On generalized metric spaces and their associated Finsler spaces II.
$g$-Landsberg spaces of scalar curvature $K$

By TOSHIO SAKAGUCHI (Yokosuka), HIDEO IZUMI (Fujisawa)
and MAMORU YOSHIDA (Fujisawa)

Dedicated to Professor Lajos Tamássy on his 70th birthday

The present paper is the continuation of “On generalized metric spaces
and their associated Finsler spaces I. Fundamental relations” (Publ. Math.,
Debrecen, Vol. 45 (1994), 187–203). So we follow the notation and terminol-
yogy of the previous paper.

In § 5 we introduce the notion of $g$-$C$-reducible condition and obtain

[A] In a $g$-$C$-reducible $M_n$ space there exists a scalar $\tau$ such that

$$C_{hj} - \frac{F}{n + 1} T_{hj} = \tau h_{hj} \quad \text{(Theorem 5.9)}.$$

[B] In a $g$-$C$-reducible and $g$-Landsberg space $M_n$, for $n > 3$ the Douglas
tensor $D_{h\, ijk}$ vanishes (Theorem 5.13).

[C] In a $g$-$C$-reducible and $g$-Landsberg space $M_n$, for $n > 3$ if the scalar
$G$ vanishes, then the space is a $g$-Berwald space (Theorem 5.17).

In § 6 we consider a space $M_n$ of scalar curvature $K$ and obtain

[D] A $g$-Landsberg space of scalar curvature $K$ is $g$-$C$-reducible (Theo-
rem 6.8).

[E] A $g$-Landsberg space $M_n$ of scalar curvature $K$ is for $n > 3$ project-
ively flat (Theorem 6.9).

[F] In a $g$-Landsberg space of scalar curvature $K$, the curvature tensor
$S_{h\, ijk}$ has the form

$$S_{hijk} = F^{-2} S(h_{hj} h_{ik} - h_{hk} h_{ij}), \quad \text{where } S \text{ is a con-
stant} \quad \text{(Theorem 6.11)}.$$

The purpose of § 7 is to prove

[G] The Finsler space associated with a $g$-Landsberg space of scalar cur-
vature $K$ is semi-$C$-reducible (Theorem 7.1).

[H] If the associated Finsler space of a $g$-Landsberg space of scalar cur-
vature $K$ is $S3$-like, then the space is an $RccM_n$ space (Theorem 7.7).
§5. A $g$-Landsberg space, $g$-$C$-reducible condition and a $g$-Berwald space

In this section we refer to some special $M_n$ spaces.

5.1. A $g$-Landsberg space.

Definition. If the tensor $P^i_{jk}$ vanishes, then the space $M_n$ is called a $g$-Landsberg space.

Theorem 5.1. ([3], Theorems 4.6, 4.7 and 4.8). A space $M_n$ is a $g$-Landsberg space, if and only if any one of the following conditions holds:

\[(a)\] $P^i_{jk} = 0,$ \hspace{1em} \[(b)\] $D^i_{jk} = 0,$ \hspace{1em} \[(c)\] $P^i_{hjk} = 0,$ \hspace{1em} \[(d)\] $C^i_{jk} = 0.$

Remark. In a $g$-Landsberg space, the connection $B\Gamma(G)$ is $h$-metrical, that is, $g_{ij/k} = g_{ij/k} = 0$ from (1.12)(a). However $g_{ij/k} = 0$ does not mean that the space is a $g$-Landsberg space (cf. (1.12)(b)).

From the Bianchi identity ([3],(2.18)(b))

\[P^i_{jkl} + C^i_{jlk} - C^i_{jkl} = 0,\]

we have the following

Lemma 5.2. In a $g$-Landsberg space we have the following relations:

\[(5.1)\] \hspace{1em} \[(a)\] $C^i_{jil/k} - j|k = 0,$ \hspace{1em} \[(b)\] $C^i_{j/k} - j|k = 0,$ \hspace{1em} \[(c)\] $C^i_{jk} = 0,$ \hspace{1em} \[(d)\] $C^i_{jk} = 0.$

Remark. The condition \[(a)\] is equivalent to the condition $P^i_{jk} = 0.$

Using (1.16) and (1.17) we get

Lemma 5.3. In a $g$-Landsberg space we have

\[(5.2)\] \hspace{1em} \[(a)\] $E^i_{hjk} = 0,$ \hspace{1em} $H^i_{hjk} = K^i_{hjk},$ \hspace{1em} $H^i_{jk} = R^i_{jk},$ \hspace{1em} \[(b)\] $H_{hijk} + H_{ijkh} + H^r_{jk} g_{hi(r)} = 0,$ \hspace{1em} \[(c)\] $F^i_{hjk} = G^i_{hjk} = G^i_{hjk},$ \hspace{1em} $G^i_{jk} = C^i_{jk},$ \hspace{1em} \[(d)\] $G^i_{hijk} + G^i_{ihjk} = g_{hi(k)j},$ \hspace{1em} $G^0_{hjk} = C^0_{hjk}.$

From the Bianchi identity ([3],(2.20)(c))

\[-S^i_{hkj} = P^i_{jkl} + P^i_{rk} C^r_{j} l + S^i_{tr} P^r_{jk} - k|l,\]

we obtain the following
Lemma 5.4. In a $g$-Landsberg space we have $S^i_{jk/l} = 0$.

Definition. A generalized metric space $M_n$ whose associated Finsler space $F^*_n(g)$ is a Landsberg space ($P^*_i{}_{jk} = 0$) is called an $LM_n$ space (abbreviation).

From (2.12)(b) and Proposition 2.3 we infer

Theorem 5.5. A $g$-Landsberg space is an $LM_n$ space if and only if the condition $C_{ij/k} = 0$ holds.

5.2. $g$-$C$-reducible condition ([33]).

Definition. A generalized metric space $M_n$ is called $g$-$C$-reducible if the following condition holds:

\[(5.3) \quad C_{hik} = F^{-1}l_iC_{hk} + \frac{1}{n+1}(C_{hik} + C_{hik} + C_{hik}).\]

Proposition 5.6. The $g$-$C$-reducible condition is characterized by

\[(5.4) \quad g_{hi(k)} = F^{-1}(l_iC_{hk} + l_hC_{ik}) + \frac{2}{n+1}(C_{hik} + C_{hik} + C_{hik}).\]

Proof. As $g_{hi(k)} = C_{hik} + C_{ihk},$ (5.3) gives (5.4). Conversely, if we put $U_{hik} := C_{hik} - \{F^{-1}l_iC_{hk} + \frac{1}{n+1}(C_{hik} + C_{hik} + C_{hik})\} = U_{khi},$ the condition (5.4) can be rewritten $U_{hik} + U_{ikh} = 0.$ Then we see

\[(U_{hik} + U_{ikh}) + (U_{ikh} + U_{khi}) - (U_{khi} + U_{hki}) = 2U_{hik} = 0.\]

Hence this gives (5.3). \[\square\]

Lemma 5.7. In a $g$-$C$-reducible $M_n$ space we have

\[(a) \quad C_{hjk} - j|k = F^{-1}l_jC_{hk} - j|k,\]
\[(b) \quad C_{hj(k)} - j|k = -C_{hjk} - j|k = F^{-1}l_jC_{hk} - j|k, \text{ (cf. (1.7)(d))},\]
\[(c) \quad g_{hj(k)} - j|k = F^{-1}l_jC_{hk} - j|k.\]

We shall give the following

Lemma 5.8. In a $g$-$C$-reducible $M_n$ space we have

\[(5.6) \quad F^p \cdot C_{hij(k)} - j|k = F^{-1}h^*_{ik}C_{hj} + \frac{1}{n+1}(T_{ik}h_{hj} - T_{hj}h_{ik}) - j|k,\]
where \( T_{ik} := F_p \cdot C_{i/(k)} \).

**Proof.** Using the following calculations:

\[
F_p \cdot F_{(k)} = F_p \cdot l_k = F h_k^r l_r = 0,
\]

\[
F_p \cdot l_{i(k)} = h^*_k = C_{ik} + h_{ik},
\]

\[
F_p \cdot C_{i(k)} = F_p \cdot (C_{i/(k)} + C_i \tau_{r k} C_r) = T_{ik} + \frac{F}{n+1} (C^2 h_{ik} + 2C_i C_k),
\]

\[
C_{j(k)} - j|k = (\log \sqrt{g})(j)(k) - j|k = 0, \quad g := \det(g_{ij}),
\]

\[
F_p \cdot h_{hi(k)} = F_p \cdot g_{hi(k)} = \frac{2F}{n+1} (C_h h_{ik} + C_i h_{hk} + C_k h_{hi}),
\]

\[
F_p \cdot h_{hj(k)} - j|k = 0, \quad C^2 := C^i C_i,
\]

and after some further calculations, we obtain (5.6) from (5.3). \( \Box \)

As the identity: \( g_{hi(j)(k)} - j|k = (C_{hij}(k) + C_{ihj}(k)) - j|k = 0 \) holds, we see from (5.6)

\[
F^2 p \cdot g_{hi(j)(k)} - j|k = h_{ik}(C_h - \frac{2F}{n+1} T_{hj}) - h_{hj}(C_i - \frac{2F}{n+1} T_{ik}) - j|k = 0.
\]

Transvecting the above equation with \( h^{ik} \), we have

\[
(n - 1)(C_{hj} - \frac{2F}{n+1} T_{hj}) - (C^i - \frac{2F}{n+1} T^i) h_{hj} = 0.
\]

Hence we have

**Theorem 5.9.** In a \( g \)-C-reducible \( M_n \) space there exists a scalar \( \tau \) such that

\[
(5.7) \quad C_{hj} - \frac{2F}{n+1} T_{hj} = \tau h_{hj}, \quad \tau := \frac{1}{n-1} (C^i - \frac{2F}{n+1} T^i).
\]

To find the curvature tensor \( S_{h \ r \ j \ k} \), we give the relation

\[
S_{h \ r \ j \ k} := g_{ir} S_{h \ r \ j} = g_{ir} (C_{h \ r \ j}(k) + C_{h \ r \ j} C_{m \ r \ k}) - j|k \quad (\text{cf. } [3], (2.12)(c))
\]

\[
= C_{hij(k)} - g_{ir(k)} C_{h \ r \ j} + C_{h \ r \ j} C_{rik} - j|k
\]

\[
= C_{hij(k)} - C_{h \ r \ j} C_{rik} - j|k.
\]

As the tensor \( S_{h \ r \ j \ k} \) is indicatric, we see

\[
F^2 S_{h \ r \ j \ k} = F^2 p \cdot (C_{hij(k)} - C_{h \ r \ j} C_{rik}) - j|k
\]

\[
= h_{ik} C_{hj} + \frac{F}{n+1} (T_{ik} h_{hj} - T_{hj} h_{ik})
\]

\[
- \frac{F^2}{(n+1)^2} (C^2 h_{hj} h_{ik} + C_i C_k h_{hj} + C_k C_j h_{ik}) - j|k.
\]
Substituting $F_{n+1} T_{ik} = \frac{1}{2} (C_{ik} - \tau h_{ik})$, we have

**Proposition 5.10.** In a $g$-$C$-reducible $M_n$ space, the curvature tensor $S_{h^{i}jk}$ has the form

\begin{align*}
(a) \quad S_{hijk} &= \frac{F-2}{2} (M_{hj} h_{ik} + M_{ik} h_{hj}) - j |k, \\
(b) \quad M_{hj} &:= C_{hj} - \frac{F^2}{(n+1)^2} (C^2 h_{hj} + 2 C_h C_j).
\end{align*}

5.3. A $g$-$C$-reducible and $g$-Landsberg space.

Let us consider the case that a space $M_n$ satisfies the conditions $P^{i}_{jk} = 0$ and (5.3). Substituting (5.3) into (5.1)(a), we have

\[ C_{hij/k}^j |k = F^{-1} C_{hj/k} + \frac{1}{n+1} (C_{h/k} h_{ij} + C_{i/k} h_{hj} + C_{j/k} h_{hi}) - j |k = 0. \]

In virtue of (5.1)(b) and (c), we get

\[ C_{h/k} h_{ij} + C_{i/k} h_{hj} - C_{h/j} h_{ik} - C_{i/j} h_{hk} = 0. \]

Transvecting the above equation with $h^{ik}$, we obtain

\[-(n-1) C_{h/j} + C_{i}^{i} h_{hj} = 0, \quad \text{or} \quad -(n-1) G_{hj} + G_{i}^{i} h_{hj} = 0 \quad \text{(cf. (5.2)(c)).}\]

Hence we have from (5.2)(c)

**Proposition 5.11.** In a $g$-$C$-reducible and $g$-Landsberg space, we have

\[ G_{hj} = C_{h/j} = G h_{hj}, \quad G := \frac{G_{i}^{i}}{n-1}. \]

From $S_{h^{i}jk/l} = 0$ and (5.8), we see

\[ M_{hj/l} h_{ik} + M_{ik/l} h_{hj} - M_{hk/l} h_{ij} - M_{ij/l} h_{hk} = 0. \]

Transvecting the above equation with $h^{ik}$, we get

\[(n-3) M_{hj/l} + M_{i/l} h_{hj} = 0. \]

Moreover, transvecting the above equation with $h^{hj}$, we get $2(n-2) M_{i/l} = 0$ and hence we obtain $(n-3) M_{hj/l} = 0$. On the other hand, for $n > 3$ (5.8)(b) and (5.9) lead us to

\[ M_{hj/k} = C_{hj/k} - \frac{2F^2 G}{(n+1)^2} (C_{h} h_{jk} + C_{j} h_{hk} + C_{k} h_{hj}) = 0. \]
Proposition 5.12. In a $g$-$C$-reducible and $g$-Landsberg space $M_n$, for $n > 3$ we have

\begin{align}
(5.10) &

(a) \quad C_{hj/k} = \frac{2F^2G}{(n+1)^2}(C_hh_{jk} + C_jh_{hk} + C_kh_{hj}), \\
&

C^i_{i/k} = \frac{2F^2GC_k}{n+1}, \\

(b) \quad G_h^i_{jk} = C_h^i_{j/k} = \frac{G}{(n+1)^2}\{2Fl_i(C_hh_{jk} + C_jh_{hk} + C_kh_{hj}) \\
&

+ (n+1)(h^i_jh_{jk} + h^i_jh_{kh} + h^i_kh_{hj})\}.
\end{align}

On the other hand, from (2.12)(b) and (2.5)(a) we see

\begin{align}
(5.11) &

P_{*hjk} := g_{*hr}P_{*rjk} = -g_{*hr}A_{jk} = -\frac{1}{2}C_{hj/k}.
\end{align}

Hence from the definition of the tensor $D^*_{h^i_{jk}}$ (cf. (6.3)(b)) we have

\begin{align}
D^*_{h^i_{jk}} &= G_h^i_{jk} + 2F^{-1}l_iP^*_{hjk} - \frac{1}{n+1}(h^i_hG_{jk} + h^i_jG_{kh} + h^i_kG_{hj}) \\
&

= C_h^i_{j/k} - F^{-1}l_iC_{hj/k} - \frac{G}{n+1}(h^i_jh_{jk} + h^i_jh_{kh} + h^i_kh_{hj}) = 0.
\end{align}

From Proposition 6.7 in §6 we obtain

Theorem 5.13. In a $g$-$C$-reducible and $g$-Landsberg space $M_n$, for $n > 3$ the Douglas tensor $D_{h^i_{jk}}$ vanishes.

Remark. In a Finsler geometry ($C_{ij} = 0$) it is known ([23],Theorem 1) that a $C$-reducible Landsberg space is a Berwald space ($G_h^i_{jk} = 0$). This fact implies $D_{h^i_{jk}} = 0$.

5.4. A $g$-Berwald space.

Definition. If the connection parameters $F^i_{jk}$ of $CT(N)$ are independent of $y^i$, that is, $F^i_{jk} = 0$, then the space is called a $g$-Berwald space (an affinely connected space).

Theorem 5.14. ([3], Lemma 4.4, Theorem 4.5). A space $M_n$ is a $g$-Berwald space if any one of the following conditions holds:

\begin{align}
(a) \quad F^i_{jk} = 0, & \quad (b) \quad C^i_{jk/l} = 0, & \quad (c) \quad g_{ij(k)/l} = 0.
\end{align}

Definition. A generalized metric space $M_n$ whose associated Finsler space $F^*_n(g)$ is a Berwald space ($*G^i_{jk} = G^i_{jk} = 0$) is called a $BM_n$ space (abbreviation).
Remark. In a $g$-Berwald space, we have

\[(a) \quad P^i_{jk} = 0 \quad (\text{cf. (1.7)(b)}), \quad (b) \quad G^i_{jkl} = 0 \quad (\text{cf. (1.14)(c)}).\]

Hence we see that a $g$-Berwald space is a $g$-Landsberg space from $(a)$ and a $BM_n$ space from $(b)$.

From (2.16)(c) we see

\[\text{Theorem 5.15. A } BM_n \text{ space is a } g\text{-Berwald space if the condition } C_{ijk}^j = 0 \text{ holds.}\]

Recently S. Bácsó, F. Ilosvay and B. Kis proved the following theorem ([16],Theorem 1): If a Landsberg space $(P^*_{ij} = 0)$ satisfies the condition $D^i_{hkj} = 0$, then the space is a Berwald space $(G^i_{hkj} = 0)$. Hence we have

\[\text{Theorem 5.16. If an } LM_n \text{ space satisfies the condition } D^i_{hkj} = 0, \text{ then the space is a } BM_n \text{ space.}\]

From (5.10)(b) and Theorem 5.14, we have

\[\text{Theorem 5.17. In a } g\text{-C-reducible and } g\text{-Landsberg space } Mn, \text{ for } n > 3 \text{ if the scalar } G \text{ vanishes, then the space is a } g\text{-Berwald space.}\]

§6. A generalized metric space of scalar curvature $K$

As the geodesics in $F^*_n(g)$ are geodesics in $M_n$, the notion of a space of scalar curvature is contained in the geometry of $M_n$. In this section we shall refer to this interesting property about which many results are known in Finsler geometry.

\[\text{Definition. A space } M_n \text{ (} n > 2 \text{) whose associated Finsler space } F^*_n(g) \text{ is of scalar curvature } K \text{ (} K \neq 0 \text{) is called a generalized metric space of scalar curvature } K \text{ and called an } M_n sc \text{ space (abbreviation). If the scalar } K \text{ is constant, we call the space a generalized metric space of constant curvature } K \text{ and call it an } M_n cc \text{ space.}\]

From the above definition an $M_n sc$ space is characterized by

\[H^i_k = F^2K^i_k \quad (\text{cf. [17]}).\]
From (1.10) we have

\[(b)\quad H^i_{jk} = F(Kl_j + \frac{1}{3} K_j)h^i_k - j|k,\]

\[\quad H^i_{jk} = Kg^*_{hj}\delta^i_k + \frac{1}{3}\{h^*_{hj} l^i_k + h^i_h l_j K_k\}
\quad + h^i_k (K_hj + l_h K_j + 2K_h l_j) - j|k,\]

\[(6.1)\]

\[\quad H^i_{hk} = (n - 1)Kg^*_{hj}
\quad + \frac{1}{3}\{(n - 2)(K_hj + l_h K_j) + (2n - 1)K_h l_j\},\]

where \(K_j := FK(j),\quad K_{hj} := Fp \cdot K_{j(h)} = FK_{j(h)} + K_h l_j = K_{jh}.\)

6.1. An \(M_ncc\) space.

It is well known (e.g. [29],[25]) that in a Finsler space of scalar curvature \(K\), the scalar \(K\) is constant if it is independent of \(y\).

From Proposition 1.2 the following is evident.

**Lemma 6.1.** An \(M_nsc\) space is an \(M_ncc\) space if \(K_j = 0\) or \(K_{hj} = 0\) holds.

Moreover we know the following

**Theorem 6.2.** A generalized metric space \(M_n\) reduces to one of constant curvature \(K\) if and only if the tensors \(H^i_{jk}, H^i_{jk}\) and \(H^i_k\) have any one of the following forms:

\[(a)\quad H^i_{jk} = K(y_j \delta^i_k - j|k)\quad (\text{cf. [29],p. 133}),\]

\[(b)\quad H^i_{jk} = A_{hj} \delta^i_k - A_{jk} \delta^i_h - j|k\quad (\text{cf. [35]}),\]

\[(c)\quad H^i_{jk} = \frac{1}{n + 1}(\delta^i_k H_j(h) - \delta^i_h H_{jk} - j|k)\quad (\text{cf. [18],Theorem 2}),\]

\[(d)\quad H^i_k = \frac{1}{n - 1}(H_0 \delta^i_k - H_k y^i)\quad (\text{cf. [36],[37]}),\]

where \(A_{hj} := \frac{1}{n^2 - 1}(nH_{hj} + H_{jh}).\)
Theorem 6.3. (e.g. [10]). A generalized metric space of scalar curvature $K$ reduces to one of constant curvature $K$ if and only if any one of the following conditions holds:

(a) $K_{/0} = 0$,  
(b) $K_{/j} = K_{/j} - F^{-1} P^h_j K_h = 0$ (cf. [26],Theorem 1),  
(c) $H_{hj} = H_{jh}$ or $H_{i/jk} = 0$ or $p \cdot H_{i/jk} = 0$,  
(d) $G_{hj/k - j|k} = 0$ or $H_{hj(k) - j|k} = 0$ (cf. [18],Theorem 1),  
(e) $p \cdot G_{h^i/jk/0} = 0$.

Theorem 6.4. ([30], Theorem 4.3). If a generalized metric space of scalar curvature $K$ satisfies the condition

$$F_{h^i/jk} := H_{h^i/jk} - (g_{hj} L^i_k + L_{hj} \delta^i_k - h^i_h L_{jk} - j|k) = 0 \quad (\text{cf. [21]},$$

$$L_{hj} := \frac{1}{(n-1)(n-2)} \{(n-1)H_{hj} - \frac{1}{2} g^{ik} H_{ik} g_{hj} + l^i_j (H_{ih} - H_{hi})\},$$

then the space is one of constant curvature $K$.

6.2. Projective(geodesic) change in $M_n$.

It is well known that in a metric space, a path (autoparallel curve) is coincident with the geodesic (extremal curve). From Finsler geometry, we know the following results:

Theorem 6.5. ([35],[25]). A generalized metric space $M_n$ is projectively flat if the Weyl tensor $W^i_{jk}$ and the Douglas tensor $D^i_{h^i/jk}$ vanish, where

$$W^i_{jk} := H^i_{jk} + \frac{1}{n+1} \{H^i_{jky^i} + \frac{1}{n-1} (nH_k + H_{k0})\delta^i_j - j|k\},$$

$$D^i_{h^i/jk} := G^i_{h^i/jk} - \frac{1}{n+1} (l^i G_{h^i/jk} + h^i_h G_{jk} + h^i_j G_{kh} + h^i_k G_{hj}),$$

where $G_{h^i/jk} := Fp \cdot G_{jk(h)} = FG_{jk(h)} + l_j G_{kh} + l_k G_{hj} + l_h G_{jk}$.

Theorem 6.6. ([34],[25]). A generalized metric space $M_n$ ($n > 2$) is one of scalar curvature $K$ if and only if the condition $W^i_{jk} = 0$ holds.

We shall show
Proposition 6.7. The following relations hold:

(a) \( W^i_{jk} = 0 \) is equivalent to \( W^{*i}_{jk} = 0 \), where
\[
W^{*i}_{jk} := H^{i}_{jk} - \frac{1}{n-2} \{(H_j - F^{-1}Hl_j)h^i_k - j|k\},
\]
\[
H := \frac{1}{n-1} H^i_i,
\]
\[
(6.3)
\]

(b) \( D^i_{jk} = 0 \) is equivalent to \( p \cdot D^i_{jk} =: D^{*i}_{jk} = 0 \), where
\[
D^{*i}_{jk} = G^i_{jk} + 2F^{-1}l^i_{P^*_{hjk}}
- \frac{1}{n+1} (h^i_h G_{jk} + h^i_j G_{kh} + h^i_k G_{hj}).
\]

Proof. For (a), from (6.1)(a) and (d) we see
\[
H_j = (n-2)FL_j + F^{-1}Hl_j, \quad L_j := Kl_j + \frac{1}{3}K_j.
\]
Hence eliminating \( L_j \) in (6.1)(b), we have
\[
H^{i}_{jk} = \frac{1}{n-2} \{(H_j - F^{-1}Hl_j)h^i_k - j|k\}.
\]
For (b), see [10], Theorem 4.4. □

6.3. A \( g \)-Landsberg space of scalar curvature \( K \).

Now we shall refer to a \( g \)-Landsberg space of scalar curvature \( K \).

First, substituting (6.1)(b) and (c) into (5.2)(b) and after some arrangement, we obtain
\[
3K(C_{hj}g_{ik} + C_{ij}g_{hk} + F_ljg_{hi(k)}) + K_{hj}h_{ik} + K_{ij}h_{hk} + FK_{j}g_{hi(k)}
+ (l_t C_{hj} + l_h C_{ij})K_k + 2l_j(K_{hj}h_{ik} + K_{ij}h_{hk} + K_{k}h_{hi}) - j|k = 0.
\]

Transvecting (6.5) with \( l^j \), we get
\[
3K\{Fg_{hi(k)} - (l_h C_{ik} + l_t C_{hk})\} + 2(K_{hj}h_{ik} + K_{ij}h_{hk} + K_{k}h_{hi}) = 0.
\]
Moreover, on transvecting (6.6) with \( h^{hi} \), we find
\[
3FKC_k + (n+1)K_k = 0, \quad \text{or} \quad K_k = -\frac{3FK}{n+1}C_k.
\]
Using (6.7), we can eliminate \( K_k \) in (6.6). As \( K \neq 0 \), we have
\[
F_{gi(h)} - (l_h C_{ik} + l_t C_{hk}) - \frac{2F}{n+1} (C_{hj}h_{ik} + C_{ij}h_{hk} + C_{k}h_{hi}) = 0.
\]
With Proposition 5.6 and Theorem 5.9 in mind, the above equation gives the following
Theorem 6.8. A $g$-Landsberg space of scalar curvature $K$ is $g$-C-reducible and there exists a scalar $\tau$ such that 
\[ C_{hj} - \frac{2F}{n + 1} T_{hj} = \tau h_{hj}. \]

From Theorems 5.13, 6.5 and 6.6, we have

Theorem 6.9. A $g$-Landsberg space $M_n$ of scalar curvature $K$ is for $n > 3$ projectively flat.

Next, operating $F p \cdot \hat{\partial}_i$ to (6.7), we have

\[ 3F[K_i C_k + K(T_{ik} + \frac{F}{n + 1}(C^2 h_{ik} + 2C_i C_k))] + (n + 1)K_{ik} = 0. \]

Moreover, substituting (6.7) into the above equation, we obtain

\[ (6.8) \quad K_{ik} = -3K\{ \frac{F}{n + 1} T_{ik} + \frac{F^2}{(n + 1)^2}(C^2 h_{ik} - C_i C_k) \}. \]

Operating $p \cdot$ to (6.5), we get

\[ 3F(C_{hj} h_{ik} - C_{ik} h_{hj}) + K_{hj} h_{ik} - K_{ik} h_{hj} + \frac{2F}{n + 1} K_j(C_{hj} h_{ik} + C_i h_{hk} + C_k h_{hi}) - j|k = 0. \]

Substituting $K_k$ in (6.7) and $K_{ik}$ in (6.8) into the above equation, we have ($K \neq 0$)

\[ (C_{hj} - \frac{F}{n + 1} T_{hj} - \frac{F^2}{(n + 1)^2} C_h C_j) h_{ik} - (C_{ik} - \frac{F}{n + 1} T_{ik} - \frac{F^2}{(n + 1)^2} C_i C_k) h_{hj} - j|k = 0. \]

On the other hand, we know $\frac{F}{n + 1} T_{hj} = \frac{1}{2}(C_{hj} - \tau h_{hj})$. Then the above equation leads us to

\[ (C_{hj} - \frac{2F^2}{(n + 1)^2} C_h C_j) h_{ik} - (C_{ik} - \frac{2F^2}{(n + 1)^2} C_i C_k) h_{hj} - j|k = 0. \]

Lastly transvecting the above equation with $h^{ik}$, we obtain

\[ (n - 1)(C_{hj} - \frac{2F^2}{(n + 1)^2} C_h C_j) - (C_{i k} - \frac{2F^2 C^2}{(n + 1)^2} h_{hj} = 0. \]

Thus we have
Theorem 6.10. In a $g$-Landsberg space of scalar curvature $K$, there exists a scalar $\beta$ such that
\begin{equation}
C_{hj} - \frac{2F^2}{(n+1)^2}C_h C_j = \beta h_{hj}, \quad \beta := \frac{1}{n-1}(C^i i - 2F^2C^2).
\end{equation}

Theorem 6.11. In a $g$-Landsberg space of scalar curvature $K$, the curvature tensor $S_{hijk}$ has the form
\begin{equation}
S_{hijk} = F^{-2}S(h_{hj}h_{ik} - h_{hk}h_{ij}) \quad \text{(called $S3$-type, cf. [27])},
\end{equation}
where $S$ is a constant.

Proof. In view of (5.8) and (6.9), we see
\[M_{hj} = C_{hj} - \frac{F^2}{(n+1)^2}(C^2h_{hj} + 2C_h C_j) = (\beta - \frac{F^2C^2}{(n+1)^2})h_{hj}.\]

Putting $M_{hj} =: Sh_{hj}$ we have $S = \beta - \frac{F^2C^2}{(n+1)^2}$, or exactly
\[S = \frac{1}{n-1}(C^i i - 2F^2C^2) - \frac{F^2C^2}{(n+1)^2} = \frac{1}{n^2 - 1}\{(n+1)C^i i - F^2C^2\}.\]

On the other hand, we see
\[S_{/k} = \frac{1}{n^2 - 1}\{(n+1)C^i i/k - 2F^2C^i C_{i/k}\}.\]

Substituting (5.10)(a) and (5.9), we obtain
\[S_{/k} = \frac{1}{n^2 - 1}(2F^2GC_k - 2F^2GC) = 0.\]

Accordingly, by means of the Ricci identity, we see from (6.1)(b)
\[S_{/j/k} - S_{/j/k} = 0 = -H^r_{jk}S_{(r)} = -F(Kl_j + \frac{1}{3}K_j)S_{(k)} - j|k,\]
and transvecting the above equation with $y^j$, we have $S_{(k)} = 0$. Therefore $S_{/k} = 0$ means that the scalar $S$ is a constant. \qed

Theorem 6.12. A $g$-Berwald space of scalar curvature $K$ is an $RccM_n$ space, which satisfies $C_i = 0$ and $C_{hij} = F^{-1}l_{i}C_{hj}$. 
§7. Semi-C-reducibility

In this section we refer to the tensor \( C_{hj} \) which vanishes in a Finsler geometry. The purpose of this section is to prove the following

**Theorem 7.1.** The Finsler space associated with a \( g \)-Landsberg space of scalar curvature \( K \) is semi-C-reducible, that is,

\[
C^*_{hjk} = \frac{p}{n+1} \left( C^*_{h} h^*_j k + C^*_{j} h^*_h k + C^*_{k} h^*_h j \right) + \frac{q}{(C^*)^2} C^*_{h} C^*_{j} C^*_{k},\quad (C^*)^2 := g^{*ij} C^*_{i} C^*_{j},
\]

where \( p + q = 1, \quad p \neq 0, \quad q \neq 0. \)

**Definition.** A Finsler space that satisfies the condition (7.1) (without asterisk mark \( * \)) is called semi-C-reducible.

**Examples.** There are many semi-C-reducible Finsler spaces: e.g.

1° A Finsler space with \( (\alpha, \beta) \)-metric is semi-C-reducible ([24],[28]).

2° If an \( R3 \)-like Finsler space satisfies the condition \( p \cdot P^i_{jk/l} - k|l = 0 \), then the space is semi-C-reducible or satisfies \( F^{-1}P_{hij/0} + FKC_{hij} = 0 \), where \( K \) is some scalar ([20],Proposition 5.5).

3° If an \( R3 \)-like Finsler space satisfies the condition \( *P^i_{jk} = 0 \), then the space is semi-C-reducible or \( S3 \)-like ([38],Theorem 4.3).

4° If a Finsler space with property \( H \) satisfies the condition \( *P^i_{jk} = 0 \) or \( p \cdot P^i_{jk/l} - k|l = 0 \), then the space is semi-C-reducible under some condition ([31],Theorems 4.2 and 4.4).

See Appendix.

7.1. Tensors \( m_i, \ m_{hj} \) and \( m_{hjk} \).

First, in a \( g \)-Landsberg space of scalar curvature \( K \), we recall the following relations:

\[
3C^*_{ijk} = C_{ijk} + C_{jki} + C_{kij} + \frac{1}{2} (C_{ij(k)} + C_{j(k)i} + C_{k(i)j}),
\]

(3.1)

\[
C_{hik} = F^{-1}l_i C_{hk} + \frac{1}{n+1} (C_{hi} h_{ik} + C_{i} h_{hk} + C_{k} h_{hi}),
\]

(5.3)

\[
C_{hj} - \frac{2F}{n+1} T_{hj} = \tau h_{hj},
\]

(5.7)

\[
C_{hj} - \frac{2F^2}{(n+1)^2} C_{h} C_j = \beta h_{hj},
\]

(6.9)

\[
S = \beta - \frac{F^2C^2}{(n+1)^2} \quad (S: \text{constant}).
\]

(6.10)
To save the complicated calculations we introduce new notations:

\[ m_i := \frac{F}{n+1} C_i, \quad m_{hj} := Fp \cdot m_{h(j)} = \frac{F}{n+1} T_{hj}, \]

then we see that the above equations are expressed by

(a) \[ C_{hj} = 2m_k m_j + \beta h_{hj}, \]
(b) \[ C_{hj} = 2m_{hj} + \tau h_{hj}, \]
(c) \[ m_{hj} = m_h m_j + \frac{\beta - \tau}{2} h_{hj}, \]
(d) \[ S = \beta - m^2, \quad m^2 := m^i m_i, \]
(e) \[ m_{hjk} := Fp \cdot C_{hjk} = m_h m_{jk} + m_j m_{hk} + m_k h_{hj}. \]

7.2. Metric tensors in \( F^*_n(g) \).

We see

\[ g^*_{hj} = g_{hj} + C_{hj} = g_{hj} + 2m_k m_j + \beta h_{hj}, \]

\[ h^*_{hj} = (1 + \beta) h_{hj} + 2m_k m_j. \]

We set \( g^{*hk} = g^{hk} + A m^h m^k + B h^{hk} \) with unknown coefficients \( A \) and \( B \), and substituting (7.3) into the definition \( g^{*hk} g^{*hj} = \delta^j_k \), we obtain

\[ B + \beta + B \beta = 0, \quad B = -\frac{\beta}{1 + \beta}, \quad 1 + B = \frac{1}{1 + \beta}; \]

\[ A(1 + \beta + 2m^2) + 2(1 + B) = 0, \]
\[ A = -\frac{2}{a(1 + \beta)}, \quad a := 1 + \beta + 2m^2. \]

Hence we have

**Lemma 7.2.** In a \( g \)-Landsberg space of scalar curvature \( K \) we have

\[ g^{*hk} = g^{hk} - \frac{2}{a(1 + \beta)} m^h m^k - \frac{\beta}{1 + \beta} h^{hk}, \]
\[ h^{*hk} = \frac{1}{1 + \beta} (h^{hk} - \frac{2}{a} m^h m^k). \]

**Remark.** The scalar \( \beta \) cannot satisfy \( 1 + \beta = 0 \). In fact, if \( 1 + \beta = 0 \) holds in (7.3), we have \( h^*_{hj} = 2m_h m_j \). The last equation means that the rank of the matrix \( (h^*_{hj}) \) must be 1. As the rank of the matrix \( (h^*_{hj}) \) is \( n - 1 \), the relation \( 1 + \beta = 0 \) cannot hold for \( n > 2 \). Moreover, the last equation of (7.4) is rewritten as \( Aa(1 + \beta) = -2 \). Accordingly we see that the scalar \( a \) cannot vanish.
7.3. Tensor $C^*_{hjk}$ in $F_n^*(g)$.

Let us carry out the following calculation:

\[
m_{ik} = \mathcal{F}p \cdot (m_{i(k)} - C_i^* r_{km}) = \mathcal{F}p \cdot m_{i(k)} - m_{rk}m^r
\]

\[
= \mathcal{F}p \cdot m_{i(k)} - (m^2 h_{ik} + 2m_{i}m_{k}).
\]

From (7.2)(c) we have

\[
\mathcal{F}p \cdot m_{i(k)} = 3m_{i}m_{k} + \frac{2m^2 + \beta - \tau}{2} h_{ik}.
\]

From (7.2)(d) and (c), we see

\[
\beta_k := \mathcal{F}p \cdot \beta_{(k)} = \mathcal{F}p \cdot (S + m^2)/(k) = 2m^i \mathcal{F}p \cdot m_{i(k)}
\]

\[
= 2m^i m_{ik} = (2m^2 + \beta - \tau)m_k,
\]

\[
\mathcal{F}p \cdot h_{hj(k)} = \mathcal{F}p \cdot g_{hj(k)} = 2m_{hjk}.
\]

Using (7.2)(a) and the above, we shall carry out the following calculation:

\[
\mathcal{F}p \cdot C_{hj(k)} = 2(\mathcal{F}p \cdot m_{h(k)m_j} + m_{h} \mathcal{F}p \cdot m_{j(k)}) + \beta_k h_{hj} + \beta \mathcal{F}p \cdot h_{hj(k)}
\]

\[
= 12m_{h} m_{j} m_{k} + (2m^2 + 3\beta - \tau)m_{hjk}.
\]

As the term $\mathcal{F}p \cdot C_{hj(k)}$ is symmetric in the indices $h, j, k$, (3.1) gives

\[
FC^*_{hjk} = m_{hjk} + \frac{1}{2} \mathcal{F}p \cdot C_{hj(k)}.
\]

Hence we have

**Lemma 7.3.** In a $g$-Landsberg space of scalar curvature $K$ we have

(7.6) \[ FC^*_{hjk} = 6m_{h} m_{j} m_{k} + bm_{hjk}, \quad b := \frac{2m^2 + 3\beta - \tau + 2}{2}. \]

*Remark.* If $b = 0$ holds, then the space $F_n^*(g)$ is called $C2$-like. Hence we assume $b \neq 0$.

7.4. Torsion vectors in $F_n^*(g)$.

Using (7.5) and (7.6) we see

\[
FC^*_j = Fh^* h k C^*_{hjk} = \frac{1}{1 + \beta} \{6m^2 + (n + 1)b - \frac{6m^2(2m^2 + b)}{a}\} m_j.
\]

Here, let us put

\[
FC^*_j = \frac{D}{1 + \beta} m_j, \quad D := (n + 1)b + \frac{6m^2(1 + \beta - b)}{a}.
\]
Moreover from (7.5) and $a - 2m^2 = 1 + \beta$, we find

$$FC^*k = Fh^*hk C^*_h = \frac{D}{(1 + \beta)^2 (1 - \frac{2m^2}{a})} m^k = \frac{D}{a(1 + \beta)} m^k.$$  

Hence we have

**Lemma 7.4.** In a $g$-Landsberg space of scalar curvature $K$ we have

$$FC^*_j = \frac{D}{1 + \beta} m_j, \quad FC^*_k = \frac{D}{a(1 + \beta)} m^k,$$

(7.7)

$$F^2(C^*)^2 = \frac{D^2 m^2}{a(1 + \beta)^2}.$$  

**Remark.** From (7.7) we see that if the scalar $D$ vanishes, then $C^*_j = 0$, which leads to $C^*_h j k = 0$ by Deicke’s Theorem. Hence we assume $D \neq 0$. When the vector $m_i$ vanishes, we see $C^*_h j k = F^{-1} l_j C^*_h k$ and $C^*_h j k = 0$, that is, the space considered reduces to an $RccM_n$ space.

**7.5. Proof of Theorem 7.1.** From (7.3) and (7.7), we see

$$h_{hj} = \frac{1}{1 + \beta} (h^*_h j - 2m_h m_j) = \frac{1}{1 + \beta} (h^*_h j - \frac{2m^2}{a(C^*)^2} C^*_h C^*_j)$$

and

$$m_k h_{hj} = \frac{F}{D} (C^*_k h^*_h j - \frac{2m^2}{a(C^*)^2} C^*_h C^*_j C^*_k),$$

$$m_h m_j m_k = \frac{F}{D} (1 + \beta) \frac{m^2}{a(C^*)^2} C^*_h C^*_j C^*_k.$$  

Hence (7.2)(e) gives

$$m_h j k = \frac{F}{D} (C^*_h h^*_j k + C^*_j h^*_h k + C^*_k h^*_h j - \frac{6m^2}{a(C^*)^2} C^*_h C^*_j C^*_k).$$  

Finally (7.6) is rewritten

$$C^*_h j k = \frac{b}{D} (C^*_h h^*_j k + C^*_j h^*_h k + C^*_k h^*_h j) + \frac{6m^2 (1 + \beta - b)}{Da(C^*)^2} C^*_h C^*_j C^*_k,$$

and we have

$$p = \frac{(n + 1)b}{D}, \quad q = \frac{6m^2 (1 + \beta - b)}{Da}, \quad p + q = 1.$$  

Thus the proof is complete.
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Remark. This theorem means that if a $g$-Landsberg space of scalar curvature $K$ satisfies $m_j = 0$ ($C_j = 0$), then the space reduces to an $RccM_n$ space.

(6.7) tells us the following

**Theorem 7.5.** A $g$-Landsberg space of constant curvature $K$ is an $RccM_n$ space.

Moreover, from (5.9) we see $G_{hj/k} = G_{hj/k} = G_{k}h_{hj}$. From the condition (d) of Theorem 6.3, we have

**Theorem 7.6.** If the scalar $G$ is constant, then the $g$-Landsberg space of scalar curvature $K$ is an $RccM_n$ space.

M. Matsumoto and C. Shibata showed ([24],(1.8)) that the curvature tensor $S_{hijk}$ of a semi-C-reducible Finsler space is expressed by (without asterisk mark *)

\[
\begin{align*}
(a) \quad S_{hijk} &= \frac{F^{-2}}{2}(M_{hj}h_{ik} + M_{ik}h_{hj}) - j|k, \\
(b) \quad M_{hj} &= -\frac{p^2F^2C^2}{(n+1)^2}h_{hj} - \frac{2pF^2}{(n+1)^2}(nq+1)C_jC_j.
\end{align*}
\]

**Remark.** A Finsler space with the tensor $S_{hijk}$ of (7.8)(a) is called $S4$-like. If the condition $nq + 1 = 0$ holds, then the tensor $S_{hijk}$ has the form (6.10) and the space is called $S3$-like.

In a $g$-Landsberg space of scalar curvature $K$, the condition $nq+1 = 0$ reduces to

\[
ab + 6m^2(1 + \beta - b) = 0,
\]

and after some rearrangement we have

\[
3\beta^2 - (14S + 3\tau + 11)\beta + 8S^2 + 2(2\tau + 3)S + \tau - 2 = 0,
\]

where the scalars $\beta$, $\tau$ and the constant $S$ exist exactly.

However, with (5.9) and (7.2)(d) in mind we have

\[
\beta_{/k} = (S + m^2)_{/k} = 2m^i m_{i/k} = \frac{2FG}{n+1} m_k, \quad \beta_{/0} = 0.
\]

Now, differentiating (7.10) by $x^k$ we find

\[
(4S - 3\beta + 1)\tau_{/k} = (14S + 3\tau - 6\beta + 11)\beta_{/k}.
\]
Hence from (7.11) we see \((4S - 3\beta + 1)\tau_0 = 0\). Put \(4S - 3\beta + 1 = a - 6m^2 = 0\), then the condition (7.9) can be rewritten as \(a(1 + \beta) = 0\), which cannot hold. Thus, from (5.7) and (5.10) we obtain
\[
\tau_0 = \frac{1}{n-1} (C^i - 2F^{-1} T^i_{i/0}) = - \frac{2F}{n^2 - 1} T^i_{i/0} = 0.
\]

On the other hand, by means of the Ricci identity we find
\[
C_{i/k(j)} - C_{i/(j)k} = -P^h_{ij} C^{h}_{j} - C^{h}_{k} C_{i/h} - P^h_{ij} C_{i/(h)} = -C^{h}_{k} C_{i/h}.
\]
Transvecting the above equation with \(\gamma^k\) we have \(C_{i/(j)0} = -C_{i/j} = -G_{i/j}\). From \(T_{ij/k} = F C_{i/(j)k} + l_i C_{j/k} + l_j C_{i/k}\) we obtain
\[
0 = T^i_{i/0} = F g^{ij} C_{i/(j)0} = -(n-1) FG.
\]
Thus we have \(G = 0\). Hence we have from Theorem 7.6

**Theorem 7.7.** If the associated Finsler space of a \(g\)-Landsberg space of scalar curvature \(K\) is \(S^3\)-like, then the space is an \(RccM_n\) space.

### 7.6. Appendix.

In the cases of \((2^\circ), (3^\circ)\) and \((4^\circ)\), the original condition is expressed by (e.g. [20], (5.12), (5.19))
\[
\begin{align*}
(a) & \quad (n-1)C_{hij} + h_{ij}(C^{r}_{k} C^{r}_{r} - C_{h} C_{k}) \\
& \quad + h_{kj} (C^{r}_{i} C^{r}_{r} - C_{i} C_{k}) - j|k = 0 \quad (*C\text{-reducible}), \\
(b) & \quad (n-2)C^{2}C_{hij} + (A - C^{2}C^{2})C^{2}(C_{h} h_{ij} + C_{i} h_{hj} + C_{j} h_{hi}) \\
& \quad + \{3C^{2}C^{2} - (n+1)A\} C_{h} C_{i} C_{j} = 0, \\
& \quad A := C_{hij} C^{h} C^{i} C^{j} \quad \text{(semi-*C-reducible)}.
\end{align*}
\]

It has been proved that (a) and (b) are equivalent to the semi-C-reducible condition ([32], Proposition 1.1). As \(C_{hij}\) satisfies \(A = cC^{2}C^{2}\) with some scalar \(c\), we find
\[
(n-2)C^{2}C_{hij} + (c-1)C^{2}(C_{h} h_{ij} + C_{i} h_{hj} + C_{j} h_{hi}) + \{3 - (n+1)c\} C_{h} C_{i} C_{j} = 0.
\]
Accordingly the scalar \(p\) is not arbitrary and the scalar \(c\) decides some property. However as \(p \neq 0\) and \(q \neq 0\), we see that \(c \neq 1\) and \(c \neq \frac{3}{n+1}\).

**Definition.** A Finsler space is called

1. a **Finsler space with \((\alpha, \beta)\)-metric** if the Finsler metric is given by \(F(x, y) = L(\alpha, \beta)\), where \(L\) is \(p\)-homogeneous of degree 1 in the two variables \(\alpha(x, y) := \sqrt{a_{ij}(x)y^{i}y^{j}}\) and \(\beta(x, y) := b_{i}(x)y^{i}\).
(2) a Finsler space with property $\mathcal{H}$, if the condition $\mathcal{H}^i_{jk} = 0$ holds ([22],[2.3],[31]), where
\[
\mathcal{H}^i_{jk} := Z^i_{jk} - \frac{1}{n-2} (Z_j h^i_k - Z_k h^i_j), \quad Z^i_{jk} := p \cdot H^i_{jk}, \quad Z_j := Z^i_{ji}.
\]

(3) a $^*P$-Finsler space, if the condition $^*P^i_{jk} := P^i_{jk} - \lambda C^i_{jk} = 0$ holds (called a $^*P$-condition, cf. [19]).

(4) a Finsler space with $F^i_{jk} = 0$, if the condition $F^i_{jk} = 0$ holds ([30]).

(5) an $R3$-like Finsler space, if the condition $C_{hijk} = 0$ (the formally Weyl conformal curvature tensor vanishes as in Riemannian geometry) holds ([20],[38]), where
\[
C_{hijk} := R_{hijk} - (L_{hj} g_{ik} + g_{ij} L_{ik} - j|k), \quad L_{hj} := \frac{1}{n-2} (R^i_{hj} - r g_{hj}).
\]

Remark. Three special Finsler spaces: a Finsler space of scalar curvature $K$, an $R3$-like Finsler space and a Finsler space with $F^i_{jk} = 0$, have the property $\mathcal{H}$ ([22],Theorems 2.4, 2.5 and 2.6).

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References


Toshio Sakaguchi, Hideo Izumi and Mamoru Yoshida: On generalized ... 


Toshio Sakaguchi
Department of Mathematics,
The National Defense Academy,
Yokosuka 239, Japan

Hideo Izumi
Fujisawa 2505–165,
Fujisawa 251, Japan

Mamoru Yoshida
Department of Mathematics,
Shonan Institute of Technology,
Fujisawa 251, Japan

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