On groups with small verbal conjugacy classes

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Abstract. Given a group $G$ and a word $w$, we denote by $G_w$ the set of all $w$-values in $G$ and by $w(G)$ the corresponding verbal subgroup. The main result of the paper is the following theorem. Let $n$ be a positive integer and let $w$ be either the lower central word $\gamma_n$ or the derived word $\delta_n$. Let $G$ be a group in which for any element $g \in G$ there exist finitely many Chernikov subgroups whose union contains $gG_w$. Then $\langle g^w(G) \rangle$ is Chernikov for all $g \in G$.

1. Introduction

Let $w$ be a word in $n$ variables, and let $G$ be a group. The verbal subgroup $w(G)$ of $G$ determined by $w$ is the subgroup generated by the set $G_w$ consisting of all values $w(g_1, \ldots, g_n)$, where $g_1, \ldots, g_n$ are elements of $G$. A word $w$ is said to be concise if whenever $G_w$ is finite for a group $G$, it always follows that $w(G)$ is finite. P. Hall asked whether every word is concise, but it was later proved that this problem has a negative solution in its general form (see [5, p. 439]). On the other hand, many important words are known to be concise. For instance, Turner-Smith [9] showed that the lower central words $\gamma_n$ and the derived words $\delta_n$ are concise; here the words $\gamma_n$ and $\delta_n$ are defined by the formulae $\gamma_1 = \delta_0 = x$, $\gamma_n = [\gamma_{n-1}, \gamma_1]$ and $\delta_n = [\delta_{n-1}, \delta_{n-1}]$. The corresponding verbal subgroups for these words are the familiar $n$th term of the lower central series of $G$ denoted by $\gamma_n(G)$ and the $n$th derived group of $G$ denoted by $G^{(n)}$.

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There are several natural ways to look at Hall’s question from a different angle. The circle of problems arising in this context can be characterized as follows.

Given a word $w$ and a group $G$, assume that certain restrictions are imposed on the set $G_w$. How does this influence the properties of the verbal subgroup $w(G)$?

If $X$ and $Y$ are non-empty subsets of a group $G$, we will write $X^Y$ to denote the set \{ $y^{-1}xy \mid x \in X, y \in Y$ \}. In [2] groups $G$ with the property that $x^{G_w}$ is finite for all $x \in G$ were called $FC(w)$-groups. Recall that $FC$-groups are precisely groups with finite conjugacy classes. The main result of [2] tells us that if $w$ is a concise word and $G$ is an $FC(w)$-group, then the verbal subgroup $w(G)$ is $FC$. Later it was shown in [1] that there exists a function $f = f(m, w)$ such that if, under the hypothesis of the above theorem, $x^{G_w}$ has at most $m$ elements for all $x \in G$, then $x^{w(G)}$ has at most $f$ elements for all $x \in G$. In relation with the above results, the following question was considered in [4].

Given a concise word $w$ and a group $G$, assume that for all $x \in G$ the subgroup $\langle x^{G_w} \rangle$ satisfies a certain finiteness condition. Is it true that a similar condition is also satisfied by $\langle x^{w(G)} \rangle$ for all $x \in G$?

Here and throughout the paper $\langle M \rangle$ denotes the subgroup generated by the set $M$. The following theorem is the main result of [4].

**Theorem 1.1.** Let $n$ be a positive integer and let $w$ be either the word $\gamma_n$ or the word $\delta_n$. Suppose that $G$ is a group in which $\langle g^{G_w} \rangle$ is Chernikov for all $g \in G$. Then $\langle g^{w(G)} \rangle$ is Chernikov for all $g \in G$ as well.

Recall that a group $G$ is Chernikov if it has a subgroup of finite index that is a direct product of finitely many groups of type $C_{p^\infty}$ for various primes $p$ (quasisicyclic $p$-groups, or Prüfer $p$-groups). By a deep result obtained independently by Shunkov [8], and KEGEL and WEHRFRITZ [3] Chernikov groups are precisely the locally finite groups satisfying the minimal condition on subgroups, that is, any non-empty set of subgroups possesses a minimal subgroup.

The purpose of the present paper is to strengthen Theorem 1.1 in the following way.

**Theorem 1.2.** Let $n$ be a positive integer and let $w$ be either the word $\gamma_n$ or the word $\delta_n$. Let $G$ be a group in which for any element $g \in G$ there exist finitely many Chernikov subgroups whose union contains $g^{G_w}$. Then $\langle g^{w(G)} \rangle$ is Chernikov for all $g \in G$. 
A proof of Theorem 1.2 in the case where \( w = \gamma \) can be obtained from the case \( w = \delta \) by simply replacing everywhere in the proof the term “\( \delta \)-commutators” by “\( \gamma \)-commutators”. That is why we do not provide an explicit proof for the case \( w = \gamma \) concentrating instead on proving Theorem 1.2 in the case \( w = \delta \).

The hypothesis in Theorem 1.2 is reminiscent of the situation considered in [7] where it was proved that if the set of \( \delta \)-commutators in a group \( G \) is contained in a union of finitely many Chernikov subgroups, then \( G^{(n)} \) is Chernikov. As a by-product of the proof of Theorem 1.2 we obtain a considerably stronger result – Corollary 2.1 in the next section says that for any word \( w \) if the set of \( w \)-values in a group \( G \) is contained in a union of finitely many Chernikov subgroups, then \( w(G) \) is Chernikov.

2. Preliminaries

Let \( G \) be a group acted on by a group \( A \). As usual, \( [G, A] \) denotes the subgroup generated by all elements of the form \( x^{-1}x^a \), where \( x \in G, a \in A \). It is well-known that \( [G, A] \) is a normal subgroup of \( G \). If \( B \) is a normal subset of \( A \) such that \( A = \langle B \rangle \), then \( [G, A] = \langle [G, b]; b \in B \rangle \). In particular, if \( A \) is cyclic, then \( [G, A] = [G, a] \), where \( a \) is a generator of \( A \).

The minimal subgroup of finite index of a Chernikov group \( T \) is called the radicable part of \( T \). Throughout the article we denote this subgroup by \( T^0 \). In general a group \( T \) is called radicable if the equation \( x^n = a \) has a solution in \( T \) for every positive integer \( n \) and every \( a \in T \). It is well-known that a periodic abelian radicable group is a direct product of quasicyclic \( p \)-subgroups. Suppose the radicable part of a Chernikov group \( T \) has index \( i \) and is a direct product of precisely \( j \) groups of type \( C_{p^\infty} \) (for various primes \( p \)). The ordered pair \((j, i)\) is called the size of \( T \). The set of all pairs \((j, i)\) is endowed with the lexicographic order. It is easy to check that if \( H \) is a proper subgroup of \( T \), the size of \( H \) is necessarily strictly less than that of \( T \). Also, if \( N \) is an infinite normal subgroup of \( T \), the size of \( T/N \) is necessarily strictly less than that of \( T \).

The following lemma is well-known (see for example [6, Part 1, Lemma 3.13]).

**Lemma 2.1.** Suppose that \( R \) is a radicable abelian normal subgroup of the group \( G \) and suppose that \( H \) is a subgroup of \( G \) such that \([R, H, \ldots, H] = 1\) for some natural number \( r \). If \( H/H' \) is periodic, then \([R, H] = 1\).
The next few lemmas can be easily deduced from the above. The interested reader can find their proofs for example in [4].

**Lemma 2.2.** In a periodic nilpotent group $G$ every radicable abelian subgroup $Q$ is central.

**Lemma 2.3.** Let $A$ be a periodic group acting on a periodic radicable abelian group $G$. Then $[G, A, A] = [G, A]$.

**Lemma 2.4.** Let $A$ be a finite group acting on a periodic radicable abelian group $G$. Then $[G, A]$ is radicable.

**Lemma 2.5.** Let $A$ be a radicable group acting on a Chernikov group $B$. Then $[B, A, A] = 1$.

**Lemma 2.6.** Let $G$ be a Chernikov group for which there exists a positive integer $m$ such that $G$ can be generated by elements of order dividing $m$. If $G^0 \leq Z(G)$, then $G$ is finite.

**Proof.** Essentially, this is Lemma 2.7 in [4].

**Lemma 2.7.** Let $G$ be a group, $y$ an element of $G$, and $x$ is a $\delta_n$-commutator for some $n \geq 0$. Then $[y, x, x]$ is a $\delta_{n+1}$-commutator.

**Proof.** This follows from the fact that $[y, x, x]$ can be written as $[x^{-y}, x]^x$.

**Lemma 2.8.** Let $G$ be a group generated by an element $g$ and an abelian radicable subgroup $S$. Suppose that $G$ has finitely many Chernikov subgroups whose union contains $gS$. Then the subgroup $\langle gS \rangle$ is Chernikov.

**Proof.** Suppose that the lemma is false and the subgroup $\langle gS \rangle$ is not Chernikov. Let $C_1, \ldots, C_k$ be finitely many Chernikov subgroups such that $gS \subseteq \bigcup C_i$. Without loss of generality we assume that the subgroups $C_1, \ldots, C_k$ are chosen in such a way that the sum of the sizes of $C_1, \ldots, C_k$ is as small as possible. In that case, of course, each subgroup $C_i$ is generated by $C_i \cap gS$. Remark that $\langle g^G \rangle = \langle g^S \rangle$ and therefore the subgroup $\langle g^S \rangle$ is normal. If all subgroups $C_1, \ldots, C_k$ are finite, then so is the set $g^S$. In that case the index $[S : C_S(g)]$ is finite. Being radicable, $S$ does not have proper subgroups of finite index and so we deduce that $g^S = g$ and $\langle g^S \rangle = \langle g \rangle$. Since $g$ is contained in a Chernikov subgroup, $g$ must be of finite order and so $\langle g \rangle$ is finite. Therefore, at least one of the subgroups $C_1, \ldots, C_k$ is infinite. Without loss of generality assume that $C_1$ is infinite. Among all infinite subgroups of $C_1$ that can be generated by elements
of $g^S$ we choose a minimal one, say $K$. Let $Y = K \cap g^S$ and so $K = \langle Y \rangle$. If $x$ is an arbitrary element in $S$, the set $Y^x$ has infinite intersection with at least one of the subgroups $C_i$. Suppose that $C_j \cap Y^x$ is infinite and set $L = \langle C_j \cap Y^x \rangle$. It is clear that $L^{x^{-1}}$ is an infinite subgroup of $K$ generated by a subset of $Y$. Because of minimality of $K$ we conclude that $L = K^x$. Thus, for any $x \in S$ there exists $j$ such that $K^x \leq C_j$. Choose $a \in K_0$. It follows that for any $x \in S$ there exists $j$ such that $a^x \leq C_j^0$. Since a radicable Chernikov group has only finitely many elements of any given order, we deduce that the class $a^S$ is finite. Taking into account that $S$ has no proper subgroups of finite index and that $a$ was taken in $K_0$ arbitrarily we now deduce that $[K_0, S] = 1$. Since $Y$ normalizes $K_0$ and since $G = \langle S, Y \rangle$, it follows that $K_0$ is normal in $G$. The size of the image of $C_1$ in $G/K_0$ is strictly less than that of $C_1$ and therefore, by induction, $(g^S)/K_0$ is Chernikov. Since also $K_0$ is Chernikov, so is $\langle g^S \rangle$. The proof is complete. \(\square\)

An idea from the proof of Lemma 2.8 can be used to significantly improve the result that if the set of $\delta_n$-commutators in a group $G$ is contained in a union of finitely many Chernikov subgroups, then $G^{(n)}$ is Chernikov [7]. We will now show that for any word $w$ if the set of $w$-values in a group $G$ is contained in a union of finitely many Chernikov subgroups, then $w(G)$ is Chernikov. In fact we have the following rather general proposition.

**Proposition 2.9.** Let $X$ be a normal subset of a group $G$ and suppose that $G$ has Chernikov subgroups $C_1, \ldots, C_k$ whose union contains $X$. Then $\langle X \rangle$ is Chernikov.

Recall that a group having an ascending central series is called hypercentral. For the proof of Proposition 2.9 we will require the following well-known lemma whose proof can be easily deduced for example from [6, Part 2, Theorem 9.23 and Corollary 2, page 125].

**Lemma 2.10.** Let $G$ be a hypercentral group generated by its quasicyclic subgroups. Then $G$ is abelian.

**Proof of Proposition 2.9.** Without loss of generality we assume that all subgroups $C_i$ are generated by elements of $X$. Let $C$ be the normal closure of the subgroups $C_1^0, \ldots, C_k^0$. It is clear that $C$ has no subgroups of finite index. If $C = 1$, then the set $X$ is finite. Since the elements of $X$ are contained in Chernikov subgroups, it follows that all elements of $X$ have finite order. In that case $\langle X \rangle$ is finite by Dietzmann’s Lemma on elements of finite order having finitely many conjugates (see [6, Part 1, p. 45]). So we assume that $C \neq 1$. In particular, we assume that $C_1^0 \neq 1$. Let $K$ be a minimal infinite subgroup of $C_1$ generated by
elements of $X$. Because of minimality, for every $x \in G$ there exists $i$ such that $K^x \leq C_i$. Let $a$ be an element of $K^0$. It follows that every conjugate $a^x$ belong to $C_{i}^0$ for some $i$. Since each subgroup $C_i^0$ has only finitely many elements of any given order, we conclude that the conjugacy class $a^G$ is finite. Since $C$ has no subgroups of finite index, $a \in Z(C)$. Thus, we have shown that $K^0 \leq Z(C)$.

Next, we can repeat the argument with $G$ replaced by $G/Z(C)$ and conclude that if $C \neq Z(C)$, then $Z_2(C) \neq Z(C)$. Thus, we see that $C$ is hypercentral. Since $C$ is generated by quasicyclic subgroups, Lemma 2.10 tells us that $C$ is abelian. Recall that every conjugate $(K^0)^x$ belongs to $C_i^0$ for some $i$. Hence, the normal closure $\langle (K^0)^x \rangle$ is Chernikov. Note that the sum of sizes of the images of $C_1, \ldots, C_k$ in the quotient $G/(K^0)^G$ is strictly smaller than that of $C_1, \ldots, C_k$. Thus, by induction, the image of $\langle X \rangle$ in $G/(K^0)^G$ is Chernikov. Therefore $\langle X \rangle$ is Chernikov, as desired.

The following corollary is now straightforward.

**Corollary 2.11.** Let $w$ be a group-word and $G$ a group in which the set of $w$-values is contained in a union of finitely many Chernikov subgroups. Then $w(G)$ is Chernikov.

### 3. Proof of Theorem 1.2

We will now assume the hypothesis of Theorem 1.2 with $w = \delta_n$. Thus, $n$ is a positive integer and $G$ is a group in which for any element $g \in G$ there exist finitely many Chernikov subgroups whose union contains $g^G$. We denote by $X$ the set of all $\delta_n$-commutators in $G$ and by $H$ the $n$th derived group of $G$. In other words, $H = \langle X \rangle$. Our goal is to prove that $(g^H)$ is Chernikov for all $g \in G$.

Let $B$ be the subgroup of $G$ generated by all subgroups of the form $[T, x]$, where $T$ is an abelian radicable subgroup, $x \in X$ and $x$ normalizes $T$.

**Lemma 3.1.** The subgroup $B$ is abelian.

**Proof.** Let $S = [T, x]$, where $T$ is an abelian radicable subgroup, $x \in X$ and $x$ normalizes $T$. By Lemma 2.4 $S$ is radicable. Lemma 2.3 shows that $S = [T, x, x]$. In view of Lemma 2.7 every element in $[T, x, x]$ is a $\delta_n$-commutator. Thus, $S$ is an abelian radicable subgroup contained in $X$. Choose an arbitrary element $g \in G$. By Lemma 2.8 $(g^S)$ is Chernikov. It follows from Lemma 2.5 that $[(g^S), S, S] = 1$. In particular $[g, S, S] = 1$ and so $S$ commutes with $S^g$. This happens for every $g \in G$ and therefore the normal subgroup $(S^G)$ is abelian. Lemma 2.2 shows that...
Lemma 3.2. The quotient $H/B$ is an FC-group.

Proof. Since every element of $H$ is a product of finitely many elements from $X$, it is sufficient to show that under the additional hypothesis that $B = 1$ the index $[H : C_H(x)]$ is finite for every $x \in X$. Thus, we assume that $B = 1$. Suppose that the lemma is false and choose $x \in X$ such that $[H : C_H(x)]$ is infinite. Set $Y = x^X$. Let $C_1, \ldots, C_k$ be finitely many Chernikov subgroups such that $Y \subseteq \bigcup C_i$. Without loss of generality we assume that the subgroups $C_1, \ldots, C_k$ are chosen in such a way that the sum of the sizes of $C_1, \ldots, C_k$ is as small as possible. In that case, of course, each subgroup $C_i$ is generated by $C_i \cap Y$.

If the subgroups $C_1, \ldots, C_k$ were all finite, then in view of the main result of [2] $[H : C_H(x)]$ would be finite. Thus, at least one of the subgroups $C_1, \ldots, C_k$ is infinite. Assume that $C_1$ is infinite and let $Y_1 = Y \cap C_1$. For any $y \in Y_1$ we have $[C_i^0, y] \leq B$. Since $B = 1$ and $C_1 = \langle Y_1 \rangle$, it follows that $C_i^0 \leq Z(C_1)$ whence, by Lemma 2.6, $C_1$ is finite, a contradiction. □

Lemma 3.3. For each $g \in G$ the image of $\langle g^H \rangle$ in $G/B$ is Chernikov.

Proof. It follows from Lemma 3.2 that $G$ is locally finite. Let us assume that $B = 1$. Then $H$ is an FC-group and, since radicable groups have no proper subgroups of finite index, all radicable subgroups of $H$ are contained in the center.

Choose $g \in G$ and let $C_1, \ldots, C_k$ be finitely many Chernikov subgroups such that $g^X \subseteq \bigcup C_i$. The subgroup $J = \langle C_1^0, \ldots, C_k^0 \rangle$ is Chernikov since it is generated by finitely many commuting Chernikov subgroups. Since $g$ has finite order, it is clear that $J_1 = \prod J^g$ is Chernikov, too. Set $M = H(g)$. We remark that $J_1$ is normal in $M$. The subgroups $C_1, \ldots, C_k$ all have finite images in $M/J_1$ and therefore the image of the verbal conjugacy class $g^X$ is finite. By [3, Lemma 2.9] the image of the conjugacy class $g^H$ is finite as well. Since $g$ is of finite order, by Dietzmann’s lemma the image of $\langle g^H \rangle$ in $M/J_1$ is finite. Since $J_1$ is Chernikov, the result follows. □

Lemma 3.4. The subgroup $[B, h]$ is Chernikov for every $h \in H$.

Proof. Suppose first that $h \in X$. Then, as we have remarked earlier, $[B, h] \subseteq X$. Let $C_1, \ldots, C_k$ be finitely many Chernikov subgroups such that $h[B, h] \subseteq \bigcup C_i$. Then $[B, h] = [B, h, h] \subseteq \bigcup (C_i \cap [B, h])$. In view of Lemma 3.1, the subgroups $C_i \cap [B, h]$ commute. Thus, $[B, h]$ is contained in a union of commuting Chernikov subgroups and hence is Chernikov itself.
We now drop the assumption that \( h \in X \). Since \( h \in H \), we can write \( h \) as a product of several elements from \( X \). Suppose that \( h = x_1 \cdots x_n \), where \( x_i \in X \). Then it is clear that \( [B, h] \leq \prod_i [B, x_i] \). Since each \([B, x_i]\) is Chernikov and all \([B, x_i]\) commute, the result follows.

**Lemma 3.5.** Let \( A \) be a subgroup of \( H \) whose image in \( G/B \) is abelian and radicable. Then \([B, A] = 1\).

**Proof.** Let \( a \in A \). Then, since \( B \) is abelian, \( A/B \) naturally acts on \([B, a]\) and of course \([B, a, A/B] = [B, a, A] \). By Lemma 3.4 the subgroup \([B, a]\) is Chernikov. According to Lemma 2.5 \([B, a, A, A] = 1\). In particular \([B, a, a, a] = 1\) and so Lemma 2.3 shows that \([B, a] = 1\). This happens for every \( a \in A \) and therefore \([B, A] = 1\).

**Lemma 3.6.** For every \( g \in G \) the subgroup \([B, g]\) is Chernikov.

**Proof.** It was mentioned in the proof of Lemma 3.1 that if \( T \) is an abelian radicable subgroup, \( x \in X \) and \( x \) normalizes \( T \), then \([T, x]\) is an abelian radicable subgroup contained in \( X \). Therefore \( B \) is the product of its subgroups \( S_1, S_2, \ldots \) each of which is contained in \( X \). Given \( g \in G \), let \( C_1, \ldots, C_k \) be finitely many Chernikov subgroups such that \( g^X \subseteq \cup C_i \) and \( B_i = C_i \cap B \) for \( i = 1, \ldots, k \). Denote by \( D \) the product of all subgroups of the form \((B_i)^g\) for \( i \leq k \) and \( j = 0, 1, \ldots \).

Since \( g \) has finite order, \( D \) is a product of finitely many commuting Chernikov subgroups and so is Chernikov itself. It is clear that \( D \) is normal in \( B(g) \).

Since each \( S_l \) is contained in \( X \), it follows that \( g^{S_l} \subseteq \cup C_i \) for every \( l = 1, 2, \ldots \). We look at the image of the class \( g^{S_l} \) in the quotient \( B(g)/D \) and conclude the image is finite since \( B \) has finite index in \( B(g) \). It follows that modulo \( D \) the element \( g \) centralizes a subgroup of finite index in \( S_l \). Taking into account that \( S_l \) has no proper subgroups of finite index we conclude that \([S_l, g] \leq D\). This happens for every \( l = 1, 2, \ldots \). Because \([B, g]\) is the product of subgroups of the form \([S_l, g]\), we have \([B, g] \leq D\).

**Lemma 3.7.** For every \( g \in G \) the subgroup \([B, \langle g^H \rangle]\) is Chernikov.

**Proof.** Choose \( g \in G \) and set \( K = \langle g^H \rangle \) and \( C = C_K(B) \). Then \( K/C \) naturally acts on \( B \) and \([B, K] = [B, K/C] \). By Lemma 3.3 the image of \( K \) in \( G/B \) is Chernikov. Let \( A \) be the subgroup of \( K \cap H \) whose image in \( G/B \) is the radicable part of the image of \( K \cap H \). By Lemma 3.5 the subgroup \( A \) is contained in \( C \). Obviously, \( K \cap H \) has finite index in \( K \) and therefore the index of \( A \) in \( K \) is finite. Thus, \( K/C \) is finite and so \([B, K]\) is a product of finitely many subgroups of the form \([B, u]\) for suitable elements \( u \in K \). By Lemma 3.6 each of the subgroups \([B, u]\) is Chernikov and the result follows.
We are now ready to complete the proof of Theorem 1.2. Choose \( g \in G \) and set \( K = \langle g^H \rangle \). By Lemma 3.7 the subgroup \([B, K]\) is Chernikov. We remark that \([B, K]\) is normal in \( HK \) and pass to the quotient \( \bar{V} = HK/[B, K] \). The image of a subgroup \( T \) of \( HK \) in \( \bar{V} \) will be denoted by \( \bar{T} \).

We have \([B, \bar{K}] = 1\). It follows from Lemma 3.3 that \( \bar{K}/Z(\bar{K}) \) is Chernikov. A theorem of Polovickii [6, Part 1, p. 129] now tells us that \( \bar{K}' \), the derived group of \( \bar{K} \), is Chernikov.

Therefore \( K' \) is Chernikov as well. The subgroup \( \langle g^X \rangle \) is generated by finitely many Chernikov subgroups and has Chernikov derived group \( \langle g^X \rangle' \). We conclude that \( \langle g^X \rangle \) is Chernikov for all \( g \in G \). The main theorem of [4] now tells us that \( \langle g^H \rangle \) is Chernikov for all \( g \in G \). The proof is now complete.

References