)^p + 3^{p-1}v^p = 2 \) and applying Lemma 2.3 we know it has no integer solutions when \( p \geq 5 \). The left equation for \( p = 3 \) can be written as \( u^6 + 2 = 9v^3 \), and no integer solutions exists since \( u^6 + 2 \equiv 2, 3 \pmod{9} \).

In case (c), equation (3) becomes \( v^p - 2^{p-3} \times 3^{2p-2}u^{2p} = 1 \), applying Theorem 1.1 of [2], we find that the equation has no nonzero integer solutions \((u, v)\) for \( p \geq 3 \).

Finally, in case (d), one has \( 3^{p-1}v^p - 2^{2p-3}u^{2p} = 1 \), also from Theorem 1.1 of [2], we know \( (u, v, p) = (\pm 4, 1, 3) \), which yields \( (x, y) = (\pm 4, \pm 6) \).

(3) \( k = 4 \)
On the Diophantine equation \((x - 1)^k + x^k + (x + 1)^k = y^n\)

From the equation \(3(x^2 + 2)^2 - 10 = y^2\) we know \(2 \nmid x\), then \(3(x^2 + 2)^2 - 10 \equiv \pm 3 \not\equiv y^2 \pmod{10}\), that is there are no integer solutions for \(p = 2\). When \(p = 3\), one has \(3(x^2 + 2)^2 - 10 = y^3\), that is \((9x^2 + 18)^2 - 270 = (3y)^3\). Applying Magma to calculate the integer points on the elliptic curve \(y^2 = x^3 - 270\), we conclude that it has no integer solutions in this case.

We proceed to prove the equation
\[
3x^4 + 12x^2 + 2 = y^p
\]
has no integer solutions for prime \(p \geq 11\). The remaining cases \(p = 5, 7\) will be treated at the end of the paper.

Let \(u = x^2 + 2, v = y\), and write equation (4) as
\[
3u^2 - 10 = v^p.
\]
It is easy to see \(\gcd(u, v) = 1\) and \(uv \neq 0\). Suppose \(p \geq 7\). To a possible solution \((u, v)\), we associate the Frey curve (see [3])
\[
E_u : Y^2 = X^3 + 6ux^2 + 30X,
\]
with conductor \(N = 2^6 \times 3^2 \times \text{rad}(10v) = 2^7 \times 3^2 \times 5 \times \text{rad}\{2, 5\}(v)\)
where \(\text{rad}\{2, 5\}(v) = \prod_{p|v, p \neq 2, 5} p\).

Then, by the result of Bennett and Skinner [3], there is a newform of level \(N(E_u)_p = 2^7 \times 3^2 \times 5 = 5760\) such that \(E_u \sim_p f\).

Let \(l\) be a prime and \(u \equiv r \pmod{l}\). Since \(u = x^2 + 2\), one has the following table:

<table>
<thead>
<tr>
<th>(l)</th>
<th>(r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>2, 3, 4, 6</td>
</tr>
<tr>
<td>11</td>
<td>0, 2, 3, 5, 6, 7</td>
</tr>
<tr>
<td>13</td>
<td>1, 2, 3, 5, 6, 11, 12</td>
</tr>
<tr>
<td>17</td>
<td>0, 1, 2, 3, 4, 6, 10, 11, 15</td>
</tr>
<tr>
<td>19</td>
<td>0, 2, 3, 6, 7, 8, 9, 11, 13, 18</td>
</tr>
</tbody>
</table>

Recall the definition of \(a_l\) and \(c_l\) in Section 3, that is \(a_l = a_l(E) = l + 1 - \#E(\mathbb{F}_l)\), and \(c_l = c_l(f)\) the Fourier coefficient of \(f\). Therefore, calculating by Pari we obtain

(i) \(7|N\) or \(a_7(E_u) \in \{0, -4\}\); (ii) \(a_{11}(E_u) \in \{0, \pm 2, -4, \pm 6\}\); (iii) \(13|N\) or \(a_{13}(E_u) \in \{\pm 2, -6\}\); (iv) \(17|N\) or \(a_{17}(E_u) \in \{2, \pm 6\}\); (v) \(a_{19}(E_u) \in \{\pm 6\}\).

For rational newforms at level 5760 numbered in Stein’s Table [13], we get a bound for \(p\) by Proposition 3.1, that is from \(p|a_l(E_u) - c_l(f)\) when \(l \nmid N\) or \(p|\pm (l + 1) - c_l(f)\) when \(l|N\). We list these bounds in the following table.
For the nonrational newforms $f_{49}, f_{50}, \ldots, f_{64}$, we use $p = l$ or $p|N_{K/Q}(a(E_u) - c_l(f))$ or $p|N_{K/Q}(\pm(l + 1) - c_l(f))$ to bound $p$.

For $f = f_{49}$, one has $c_{13}^2(f) = -20 = 0$, $c_{17}^2(f) = -20 = 0$. Take $l = 13$, then $N_{K/Q}(a(E_u) - c_l(f)) = \pm16$, $N_{K/Q}(\pm(l + 1) - c_l(f)) = 2^4 \times 11$, which implies $p \leq 5$ or $p = 11, 13$. Take $l = 17$, then $N_{K/Q}(a(E_u) - c_l(f)) = \pm16$, $N_{K/Q}(\pm(l + 1) - c_l(f)) = 2^4 \times 19$, which implies $p \leq 5$ or $p = 17, 19$. Combining these two bounds yields $p \leq 5$.

For $f = f_{56}, f_{60}, f_{61}$, take $l = 13, 17$, and for the left 12 nonrational newforms take $l = 7, 13$, then the same argument as $f_{49}$, we get $p \leq 5$.

From the discussion above, we know there is no newform of level 5760 corresponding to $E_u$ when $p \geq 11$. It remains to deal with the prime $p = 5, 7$. We prove that there are no integer solutions to equation

$$3x^2 - 10 = y^p$$

for $p = 5, 7$.

We discuss the case $p = 5$ in detail. By Lemma 2.4 we get

$$(\sqrt{3}x + \sqrt{10})^2 = (11 + 2\sqrt{30})^i(a + b\sqrt{30})^5$$

for some integers $a, b, i$ with $-2 \leq i \leq 2$. Replacing $x$ by $-x$, we only need to consider the cases $0 \leq i \leq 2$.

If $i = 0$, expanding both sides of equation (5) we obtain

$$2x = 5a^4b + 300a^2b^3 + 900b^5,$$

so that $5|x$, an impossibility.

If $i = 1$, equation (5) can be written as

$$(\sqrt{3}x + \sqrt{10})^2 = (11 + 2\sqrt{30})(a + b\sqrt{30})(a + b\sqrt{30})^4,$$

thus

$$(11 + 2\sqrt{30})(a + b\sqrt{30}) = (\sqrt{3}u + \sqrt{10}v)^2$$

for some integers $u, v$. Expanding this equality we get

\[
\begin{cases}
11a + 60b = 3u^2 + 10v^2 \\
2a + 11b = 2uv,
\end{cases}
\]
On the Diophantine equation $(x - 1)^k + x^k + (x + 1)^k = y^n$ that is
\[
\begin{align*}
a &= 33u^2 + 110v^2 - 120uv \\
b &= -6u^2 - 20v^2 + 22uv.
\end{align*}
\]
Substitution into
\[
\sqrt{3}x + \sqrt{10} = (\sqrt{3}u + \sqrt{10}v)(a + b\sqrt{30})^2
\]
yields the Thue equation
\[
-1188u^5 + 10845u^4v - 39600u^3v^2 + 72300u^2v^3 - 66000uv^4 + 24100v^5 = 1.
\]
According to Magma one obtains no integer solutions.

If $i = 2$, we write equation (5) as
\[
(\sqrt{3}x + \sqrt{10})^2 = (11 + 2\sqrt{30})^2(a + b\sqrt{30})(a + b\sqrt{30})^4,
\]
and then
\[
a + b\sqrt{30} = (\sqrt{3}u + \sqrt{10}v)^2
\]
for some integers $u, v$, therefore
\[
\sqrt{3}x + \sqrt{10} = (11 + 2\sqrt{30})(\sqrt{3}u + \sqrt{10}v)^5.
\]
Expanding the right hand side of the equation yields the Thue equation
\[
54u^5 + 495u^4v + 1800u^3v^2 + 3300u^2v^3 + 3000uv^4 + 1100v^5 = 1
\]
and again we find no integer solutions after appealing to Magma.

For the case $p = 7$, the same argument as in case $p = 5$, solving the corresponding Thue equations, we know the equation $3x^2 - 10 = y^7$ has no integer solutions. From the discussion above, this completes the proof of Theorem 1.1. □

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References

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