On \(p\)-hypercyclically embedded subgroups of finite groups

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Abstract. Let \(G\) be a finite group and \(p\) a prime. A normal subgroup \(E\) of \(G\) is said to be \(p\)-hypercyclically embedded in \(G\) if every \(p\)-chief factor of \(G\) below \(E\) is cyclic. We say that a subgroup \(H\) of \(G\) is generalized \(S\Phi\)-supplemented in \(G\) if \(G\) has a subnormal subgroup \(T\) such that \(G = HT\) and \((H \cap T)_{sG}/_{sG} \leq \Phi(H/_{sG})\), where \(H_{sG}\) is the subgroup of \(H\) generated by all those subgroups of \(H\) which are \(s\)-permutable in \(G\). In this paper, some new characterizations of \(p\)-hypercyclically embeddability of normal subgroups of a finite group are obtained based on the assumption that some primary subgroups are generalized \(S\Phi\)-supplemented in \(G\).

1. Introduction

Throughout this paper, all groups considered are finite. \(G\) always denotes a group, \(p\) denotes a prime, and \(|G|_p\) denotes the order of a Sylow \(p\)-subgroup of \(G\).

A normal subgroup \(E\) of \(G\) is said to be hypercyclically embedded (resp. \(p\)-hypercyclically embedded) in \(G\) if every chief factor (resp. \(p\)-chief factor) of \(G\) below \(E\) is cyclic. The hypercyclically embedded subgroups have a great influence on the structure of a group, and some important classes of groups can be characterized in terms of hypercyclically embedded subgroups. For example, if all subgroups of \(G\) of prime order or order 4 are hypercyclically embedded in \(G\),

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then $G$ is supersolvable (Huppert [12], Doerk [5], see also [23]). A group $G$ is quasisupersoluble (i.e. for every non-cyclic chief factor $H/K$ of $G$, every automorphism of $H/K$ induced by an element of $G$ is inner) if and only if it has a normal hypercyclically embedded subgroup $E$ such that $G/E$ is semisimple (see [10, Theorem C]). Some recent results in this topic can be found in, for example, [2], [9], [11], [22], [23], [24], [25].

Recall that a subgroup $H$ of $G$ is said to be $s$-permutable in $G$ if $H$ permutes with every Sylow subgroup of $G$. A subgroup $H$ of $G$ is said to be weakly $s$-permutable in $G$ [21] if $G$ has a subnormal subgroup $T$ such that $G = HT$ and $H \cap T \leq H_{sG}$, where $H_{sG}$ is the subgroup of $H$ generated by all those subgroups of $H$ which are $s$-permutable in $G$. A subgroup $H$ of a group $G$ is called $S\Phi$-supplemented [17] (or $\Phi$-$s$-supplemented [16]) in $G$ if there exists a subnormal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T \leq \Phi(H)$, where $\Phi(H)$ is the Frattini subgroup of $H$. Note that $H_{sG}$ is normal in $H$. We now introduce the following concept which is closely related to the above two concepts.

**Definition 1.1.** A subgroup $H$ of $G$ is said to be generalized $S\Phi$-supplemented in $G$ if there exists a subnormal subgroup $T$ of $G$ such that $G = HT$ and $(H \cap T)H_{sG}/H_{sG} \leq \Phi(H/H_{sG})$.

It is easy to see that weakly $s$-permutable subgroups and $S\Phi$-supplemented subgroups of $G$ are all generalized $S\Phi$-supplemented in $G$. But the following examples show that the converse does not hold in general.

**Example 1.2.** Let $G = Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle$ and $H = \langle b^2 \rangle$. Then, clearly, $H$ is $s$-permutable in $G$ and $H$ has the unique supplement $G$ in $G$. Hence $H$ is generalized $S\Phi$-supplemented in $G$. But $H$ is not $\Phi$-$s$-supplemented in $G$ because $\Phi(H) = 1$.

**Example 1.3.** Let $G = S_5$ be the symmetric group of degree 5 and $H = \langle (1234) \rangle$. Then $H_{sG} = H_G = 1$. Since $G = HA_5$ and $H \cap A_5 = \Phi(H) = \langle (13)(24) \rangle$, $H$ is generalized $S\Phi$-supplemented in $G$, but $H$ is not weakly $s$-permutable in $G$.

A class of groups $\mathcal{F}$ is called a formation if it is closed under taking homomorphic images and subdirect products. The $\mathcal{F}$-residual of $G$, denoted by $G^\mathcal{F}$, is the smallest normal subgroup of $G$ with quotient in $\mathcal{F}$. Let $Z_\mathcal{F}(G)$ (resp. $Z_{p\mathcal{F}}(G)$) denote the $\mathcal{F}$-hypercentre (resp. $p\mathcal{F}$-hypercentre) of $G$, that is, the product of all normal subgroups $H$ of $G$ such that all chief factors (resp. $p$-chief factors) $L/K$ of $G$ below $H$ is $\mathcal{F}$-hypercentral (i.e. $L/K \times G/C_G(L/K) \in \mathcal{F}$ (see [8, Chap. 1])). Let $\mathcal{U}$ denote the classes of all supersoluble groups. Then $Z_{\mathcal{U}}(G)$ (resp. $Z_{p\mathcal{U}}(G)$) is the product of all normal hypercyclically embedded (resp. $p$-hypercyclically
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Moreover, the generalized Fitting subgroup $F^*$ of $G$ is the maximal quasinilpotent subgroup (resp. the maximal $p$-quasinilpotent subgroup) of $G$ (for details, see [14, Chap. X] and [15]).

In the present paper, we will give a new characterization of p-hypercyclically embedded subgroups of $G$ by using the generalized $S\Phi$-supplemented subgroups.

Our main result is the following.

**Theorem 1.4.** Let $E$ and $X$ be normal subgroups of $G$ such that $F^*_p(E) \leq X \leq E$ and $P$ a Sylow $p$-subgroup of $P$ with $|P| = |D|$ and all cyclic subgroups of $P$ of order 4 (when $P$ is a non-abelian 2-group and $|D| = 2$) are generalized $S\Phi$-supplemented in $G$.

The following example illustrates that the converse of Theorem 1.4 does not hold.

**Example 1.5.** Let $G = \langle a, b \mid a^5 = 1, b^4 = 1, b^{-1}ab = a^2 \rangle$ and $H = \langle b^2 \rangle$.

Then clearly, $G$ is 2-supersoluble, $H \leq G$ and $|H| = |D| = 2$. If $H$ is generalized $S\Phi$-supplemented in $G$, then there exists a subnormal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T = 1$. This implies that $\langle b \rangle \leq \langle b \rangle \cap T$, and so $H \leq \langle b \rangle \leq T$.

**Proposition 1.6.** Let $P$ be a normal $p$-subgroup of $G$. If $P$ has a subgroup $D$ such that $1 < |D| < |P|$ and all cyclic subgroups of $P$ of order 4 (when $P$ is a non-abelian 2-group and $|D| = 2$) are generalized $S\Phi$-supplemented in $G$, then $P \leq Z_pU(G)$.

**Proposition 1.7.** Let $E$ be a normal subgroup of $G$ and $P$ a Sylow $p$-subgroup of $E$. If all cyclic subgroups of $P$ of order $p$ or 4 (when $P$ is a non-abelian 2-group) are generalized $S\Phi$-supplemented in $G$, then $E \leq Z_pU(G)$.

**Proposition 1.8.** Let $E$ be a normal subgroup of $G$ and $P$ a Sylow $p$-subgroup of $E$. If every maximal subgroup of $P$ is generalized $S\Phi$-supplemented in $G$, then either $E \leq Z_pU(G)$ or $|E| = |P|$. Note that Propositions 1.6–1.8 are independently interesting since they cover main results of many papers among which one can find recent publications (for example, [7], [16], [19]). We prove Theorem 1.4 and Propositions 1.6–1.8 in Section 3. Some unexplained notation and terminology are standard, as in [6], [7], [8].
2. Preliminaries

Lemma 2.1 (see [8, Chap. 1, Lemma 5.34]). Let $H \leq G$, $K \leq G$ and $N \trianglelefteq G$.

(1) If $H$ is s-permutable in $G$, then $H$ is subnormal in $G$.

(2) If $H$ is s-permutable in $G$, then $HN/N$ is s-permutable in $G/N$.

(3) If $H$ is a $p$-group, then $H$ is s-permutable in $G$ if and only if $O^p(G) \leq N_G(H)$.

(4) If $H$ is s-permutable in $G$, then $H \cap K$ is s-permutable in $K$.

Lemma 2.2 (see [21, Lemma 2.8] or [8, Chap. 3, Lemma 3.6]). Let $H \leq K \leq G$.

(1) $HsG$ is an s-permutable subgroup of $G$;

(2) $HsG \leq HsK$;

(3) If $H \trianglelefteq G$, then $(K/H)s(G/H) = KsG / H$.

Lemma 2.3. Let $H \leq K \leq G$ and $N \trianglelefteq G$. Suppose that $H$ is generalized $S\Phi$-supplemented in $G$. Then:

(1) $H$ is generalized $S\Phi$-supplemented in $K$.

(2) If either $N \leq H$ or $(|H|, |N|) = 1$, then $HN/N$ is generalized $S\Phi$-supplemented in $G/N$.

Proof. By the hypothesis, $G$ has a subnormal subgroup $T$ such that $G = HT$ and $(H \cap T)HsG / HsG \leq \Phi(H/HsG)$. Let $V/HsG = \Phi(H/HsG)$.

(1) By Dedekind’s identity, $K = H(T \cap K)$. Then by Lemma 2.2(2), $HsG \leq HsK$, and so $(H \cap T)HsK / HsK \leq VHsK / HsK \leq \Phi(H/HsK)$. Hence $H$ is generalized $S\Phi$-supplemented in $K$.

(2) Clearly, $G/N = (HN/N)(TN/N)$ and $HsG N / N \leq (HN/N)sG = (HN)sG / N$ by Lemma 2.2(3). Also, by Lemma 2.1(4), $(HN)sG = ((HN)sG \cap H)N \leq HsG N$. This implies that $(HN)sG = HsG N$. Since either $N \leq H$ or $(|H|, |N|) = 1$, $HN \cap TN = (H \cap T)N$, and so $(HN \cap TN)(HN)sG / (HN)sG \leq VHsG N \leq \Phi(HN/HsG N)$. This shows that $HN/N$ is generalized $S\Phi$-supplemented in $G/N$. \hfill \Box

Let $P$ be a p-group. If $P$ is not a non-abelian 2-group, then we use $\Omega(P)$ to denote the subgroup $\Omega_1(P)$. Otherwise, $\Omega(P) = \Omega_2(P)$.

Lemma 2.4 (see [4, Lemma 2.12]). Let $P$ be a normal $p$-subgroup of $G$ and $C$ a Thompson critical subgroup of $P$ (see [7, p. 186]). If $P/\Phi(P) \leq Z_2(G/\Phi(P))$ or $C \leq Z_2(G)$ or $\Omega(P) \leq Z_2(G)$, then $P \leq Z_2(G)$. 

Lemma 2.5 (see [3, Lemma 2.10]). Let $C$ be a Thompson critical subgroup of a nontrivial $p$-group $P$.

1. If $p$ is odd, then the exponent of $\Omega_1(C)$ is $p$.
2. If $P$ is an abelian 2-group, then the exponent of $\Omega_1(C)$ is 2.
3. If $p = 2$, then the exponent of $\Omega_2(C)$ is at most 4.

Lemma 2.6 (see [1, Theorem 2.1.6]). Let $G$ be a $p$-supersoluble group. Then the derived subgroup $G'$ of $G$ is $p$-nilpotent. In particular, if $O_p'(G) = 1$, then $G$ has a unique Sylow $p$-subgroup.

Lemma 2.7 (see [25, Lemma 2.13]). Let $\mathcal{F}$ be a formation and $E$ a normal subgroup of $G$. Then $E \leq \mathcal{Z}_p \mathcal{F}(G)$ if and only if $F^*_p(E) \leq \mathcal{Z}_p \mathcal{F}(G)$.

Lemma 2.8 (see [20, Lemma 2.6]). Let $V$ be an $s$-permutable subgroup of $G$ of order 4.

1. If $V = A \times B$, where $|A| = |B| = 2$ and $A$ is $s$-permutable in $G$, then $B$ is $s$-permutable in $G$.
2. If $V = \langle x \rangle$ is cyclic, then $\langle x^2 \rangle$ is $s$-permutable in $G$.

Lemma 2.9 (see [22, Theorem B]). Let $\mathcal{F}$ be any formation and $E$ a normal subgroup of $G$. If $F^*(E) \leq \mathcal{Z}_\mathcal{F}(G)$, then $E \leq \mathcal{Z}_\mathcal{F}(G)$.

3. Proof of main results

For a $p$-subgroup $H$ of $G$, we know that $\Phi(H/H_{sG}) = \Phi(H)H_{sG}/H_{sG}$ (see [13, Chap. 3, Theorem 3.14(c)]). Therefore, if $H$ is a generalized $S\Phi$-supplemented $p$-subgroup of $G$, then there exists a subnormal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T \leq \Phi(H)H_{sG}$.

Proof of Proposition 1.6. Suppose that the assertion is false and let $(G, P)$ be a counterexample for which $|G| + |P|$ is minimal. Then:

1. $|D| > p$.

If $|D| = p$, then by the hypothesis, every cyclic subgroup of $P$ of order $p$ or 4 (when $P$ is a non-abelian 2-group) is generalized $S\Phi$-supplemented in $G$. Let $P/R$ be a chief factor of $G$. Clearly, $(G, R)$ satisfies the hypothesis of the proposition. The choice of $(G, P)$ implies that $R \leq Z_d(G)$. If $|P/R| = p$, then $P \leq Z_d(G)$, a contradiction. Hence $|P/R| > p$. Suppose that $L \leq G$ and $L < P$. Then, similarly as above, we have that $L \leq Z_d(G)$. If $L \nleq R$, then $P = RL \leq Z_d(G)$, a contradiction. Hence $L \leq R$. This shows that $G$ has a unique normal subgroup $R$. 


such that $P/R$ is a chief factor of $G$. Let $C$ be a Thompson critical subgroup of $P$. Note that $C$ is characteristic in $P$ (see [7, Chap. 5, Theorem 3.11]). If $\Omega(C) < P$, then $\Omega(C) \leq R \leq Z_\Omega(G)$. It follows from Lemma 2.4 that $P \leq Z_\Omega(G)$, which is impossible. Hence $P = C = \Omega(C)$. Then by Lemma 2.5, the exponent of $P$ is $p$ or $4$ (when $P$ is a non-abelian 2-group).

Obviously, $P/R \cap Z(G_p/R) > 1$, where $G_p$ is a Sylow $p$-subgroup of $G$. Suppose that $V/R \leq P/R \cap Z(G_p/R)$ and $|V/R| = p$. Let $x \in V \setminus R$ and $H = \langle x \rangle$. Then $V = HR$ and $|H| = p$ or $4$. If $H = H_{sG}$, then by Lemma 2.2(1), $H$ is $s$-permutable in $G$, and so $V/R = HR/R \leq G/R$ by Lemma 2.1(2)(3). But since $P/R$ is a chief factor of $G$, we have that $P = V$. It follows that $P/R = V/R$ is cyclic, and so $P \leq Z_\Omega(G)$, a contradiction. Hence $H \neq H_{sG}$ and so $H_{sG} \leq \Phi(H)$.

By the hypothesis, there exists a subnormal subgroup $T$ such that $P/R \leq \Phi(T)$, which is impossible. Hence $P \leq Z_\Omega(G)$. Then by Lemma 2.5, the exponent of $P$ is $p$ or $4$ (when $P$ is a non-abelian 2-group).

Obviously, $P/R \cap Z(G_p/R) > 1$, where $G_p$ is a Sylow $p$-subgroup of $G$. Suppose that $V/R \leq P/R \cap Z(G_p/R)$ and $|V/R| = p$. Let $x \in V \setminus R$ and $H = \langle x \rangle$. Then $V = HR$ and $|H| = p$ or $4$. If $H = H_{sG}$, then by Lemma 2.2(1), $H$ is $s$-permutable in $G$, and so $V/R = HR/R \leq G/R$ by Lemma 2.1(2)(3). But since $P/R$ is a chief factor of $G$, we have that $P = V$. It follows that $P/R = V/R$ is cyclic, and so $P \leq Z_\Omega(G)$, a contradiction. Hence $H \neq H_{sG}$ and so $H_{sG} \leq \Phi(H)$.

By the hypothesis, there exists a subnormal subgroup $T$ such that $P/R \leq \Phi(T)$, which is impossible. Hence $P \leq Z_\Omega(G)$. Then by Lemma 2.5, the exponent of $P$ is $p$ or $4$ (when $P$ is a non-abelian 2-group).

Obviously, $P/R \cap Z(G_p/R) > 1$, where $G_p$ is a Sylow $p$-subgroup of $G$. Suppose that $V/R \leq P/R \cap Z(G_p/R)$ and $|V/R| = p$. Let $x \in V \setminus R$ and $H = \langle x \rangle$. Then $V = HR$ and $|H| = p$ or $4$. If $H = H_{sG}$, then by Lemma 2.2(1), $H$ is $s$-permutable in $G$, and so $V/R = HR/R \leq G/R$ by Lemma 2.1(2)(3). But since $P/R$ is a chief factor of $G$, we have that $P = V$. It follows that $P/R = V/R$ is cyclic, and so $P \leq Z_\Omega(G)$, a contradiction. Hence $H \neq H_{sG}$ and so $H_{sG} \leq \Phi(H)$.

By the hypothesis, there exists a subnormal subgroup $T$ such that $P/R \leq \Phi(T)$, which is impossible. Hence $P \leq Z_\Omega(G)$. Then by Lemma 2.5, the exponent of $P$ is $p$ or $4$ (when $P$ is a non-abelian 2-group).

Suppose that $p|D| = |P|$. By the hypothesis, every maximal subgroup of $P$ is generalized $S\Phi$-supplemented in $G$. Let $N$ be a minimal normal subgroup of $G$ contained in $P$. Then by Lemma 2.3(2), $(G/N, P/N)$ satisfies the hypothesis of the proposition. The choice of $(G, P)$ yields that $P/N \leq Z_\Omega(G/N)$. If $|N| = p$, then $P \leq Z_\Omega(G)$, which is impossible. Hence $|N| > p$. Suppose that $G$ has another minimal normal subgroup $L$ contained in $P$ such that $N \neq L$. With a similar discussion as above, we have that $P/L \leq Z_\Omega(G/L)$. It follows that $NL/L \leq Z_\Omega(G/L)$, and so $|N| = p$, a contradiction. Thus $G$ has a unique minimal normal subgroup $N$ contained in $P$.

If $\Phi(P) = 1$, then $P$ is elementary abelian. Let $N_1$ be a maximal subgroup of $N$ such that $N_1$ is normal in some Sylow $p$-subgroup $G_p$ of $G$, and let $S$ be a complement of $N$ in $P$. Then $P_1 = N_1 S$ is a maximal subgroup of $P$. By [13, Chap. 3, Lemma 3.3], $\Phi(P_1) \leq \Phi(P) = 1$. Therefore, there exists a subnormal subgroup $T$ of $G$ such that $G = P_1 T$ and $P_1 \cap T \leq (P_1)_{sG}$. Then $G = PT$ and $P = P_1 (P \cap T)$. It is easy to see that $T \neq P \cap T \trianglelefteq G$. Hence $N \leq P \cap T$, and so $P_1 \cap N \leq P_1 \cap T \leq (P_1)_{sG}$. It follows that $N_1 = P_1 \cap N = (P_1)_{sG} \cap N$ is $s$-permutable in $G$. By Lemma 2.1(3), $N_1 \leq G$, and so $|N| = p$, a contradiction. Thus $\Phi(P) \neq 1$. Then $N \leq \Phi(P)$. Since $P/N \leq Z_\Omega(G/N)$, $P/\Phi(P) \leq Z_\Omega(G/\Phi(P))$. Applying Lemma 2.4, we obtain that $P \leq Z_\Omega(G)$. The contradiction completes the proof of (2).
(3) Final contradiction.

We shall show that all subgroups $H$ of $P$ with $|H| = |D|$ are $s$-permutable in $G$. By the hypothesis, $G$ has a normal subgroup $T$ such that $G = HT$ and $H \cap T \leq \Phi(H)H_G$. If $T < G$, then there exists a normal subgroup $M$ of $G$ such that $T \leq M$ and $|G : M| = p$. Since $|P : P \cap M| = |PM : M| = p$, $P \cap M$ is a maximal subgroup of $P$ and so $|D| < |P \cap M|$ by (2). Clearly, $P \cap M \leq G$. Then $(G, P \cap M)$ satisfies the hypothesis of the proposition. The choice of $(G, P)$ yields that $P \cap M \leq Z_{\Omega}(G)$. Consequently, $P \leq Z_{\Omega}(G)$, which is impossible. Hence $T = G$. This implies that $H = H_G$ is $s$-permutable in $G$ by Lemma 2.2(1). Then by [24, Theorem], $P \leq Z_{\Omega}(G)$. The final contradiction ends the proof. □

Proof of Proposition 1.7. Suppose that the assertion is false and let $(G, E)$ be a counterexample for which $|G| + |E|$ is minimal. We now proceed via the following steps.

1. $O_{p'}(E) = 1$.

If $O_{p'}(E) \neq 1$, then by Lemma 2.3(2), $(G/O_{p'}(E), E/O_{p'}(E))$ satisfies the hypothesis of the proposition. The choice of $(G, E)$ implies that $E/O_{p'}(E) \leq Z_{\Omega}(G/O_{p'}(E)) = Z_{\Omega}(G)/O_{p'}(E)$, and so $E \leq Z_{\Omega}(G)$, a contradiction. Hence $O_{p'}(E) = 1$.

2. $E \leq G$.

Suppose that $E < G$. Then by Lemma 2.3(1), $(E, E)$ satisfies the hypothesis of the proposition. The choice of $(G, E)$ implies that $E$ is $p$-supersoluble. By (1) and Lemma 2.6, we see that $P \leq E$. Thus $P \leq G$. Then by Proposition 1.6, we have $P \leq Z_{\Omega}(G)$. Consequently, $E \leq Z_{\Omega}(G)$, which is absurd. Therefore, $E = G$.

3. $Z_{\Omega}(G)$ is the unique normal subgroup of $G$ such that $G/Z_{\Omega}(G)$ is a chief factor of $G$, $G^{\Omega} = G$ and $O_{p}(G) = Z(G) = Z_{\Omega}(G)$ is the Sylow $p$-subgroup of $Z_{\Omega}(G)$.

Let $G/K$ be a chief factor of $G$. Obviously, $(G, K)$ satisfies the hypothesis of the proposition. By the choice of the $(G, E)$, $K \leq Z_{\Omega}(G)$, and so $K = Z_{\Omega}(G)$. This shows that $Z_{\Omega}(G)$ is the unique normal subgroup of $G$ such that $G/Z_{\Omega}(G)$ is a chief factor of $G$. By Proposition 1.6, $O_{p}(G) \leq Z_{\Omega}(G) \leq Z_{\Omega}(G)$.

Proof of Proposition 1.7. Suppose that the assertion is false and let $(G, E)$ be a counterexample for which $|G| + |E|$ is minimal. We now proceed via the following steps.

1. $O_{p'}(E) = 1$.

If $O_{p'}(E) \neq 1$, then by Lemma 2.3(2), $(G/O_{p'}(E), E/O_{p'}(E))$ satisfies the hypothesis of the proposition. The choice of $(G, E)$ implies that $E/O_{p'}(E) \leq Z_{\Omega}(G/O_{p'}(E)) = Z_{\Omega}(G)/O_{p'}(E)$, and so $E \leq Z_{\Omega}(G)$, a contradiction. Hence $O_{p'}(E) = 1$.

2. $E \leq G$.

Suppose that $E < G$. Then by Lemma 2.3(1), $(E, E)$ satisfies the hypothesis of the proposition. The choice of $(G, E)$ implies that $E$ is $p$-supersoluble. By (1) and Lemma 2.6, we see that $P \leq E$. Thus $P \leq G$. Then by Proposition 1.6, we have $P \leq Z_{\Omega}(G)$. Consequently, $E \leq Z_{\Omega}(G)$, which is absurd. Therefore, $E = G$.

3. $Z_{\Omega}(G)$ is the unique normal subgroup of $G$ such that $G/Z_{\Omega}(G)$ is a chief factor of $G$, $G^{\Omega} = G$ and $O_{p}(G) = Z(G) = Z_{\Omega}(G)$ is the Sylow $p$-subgroup of $Z_{\Omega}(G)$.

Let $G/K$ be a chief factor of $G$. Obviously, $(G, K)$ satisfies the hypothesis of the proposition. By the choice of the $(G, E)$, $K \leq Z_{\Omega}(G)$, and so $K = Z_{\Omega}(G)$. This shows that $Z_{\Omega}(G)$ is the unique normal subgroup of $G$ such that $G/Z_{\Omega}(G)$ is a chief factor of $G$. By Proposition 1.6, $O_{p}(G) \leq Z_{\Omega}(G) \leq Z_{\Omega}(G)$.

\[ \text{Z}_d(G) \leq Z(G). \] Since \( O_{p'}(Z(G)) \leq O_{p'}(G) = 1 \) by (1) and (2), \( Z(G) \leq O_p(G) \). Therefore, \( O_p(G) = Z(G) = \text{Z}_d(G) \).

4 Final contradiction.

By (3), we have that \( G' = G \). If \( P \) is abelian, then by (3) and [13, Chap. VI, Theorem 14.3], \( Z(G) = 1 \). Hence by (3), \( Z_{\Phi}(G) \) is a \( p' \)-group. Then by (1) and (2), \( Z_{\Phi}(G) = 1 \), and so \( G \) is simple by (3) again. Let \( x \) be an element of \( G \) of order \( p \).

Then by the hypothesis, \( G \) has a subnormal subgroup \( T \) such that \( G = \langle x \rangle T \) and \( \langle x \rangle \cap T \leq \langle x \rangle \Phi(G) \). In this case, clearly, \( T = G \) and so \( \langle x \rangle \) is \( s \)-permutable in \( G \) by Lemma 2.2(1). Then \( \langle x \rangle \) is subnormal in \( G \) by Lemma 2.1(1). So \( G = \langle x \rangle \), which is impossible. Thus \( P \) is non-abelian.

By [13, Chap. IV, Satz 5.5], we see that there exists a cyclic subgroup \( H \) of \( P \) of order \( p \) or \( 4 \) which is not contained in \( Z(G) \). Then by the hypothesis, \( H \) is generalized \( \Phi \)-supplemented in \( G \). Thus \( G \) has a subnormal subgroup \( T \) such that \( G = HT \) and \( H \cap T \leq \Phi(H)H_{SG} \). If \( T < G \), then \( G \) has a normal subgroup \( M \) such that \( T \leq M \) and \( |G : M| = p \). It is easy to see that \( (G, M) \) satisfies the hypothesis of the proposition. The choice of \( (G, E) \) implies that \( M \leq \text{Z}_{\Phi}(G) \), and so \( G \leq \text{Z}_{\Phi}(G) \), which is impossible. Hence \( T = G \). Then \( H = H_{SG} \) is \( s \)-permutable in \( G \) by Lemma 2.2(1). Since \( H \nsubseteq Z(G) \) and \( Z(G) \) is the Sylow \( p' \)-subgroup of \( \text{Z}_{\Phi}(G) \) by (3), \( H \nsubseteq \text{Z}_{\Phi}(G) \). Hence by (3) and Lemma 2.1(3), we have that \( G = (\text{H}_{\text{Z}_{\Phi}(G)})^G = (\text{H}_{\text{Z}_{\Phi}(G)})^P \leq \text{PZ}_{\Phi}(G) \). But since \( G/\text{Z}_{\Phi}(G) \) is a chief factor of \( G \), \( |G/\text{Z}_{\Phi}(G)| = p \). This shows that \( G \) is \( p \)-supersoluble, a contradiction. This completes the proof.

Proof of Proposition 1.8. Suppose that the assertion is false and let \( (G, E) \) be a counterexample for which \( |G| + |E| \) is minimal. Then:

1. \( O_{p'}(E) = 1 \) and \( E = G \).

See steps (1) and (2) in the proof of Proposition 1.7.

2. Let \( N \) be a minimal normal subgroup of \( G \). Then either \( G/N \) is \( p \)-supersoluble or \( |G/N|_p = p \).

Suppose that \( M/N \) is a maximal subgroup of \( PN/N \). Then there exists a maximal subgroup \( P_1 \) of \( P \) such that \( M = P_1N \) and \( P \cap N = P_1 \cap N \). By the hypothesis, \( G \) has a subnormal subgroup \( T \) such that \( G = P_1T \) and \( P_1 \cap T \leq \Phi(P_1)(P_1)_{SG} \). Clearly, \( (|N : P_1 \cap N|, |N : T \cap N|) = 1 \). Hence \( N = (P_1 \cap N)(T \cap N) \), and so \( P_1N \cap TN = (P_1 \cap T)N \). By discussing similarly as in the proof of Lemma 2.3(2), \( M/N = P_1N/N \) is generalized \( \Phi \)-supplemented in \( G/N \). This shows that \( (G/N, G/N) \) satisfies the hypothesis of the proposition. The choice of \( (G, E) \) implies that either \( G/N \) is \( p \)-supersoluble or \( |G/N|_p = p \). Hence (2) holds.
(3) If $PN < G$, then $N \leq O_p(G)$.

By Lemma 2.3(1), $(PN, PN)$ satisfies the hypothesis of the proposition, and so the choice of $(G, E)$ implies that either $PN$ is $p$-supersoluble or $|PN|_p = p$. Then by (1), $N \leq O_p(G)$.

(4) $N$ is the unique minimal normal subgroup of $G$.

Let $N$ and $L$ be two distinct minimal normal subgroups of $G$. By (2), we may discuss the following three possible cases.

(i) If $G/N$ and $G/L$ are all $p$-supersoluble, then $G$ is $p$-supersoluble, a contradiction.

(ii) Without loss of generality, we may assume that $G/N$ is $p$-supersoluble and $|G/L|_p = p$. Since $LN/N$ is a minimal normal subgroup of $G/N$ and $p||L|$ by (1), $|L| = |LN/N| = p$, and so $|P| = p^2$. Then by (1), $|N|_p = |P \cap N| = p$ and $N$ is a non-abelian simple group. Let $N_1 = P \cap N$. Then $(N_1)_G = 1$ by Lemma 2.1(1). By the hypothesis, $N_1$ is generalized $S\Phi$-supplemented in $G$. Thus $G$ has a subnormal subgroup $T$ such that $G = N_1 T$ and $N_1 \cap T = 1$. Thus $T \leq G$. It follows that either $N \cap T = 1$ or $N \leq T$. For the former case, we have $N = N \cap N_1 T = N_1$, a contradiction. For the latter case, it follows that $N_1 = 1$, which is impossible.

(iii) Suppose that $|G/N|_p = p$ and $|G/L|_p = p$. Without loss of generality, we may assume that $N$ and $L$ are non-abelian simple groups. Then $P = (P \cap N)(P \cap L)$, and so $|P| = p^2$. Then with a similar discussion as above, we can derive a contradiction. Hence (4) holds.

(5) $N \not\leq \Phi(P)$.

Suppose that $N \leq \Phi(P)$. Then $N \leq \Phi(G)$. By (2), either $G/N$ is $p$-supersoluble or $|G/N|_p = p$. But the former case is clearly impossible. Hence we may assume that $|G/N|_p = p$. Then $|P/N| = p$. This implies that $P$ is cyclic, and so $|N| = p$. Then $|P| = p^2$. We show that $G/N$ is a non-abelian simple group. Let $A/N = O_p(G/N)$. Then $A \cap P \leq N \leq \Phi(P)$, and so $A$ is $p$-nilpotent by [13, Chap. IV, Satz 4.7]. It follows from (1) that $A = N$. Thus $O_p(G/N) = 1$. Suppose that $K/N$ is a chief factor of $G$. Then $|K/N|_p = p$, and so $P \leq K$. Obviously, $(G, K)$ satisfies the hypothesis of the proposition. If $K \leq G$, the choice of $(G, E)$ yields that $K \leq Z_{p\Phi}(G)$. Thus $G$ is $p$-supersoluble. This contradiction shows that $G = K$. Then $G/N$ is a non-abelian simple group. Since $|N| = p$, $G/C_G(N)$ is abelian, and so $C_G(N) = G$. It follows that $N \leq Z(G)$, which contradicts [13, Chap. VI, Satz 14.3].
With a similar argument as above, we have that $(P|N) = 1$.

Suppose that $O_p(G) \neq 1$. By (4), $N \leq O_p(G)$. If $G/N$ is $p$-supersoluble, then $N \not\leq \Phi(G)$. Therefore there exists a maximal subgroup $M$ of $G$ such that $G = NM$ and $N \cap M = 1$. Since $P = N(P \cap M)$, $P$ has a maximal subgroup $P_1$ containing $P \cap M$ and $P = NP_1$. If $(P_1)_{sG} \neq 1$, then by (4), Lemma 2.1(1) and Lemma 2.2(1), $N \leq ((P_1)_{sG})^G = ((P_1)_{sG})^P \leq P_1$, a contradiction. Thus $(P_1)_{sG} = 1$. Then by the hypothesis, there exists a subnormal subgroup $T$ of $G$ such that $G = P_1T$ and $P_1 \cap T \leq \Phi(P_1)$. Note that $N \leq O^p(G) \leq T$ by (4). Thus $P_1 \cap N \leq \Phi(P_1)$. This induces that $P_1 = (P_1 \cap N)(P \cap M) = P \cap M$. Hence $P_1 \cap N = 1$, and so $|N| = p$, a contradiction. Now assume that $|G/N|_p = p$. Then $|P/N| = p$. By (5), $P$ has a maximal subgroup $P_2$ such that $P = P_2N$. With a similar argument as above, we have that $(P_2)_{sG} = 1$. Therefore, by the hypothesis, there exists a subnormal subgroup $T$ of $G$ such that $G = P_2T$ and $P_2 \cap T \leq \Phi(P_2)$. Then clearly, $N \leq T$, and so $|G : T| = p$. This implies that $T \leq G$ and $T/N$ is a $p'$-group. Thus $G/N$ is $p$-supersoluble. This case has been dealt with in the above. Hence we have (6).

(7) Final contradiction.

By (3) and (6), we have that $G = PN$. If $P \leq N$, then $G$ is a non-abelian simple group. Let $P_1$ be a maximal subgroup of $P$. Then $P_1$ is generalized $S\Phi$-supplemented in $G$. It follows that $P_1 = (P_1)_{sG}$ is $s$-permutable in $G$ by Lemma 2.2(1), and so $P_1 = 1$ by Lemma 2.1(1). Thus $|G|_p = |P| = p$. This contradiction shows that $P$ has a maximal subgroup $P_2$ such that $P \cap N \leq P_2$. Then $(P_2)_{sG} = 1$ by (6), Lemma 2.1(1) and Lemma 2.2(1). Hence, by the hypothesis, $G$ has a subnormal subgroup $T$ such that $G = P_2T$ and $P_2 \cap T \leq \Phi(P_2) \leq \Phi(P)$. By [21, Lemma 2.5(7)], we have $O^p(G) \leq T$. Hence by (4), $N \leq O^p(G) \leq T$, and thereby $P \cap N = P_2 \cap N \leq \Phi(P)$. Then by [13, Chap. IV, Satz 4.7], $N$ is $p$-nilpotent, and so $N$ is a $p$-group by (1), which contradicts (6). The proof is thus completed.

**Proof of Theorem 1.4.** Suppose that the result is false and let $(G, E)$ be a counterexample for which $|G| + |E|$ is minimal. We now proceed via the following steps.

(1) $O_p'(E) = 1$ and $X = E = G$.

Suppose that $X < E$. Then clearly, $F^*_p(X) = F^*_p(E)$. Hence $(G, X)$ satisfies the hypothesis of the theorem. The choice of $(G, E)$ implies that $F^*_p(X) \leq X \leq Z_{p^l}(G)$, and so $E \leq Z_{p^{l+1}}(G)$ by Lemma 2.7. This contradiction shows that $X = E$. With a similar argument as in steps (1) and (2) in the proof of Proposition 1.7, we have that $O_p'(E) = 1$ and $E = G$. 

\[ \Box \]
(2) \( p < |D| < |P|/p. \)

It follows immediately from Propositions 1.7 and 1.8.

(3) If \( H \leq P \) and \( |H| = |D| \), then \( H \) is \( s \)-permutable in \( G \).

By the hypothesis, \( G \) has a subnormal subgroup \( T \) such that \( G = HT \) and \( H \cap T \leq \Phi(H)H_{sG} \). If \( T < G \), then there exists a normal subgroup \( M \) of \( G \) such that \( T \leq M \) and \( |G : M| = p \). Hence by (2), \( (G, M) \) satisfies the hypothesis of the theorem. The choice of \( (G, E) \) implies that \( M \leq Z_p(G) \), and so \( G \) is \( p \)-supersoluble, a contradiction. Thus \( T = G \). It follows that \( H = H_{sG} \) is \( s \)-permutable in \( G \) by Lemma 2.2(1).

(4) Final contradiction.

Let \( N \) be a minimal normal subgroup of \( G \). Then by (1), \( p \mid |N| \). If \( N \notin O_p(G) \), then we may take a subgroup \( H \) of \( P \) such that \( |H| = |D| \) and \( H \cap N = 1 \). By (3) and Lemma 2.1(1), \( H \cap N \leq O_p(N) = 1 \), a contradiction. Hence \( N \leq O_p(G) \). If \( |N| > |D| \), then \( N \) has a subgroup \( H \) such that \( H \leq P \) and \( |H| = |D| \).

By (3) and Lemma 2.1(3), \( H \trianglelefteq G \), a contradiction. Now assume that \( |N| = |D| \). Then by (2), there exists a subgroup \( V \) of \( P \) such that \( N < V < P \), \( V \trianglelefteq P \) and \( |V : N| = p \). If \( \Phi(V) = N \), then \( V \) is cyclic, and so \( |N| = p \), which contradicts (2). Thus \( \Phi(V) < N \). It follows that \( N \) has a subgroup \( N_1 \) such that \( \Phi(V) \leq N_1 < N \), \( N_1 \leq P \) and \( |N : N_1| = p \). Then \( V \) has a subgroup \( H \) such that \( |H| = |D| \) and \( H \cap N = N_1 \). By (3), \( N_1 \) is \( s \)-permutable in \( G \), and so \( N_1 \trianglelefteq G \) by Lemma 2.1(3).

Thus \( N_1 = 1 \), which implies that \( |N| = |D| = p \), which contradicts (2). Therefore, we have that \( |N| < |D| \).

If \( p > 2 \) or \( p = 2 \) and \( P/N \) is abelian or \( p = 2 \) and \( |D| > 2|N| \), then by Lemma 2.3(2), we see that \((G/N, G/N)\) satisfies the hypothesis of the theorem. Now assume that \( p = 2 \), \( P/N \) is non-abelian and \( |D| = 2|N| \). Then \( P \) is non-abelian. By (3) and Lemma 2.1(2), all subgroups of \( P/N \) of order 2 are \( s \)-permutable in \( G/N \). Let \( L/N \) be a cyclic subgroup of order 4 of \( P/N \). If \( N \leq \Phi(L) \), then \( L \) is cyclic, and so \( |D| = 2|N| = 4 \). By (3), all subgroups of \( P \) of order 4 are \( s \)-permutable in \( G \). For any subgroup \( K \) of \( P \) of order 2 with \( K \neq N \), \( NK \) is \( s \)-permutable in \( G \). Thus by Lemma 2.8, \( K \) is \( s \)-permutable in \( G \). Now by Proposition 1.7, we have that \( G \) is \( p \)-supersoluble, a contradiction.

Hence we may assume that \( N \notin \Phi(L) \). Then there exists a maximal subgroup \( L_1 \) of \( L \) such that \( L = L_1N \). Since \( |L_1| = |D| \), \( L/N = L_1N/N \) is \( s \)-permutable in \( G/N \) by (3) and Lemma 2.1(2). This shows that \((G/N, G/N)\) also satisfies the hypothesis of the theorem. Hence, by the choice of \((G, E)\), \( G/N \) is \( p \)-supersoluble. Then clearly, \( N \) is the unique normal subgroup of \( G \) and \( N \notin \Phi(G) \). It follows that \( G \) has a maximal subgroup \( M \) such that \( G = N \times M \). Since \( O_p(G) \cap M = 1 \),
\[ N = O_p(G), \] and so \( |N| \geq |D| \) by (3) and Lemma 2.1(1). The final contradiction completes the proof. □

4. Further applications

By Theorem 1.4, we can prove the following corollaries.

**Corollary 4.1.** Let \( E \) be a normal subgroup of \( G \) and \( P \) a Sylow \( p \)-subgroup of \( E \), where \( (|E|, p - 1) = 1 \). If \( P \) has a subgroup \( D \) such that \( 1 < |D| < |P| \), and all subgroups \( H \) of \( P \) with \( |H| = |D| \) and all cyclic subgroups of \( P \) of order 4 (when \( P \) is a non-abelian \( 2 \)-group and \( |D| = 2 \)) are generalized \( S\Phi \)-supplemented in \( G \), then \( E \) is \( p \)-nilpotent.

**Proof.** By Theorem 1.4, \( E \leq Z_{p^{\phi}}(G) \), and so \( E \) is \( p \)-supersoluble. Since \( (|E|, p - 1) = 1 \), we see that \( E \) is \( p \)-nilpotent. □

**Corollary 4.2.** Let \( E \) and \( X \) be normal subgroups of \( G \) such that \( F^*(E) \leq X \leq E \). If for any non-cyclic Sylow subgroup \( P \) of \( X \), \( P \) has a subgroup \( D \) such that \( 1 < |D| < |P| \), and all subgroups \( H \) of \( P \) with \( |H| = |D| \) and all cyclic subgroups of \( P \) of order 4 (when \( P \) is a non-abelian \( 2 \)-group and \( |D| = 2 \)) are generalized \( S\Phi \)-supplemented in \( G \), then \( E \leq Z_{p^{\phi}}(G) \).

**Proof.** By Lemma 2.3(2) and Corollary 4.1, we have that \( X \) has a Sylow tower of supersoluble type. If \( P \) is cyclic, then clearly, \( X \leq Z_{p^{\phi}}(G) \). Now assume that \( P \) is non-cyclic. Then by Theorem 1.4, \( X \leq Z_{p^{\phi}}(G) \) also holds. Therefore, \( F^*(E) \leq X \leq Z_{p^{\phi}}(G) \), and so \( E \leq Z_{p^{\phi}}(G) \) by Lemma 2.9. □

**Corollary 4.3.** Let \( E \) be a normal subgroup of \( G \) such that \( G/E \) is \( p \)-nilpotent and \( P \) a Sylow \( p \)-subgroup of \( E \) such that \( N_G(P) \) is \( p \)-nilpotent. If \( P \) has a subgroup \( D \) such that \( 1 < |D| < |P| \), and all subgroups \( H \) of \( P \) with \( |H| = |D| \) and all cyclic subgroups of \( P \) of order 4 (when \( P \) is a non-abelian \( 2 \)-group and \( |D| = 2 \)) are generalized \( S\Phi \)-supplemented in \( G \), then \( G \) is \( p \)-nilpotent.

**Proof.** Suppose that the result is false and let \((G, E)\) be a counterexample for which \(|G| + |E|\) is minimal. Assume that \( O_{p'}(E) \neq 1 \). Since

\[ N_{G/O_{p'}(E)}(PO_{p'}(E)/O_{p'}(E)) = N_{G}(P)O_{p'}(E)/O_{p'}(E), \quad (G/O_{p'}(E), E/O_{p'}(E)) \]

satisfies the hypothesis of the corollary by Lemma 2.3(2). The choice of \((G, E)\) implies that \( G/O_{p'}(E) \) is \( p \)-nilpotent, and so \( G \) is \( p \)-nilpotent, a contradiction.
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Hence $O_p(E) = 1$. Note that by Theorem 1.4, $E$ is $p$-supersoluble. Then by Lemma 2.6, $P \unlhd G$. Hence $G = N_G(P)$ is $p$-nilpotent, a contradiction. □

Note that Corollaries 4.1–4.3 generalize many known results, for example, [16, Theorems 3.1, 3.6, 3.11, 4.1, 4.3 and 4.4], [17, Theorems 3.1–3.5], [18, Theorems 1.2 and 1.3], [19, Theorems 3.1 and 3.2], [20, Theorem 1.4], [21, Theorems 1.3 and 1.4], [24, Theorem]. Moreover, we point out that [16, Theorem 3.9] and [19, Theorem 3.3] follow directly from Proposition 1.8.

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