Finite groups with restrictions on normal subgroups

By VICTOR S. MONAKHOV (Gomel) and IRINA L. SOKHOR (Gomel)

Abstract. We investigate the structure of finite soluble groups all whose proper normal subgroups belong to some class of groups, namely a Fitting class and the class of all supersoluble groups.

1. Introduction

All groups in this paper are finite.

Let $\mathfrak{F}$ be a class of groups. We say that a group $G$ is a minimal non-$\mathfrak{F}$-group or a $\mathfrak{F}$-critical group if $G \notin \mathfrak{F}$ but all proper subgroups of $G$ belong to $\mathfrak{F}$. G. Miller and H. Moreno [12] studied minimal non-abelian groups in 1903. Minimal non-nilpotent groups were first investigated by O. Schmidt [18]. Such groups are also called Schmidt groups, and their properties are well known [14], [17]. B. Huppert [7, Theorem 22], K. Doerk [4] and V. Nagrebeckij [16] studied minimal non-supersoluble groups, see also [2].

It is natural to study groups in which only some proper subgroups belong to a class $\mathfrak{F}$, for instance, normal subgroups or subgroups of prime index.

S. Levischenko [10] studied groups whose subgroups of non-prime index are nilpotent. V. Monakhov [13] described the structure of groups whose subgroups of non-prime index are supersoluble. G. Malanina and G. Shevcov [11] investigated the structure of a soluble group such that its commutator subgroup is nilpotent and all proper normal subgroups are supersoluble. L. Kazarin and Y. Korzukov [9] obtained the description of a soluble group with trivial Frattini subgroup and a supersoluble normal subgroup of prime index.

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In this paper we investigate soluble groups all whose proper normal subgroups belong to a Fitting class. We also obtain new properties of a soluble group all whose proper normal subgroups are supersoluble.

2. Preliminaries

By $A \leq B$ we denote that $A$ is a subgroup of a group $B$. If $A \leq B$ and $A \neq B$, then $A$ is called a proper subgroup of $B$ and denoted by $A < B$. Let $A$ and $B$ be subgroups of a group $G$, and $G = AB$. If $G \neq AB_1$ for every proper subgroup $B_1$ of $B$, then $B$ is called a minimal supplement to $A$ in $G$. By $G_p$ we denote a Sylow $p$-subgroup of a group $G$; $G'_p$ denotes a Hall $p'$-subgroup of $G$. We also use $\pi(G)$ to denote the set of all prime divisors of $|G|$.

Let $\pi_{ind}(G)$ be the set of all primes $p$ such that in $G$ there exists a normal subgroup of index $p$. If $G$ is a soluble group, then $\pi_{ind}(G) \neq \emptyset$.

By $A$, $N$ and $U$ we denote the class of all abelian, nilpotent and supersoluble groups, respectively. We use the term ”$F$-group” to denote a group belonging to a class of groups $F$. A class of groups $F$ is said to be $s$-closed if every subgroup of $G$ belongs to $F$ whenever $G \in F$. A class of groups $F$ is said to be $s_n$-closed if every subnormal subgroup of $G$ belongs to $F$ whenever $G \in F$.

A class of groups $F$ is a Fitting class if $F$ has the following two properties:

(i) if $G \in F$ and $H$ is a normal subgroup of $G$, then $H \in F$;
(ii) if $N_1$ and $N_2$ are normal subgroups of $G$ and $N_1, N_2 \in F$, then $N_1 N_2 \in F$.

If $\mathfrak{F}$ is a Fitting class, then the subgroup $G_{\mathfrak{F}} = \langle N \triangleleft G : N \in \mathfrak{F} \rangle$ is the largest normal $\mathfrak{F}$-subgroup of $G$, and it is called the $\mathfrak{F}$-radical of $G$.

A formation is a class of groups $\mathfrak{F}$ with the following two properties:

(i) if $G \in \mathfrak{F}$ and $N \triangleleft G$, then $G/N \in \mathfrak{F}$;
(ii) if $N_1$ and $N_2$ are normal subgroups of $G$ and $G/N_1, G/N_2 \in \mathfrak{F}$, then $G/N_1 \cap N_2 \in \mathfrak{F}$.

If $\mathfrak{F}$ is a formation, then the subgroup $G^\mathfrak{F} = \bigcap \{N \triangleleft G : G/N \in \mathfrak{F} \}$ is the smallest normal subgroup of $G$ with quotient in $\mathfrak{F}$, and it is called the $\mathfrak{F}$-residual of $G$.

A Fitting class which is also a formation is said to be a Fitting formation.

If $\mathfrak{X}$ and $\mathfrak{F}$ are classes of groups, we define their class product $\mathfrak{X} \mathfrak{F}$ as $\mathfrak{X} \mathfrak{F} = \{G \in \mathfrak{G} : N \triangleleft G, N \in \mathfrak{X} \Rightarrow G/N \in \mathfrak{F} \}$. Here $\mathfrak{G}$ denotes the class of all groups. Let $\mathfrak{X}$ be a class of group and $\mathfrak{F}$ a formation. We define $\mathfrak{X} \circ \mathfrak{F} = \{G \in \mathfrak{G} : G^\mathfrak{F} \in \mathfrak{X} \}$ and call $\mathfrak{X} \circ \mathfrak{F}$ the formation product of $\mathfrak{X}$ with $\mathfrak{F}$. Clearly, if $\mathfrak{X}$ is an $s_n$-closed class
of group, then $X\mathcal{F} = X \circ \mathcal{F}$. Therefore, if $X$ and $\mathcal{F}$ are formations, then $X\mathcal{F} = X \circ \mathcal{F}$ [5, p. 337].

All unexplained notations and terminology are standard. The reader is referred to [5], [8] if necessary.

We need the following results.

**Lemma 1** (see e.g. [3], [5]). Let $X$ and $\mathcal{F}$ be formations, let $G$ be a group, $H \leq G$ and $K \triangleleft G$. Then

1. $G \in \mathcal{F}$ if and only if $G \mathcal{F} = 1$;
2. if $X \subseteq \mathcal{F}$, then $G \mathcal{F} \leq G^X$; in particular, $G^u \leq G^\mathcal{N} \leq G^\mathcal{A} = G'$;
3. $(G/K)\mathcal{F} = G^\mathcal{F} K/K$;
4. $X\mathcal{F}$ is a formation;
5. $G^{(X\mathcal{F})} = (G^X)^\mathcal{F}$.

**Lemma 2.** Let $\mathcal{F}$ be an $s_n$-closed class and $G$ a soluble group. Then the following statements are equivalent:

1. every proper normal subgroup of $G$ belongs to $\mathcal{F}$;
2. every normal subgroup of $G$ of prime index belongs to $\mathcal{F}$;
3. the commutator subgroup $G'$ and every proper subgroup of $G$ containing $G'$ belong to $\mathcal{F}$.

**Proof.** Obviously, (1) implies (2) and (3).

Now we check that (1) follows from (2). Let $N$ be a proper normal subgroup of a soluble group $G$. In $G$ there exists a maximal normal subgroup $M$ such that $N \subseteq M$ and $|G : M| = p$ for some $p \in \pi(G)$. Then by the choice of $G$, $M \in \mathcal{F}$ and $N \in \mathcal{F}$ since $\mathcal{F}$ is an $s_n$-closed class.

Finally we prove that (1) follows from (3). Let $N$ be a proper normal subgroup of a soluble group $G$. There exists a maximal normal subgroup $M$ of $G$ such that $N \subseteq M$ and $|G : M| = p$ for some $p \in \pi(G)$. The quotient group $G/M$ is abelian and $G' \leq M$. Hence by the hypothesis, $M \in \mathcal{F}$ and $N \in \mathcal{F}$ since $\mathcal{F}$ is an $s_n$-closed class. □

**Lemma 3.** Let $\mathcal{F}$ be an $s_n$-closed class and $G$ a soluble group. Every proper normal subgroup of $G$ belongs to $\mathcal{F}$ if and only if $G = G'H$, where $H$ is a minimal supplement to $G'$ in $G$, and $G'K \in \mathcal{F}$ for every proper subgroup $K$ of $H$.

**Proof.** Let $\mathcal{F}$ be an $s_n$-closed class and $G$ a soluble group. Suppose that every proper normal subgroup of $G$ belongs to $\mathcal{F}$, and let $H$ be a minimal supplement to $G'$ in $G$. If $K$ is a proper subgroup of $H$, then $G'K$ is a proper normal subgroup of $G$. Therefore $G'K \in \mathcal{F}$. 

Conversely, suppose \( G = G'H \), where \( H \) is a minimal supplement to \( G' \) in \( G \), and \( G'K \in \mathcal{F} \) for every proper subgroup \( K \) of \( H \). If \( X \) is a proper subgroup of \( G \) such that \( G' \subseteq X \), then \( X = G'(X \cap H) \) and \( X \in \mathcal{F} \) since \( X \cap H < H \). By Lemma 2, every proper normal subgroup of \( G \) belongs to \( \mathcal{F} \).

\[ \square \]

**Lemma 4.** If every two elements of a group \( G \) generate a supersoluble subgroup, then \( G \) is supersoluble \([8, VI.9.18]\). 

**Lemma 5.** Let \( A \) and \( B \) be normal supersoluble subgroups of \( G \) and let \( G = AB \).

1. If the commutator subgroup \( G' \) is nilpotent, then \( G \) is supersoluble \([1]\).
2. If the indices \( |G : A| \) and \( |G : B| \) are relatively prime, then \( G \) is supersolvable \([6]\).

3. Soluble groups whose normal subgroups belong to a Fitting class

**Theorem 1.** Let \( \mathcal{F} \) be a Fitting class and \( G \) a soluble non-\( \mathcal{F} \)-group. Every proper normal subgroup of \( G \) belongs to \( \mathcal{F} \) if and only if the following two conditions hold:

1. \( \pi_{\text{ind}}(G) = \{p\} \) for some \( p \in \pi(G) \) and \( G = G'(x) \), where \( \langle x \rangle \) is a minimal supplement to \( G' \) in \( G \) and \( x \in G_p \);
2. \( G' = G^p \); \( G^p = G'(xp) \) and \( |G : G^p| = p \).

**Proof.** Let \( G \) be a soluble non-\( \mathcal{F} \)-group such that every proper normal subgroup of \( G \) belongs to a Fitting class \( \mathcal{F} \). By \([5, \text{Lemma II.2.10(a)}]\), \( G \) has a unique maximal normal subgroup \( M \). Therefore all proper normal subgroups of \( G \) are contained in \( M \), \( |G : M| = p \) and \( \pi_{\text{ind}}(G) = \{p\} \). In particular, \( G^p = M \) and \( G_p = G \). If \( G^p \cap G_p \neq G \), then \( G^p \cap G_p \) is a proper normal subgroup of \( G \), and \( G^p \cap G \leq G^p \cap G_p \), a contradiction. Hence \( G^p = G \) and \( G/G^p \) is a \( p \)-group. Since \( G/G^p \) has a unique maximal normal subgroup, \( G/G^p \) is a cyclic \( p \)-group. Let \( X \) be a subgroup of minimal order in \( G_p \) such that \( G = G^p X \). Then \( G^p \cap X \leq \Phi(X) \) \([15, 3.21]\). Since \( G/G^p \simeq X/G^p \cap X \) is a cyclic \( p \)-group, it follows that \( X = \langle x \rangle \) is also a cyclic \( p \)-group \([15, 3.20]\). Consequently, \( G = G^p \langle x \rangle \), where \( x \in G_p \). Since \( G^p \langle x^p \rangle \) is a proper normal subgroup of \( G \) of prime index \( p \), we have \( G^p \langle x^p \rangle = M = G^p \).

Note \( G^p \leq G' \). On the other hand, \( G' \leq G^p \), since \( G/G^p \) is a cyclic \( p \)-group. Thus, \( G^p = G' \).
Conversely, let $\mathfrak{F}$ be a Fitting class, $G \, a \, \text{soluble non-$\mathfrak{F}$-group satisfying (1) and (2). Then $G_{\mathfrak{F}}$ is the unique maximal normal subgroup of $G$ and so every proper normal subgroup of $G$ belongs to $\mathfrak{F}$.}

Let $r$ be a prime. A group $G$ is $r$-closed if its Sylow $r$-subgroup is normal. The class of all $r$-closed groups is an $s$-closed Fitting formation, and it coincides with the product $\mathfrak{N}, \mathfrak{E}_r$. Here $\mathfrak{N}$ is the class of all $r$-groups, $\mathfrak{E}_r$ is the class of all groups whose order is prime to $r$.

Taking $\mathfrak{F} = \mathfrak{N}, \mathfrak{E}_r$ in Theorem 1, we obtain

**Corollary 1.** Let $r$ be a prime and $G$ a non-$r$-closed soluble group. Every proper normal subgroup of $G$ is $r$-closed if and only if $G = G'(x)$, where $x \in G_r$, $(x)$ is a minimal supplement to $G'$ in $G$, $G'(x')$ is $r$-closed, and $|G : G'(x')| = r$.

**Proof.** Let $r$ be a prime and $G$ a non-$r$-closed soluble group such that every proper normal subgroup of $G$ is $r$-closed. Then every proper normal subgroup of $G$ belongs to $\mathfrak{N}, \mathfrak{E}_r$. Since $\mathfrak{N}, \mathfrak{E}_r$ is a Fitting formation, we can use Theorem 1. By Theorem 1(1), $G = G'(x)$, where $x \in G_p$ for some $p \in \pi(G)$. If $p \neq r$, then $|G : G'|$ is prime to $r$ and $G$ is $r$-closed since $G'$ is $r$-closed, contrary to the choice of $G$. Hence $p = r$. In view of Theorem 1(2), $G'(x')$ is $r$-closed and $|G : G'(x')| = r$.

Conversely, let $r$ be a prime and $G$ a non-$r$-closed soluble group such that $G = G'(x)$, where $x \in G_r$, and $G'(x')$ is $r$-closed. Then by Lemma 3, every proper normal subgroup of $G$ is $r$-closed. 

A group $G$ is $r$-nilpotent if there exists a Hall $r'$-subgroup $G_{r'}$ which is normal in $G$. The class of all $r$-nilpotent groups is an $s$-closed Fitting formation, and it coincides with the product $\mathfrak{E}_r, \mathfrak{N}_r$.

Taking $\mathfrak{F} = \mathfrak{E}_r, \mathfrak{N}_r$ in Theorem 1, we obtain

**Corollary 2.** Let $r$ be a prime and $G$ a non-$r$-nilpotent soluble group. Every proper normal subgroup of $G$ is $r$-nilpotent if and only if $G = G'(x)$, where $x \in G_p$ for some $p \in \pi(G) \setminus \{r\}$, $(x)$ is a minimal supplement to $G'$ in $G$, and $G'(x^p)$ is $r$-nilpotent.

**Proof.** Since $\mathfrak{E}_r, \mathfrak{N}_r$ is a Fitting formation, we can use Theorem 1. By Theorem 1(1) $G = G'(x)$, where $x \in G_p$ for some $p \in \pi(G)$. If $p = r$, then $G_{r'} \leq G'$. Since $G'$ $r$-nilpotent, we obtain $G_{r'}$ is normal in $G$ and $G$ is $r$-nilpotent. This contradicts to the choice of $G$. Therefore $p \neq r$. The conclusion follows from Theorem 1. 

\[ \square \]
Example 1. $A_4, E_4$ and 1 are the proper normal subgroups of the symmetric group $S_4$ of order 4. Therefore all proper normal subgroups of $S_4$ are 2-closed and 3-nilpotent. At the same time the commutator subgroup $(S_4)' = A_4$ is not a Hall subgroup.

A group $G$ is $r$-decomposable if it is $r$-closed and $r$-nilpotent. A class of all $r$-decomposable groups is an $s$-closed Fitting formation, and it coincides with the intersection $\mathfrak{N}_r, \mathfrak{E}_r \cap \mathfrak{E}_r, \mathfrak{N}_r$.

Taking $\mathfrak{F} = \mathfrak{N}_r, \mathfrak{E}_r \cap \mathfrak{E}_r, \mathfrak{N}_r$ in Theorem 1, we obtain

**Corollary 3.** Let $r$ be a prime and $G$ a non-$r$-decomposable soluble group. Every proper normal subgroup of $G$ is $r$-decomposable if and only if $G = [G'(x), G'(x^p)]$ is $r$-decomposable, and either $G$ is $r$-closed and $p \neq r$ or $G$ is $r$-nilpotent and $p = r$.

**Proof.** Since every proper normal subgroup of $G$ is $r$-closed and $r$-nilpotent, we can use Corollary 1 and 2. Hence $G = G'(x)$, where $x \in G_p$ for some $p \in \pi(G)$, and $G'(x^p)$ is $r$-decomposable. If $G$ is $r$-nilpotent, then it is non-$r$-closed and by Corollary 1 $p = r$. If $G$ is non-$r$-nilpotent, then, by Corollary 2 $p \neq r$ and by Corollary 1, $G$ is $r$-closed. \qed

Taking $\mathfrak{F} = \mathfrak{N}$ in Theorem 1, we obtain

**Corollary 4.** Let $G$ be a soluble non-nilpotent group. Every proper normal subgroup of $G$ is nilpotent if and only if the following two conditions hold:

1. $\pi_{ind}(G) = \{p\}$ for some $p \in \pi(G)$ and $G = [G'(x)]$, where $\langle x \rangle$ is a Sylow $p$-subgroup of $G$;
2. $G' = G^{\mathfrak{N}}$ and $F(G) = [G'(x^p)]$.

**Proof.** Let $G$ be a soluble non-nilpotent group all whose proper normal subgroup are nilpotent. When $\mathfrak{F} = \mathfrak{N}$, we can use Theorem 1. Since a Hall $p'$-subgroup $G_{p'}$ of $G$ is contained in $G^{\mathfrak{N}}$ and $G^{\mathfrak{N}}$ is nilpotent, it implies $G_{p'}$ is normal in $G$. Consequently, the quotient group $G/G_{p'}$ is a $p$-group and $G^{\mathfrak{N}} = G_{p'}$. Hence $G = [G^{\mathfrak{N}}(x)]$ and $\langle x \rangle$ is a Sylow $p$-subgroup of $G$. The conclusion follows from Theorem 1. \qed

4. Soluble groups with supersoluble normal subgroups

It is well known that the class $\mathfrak{U}$ is not a Fitting class. Therefore, we can not use Theorem 1 for $\mathfrak{F} = \mathfrak{U}$. 
Lemma 6. If every proper normal subgroup of a soluble non-supersoluble group $G$ is supersoluble, then the following statements hold:

1. $G/G^n$ is a $p$-group for some $p \in \pi(G)$; in particular, $\pi_{ind}(G) = \{p\}$;
2. $G/G'$ is cyclic or a direct product of two cyclic $p$-groups;
3. if the quotient group $G/G'$ is non-cyclic, then
   1. $G'$ is non-nilpotent;
   2. $G^{\text{id}} = (G')^{\text{ind}} \leq G''$;
   3. $G^{\text{p}} \leq F(G) < G'$; in particular, $G/G^{\text{p}}$ is non-abelian.

Proof. (1) Let $G$ be a soluble non-supersoluble group all whose proper normal subgroups are supersoluble. Suppose that $G/G^{\text{p}}$ is not primary. If $A/G^{\text{p}}$ is a Sylow $p$-subgroup and $B/G^{\text{p}}$ is a Hall $p'$-subgroup of $G/G^{\text{p}}$, then $A$ and $B$ are normal proper subgroups of $G$. Therefore, $A$ and $B$ are supersoluble. Thus $G = AB$ and $G$ is supersoluble by Lemma 5(2). This contradiction implies that $G/G^{\text{p}}$ is a $p$-group for some $p \in \pi(G)$. In particular, $\pi_{ind}(G) = \{p\}$.

(2) Since $G^{\text{p}} \leq G'$, $G/G'$ is an abelian $p$-group. Therefore, $G/G'$ can be decomposed as a direct product of cyclic $p$-subgroups:

$$G/G' = \langle a_1 \rangle \times \langle a_2 \rangle \times \ldots \times \langle a_t \rangle.$$  \hspace{1cm} (1)

If $\langle x, y \rangle G'$ is supersoluble for all $x, y \in G$, then $G$ is also supersoluble by Lemma 4. This contradiction implies that the subgroup $\langle x, y \rangle G'$ is not supersoluble for some $x, y \in G$. Since $\langle x, y \rangle G'$ is normal in $G$, it follows that $\langle x, y \rangle G'$ coincides with $G$ and $G/G'$ has at most two generators. Thus $t \leq 2$ in (1).

(3) Suppose that $G/G'$ is not cyclic. Let $A/G'$ and $B/G'$ be different subgroups of index $p$. Then $G = AB$, subgroups $A$ and $B$ are normal in $G$ and supersoluble. In view of Lemma 5(1), $G'$ is non-nilpotent. This implies that $(G')^{\text{p}} \neq 1$ by Lemma 1(1).

By Lemma 1(5) $G^{\text{p}} = (G^{\text{p}})^{\text{ind}} = (G')^{\text{ind}}$. As $G' \subseteq G^{\text{p}}$, it follows that $G^{\text{p}} = (G')^{\text{p}} \leq G^{\text{id}}$. Since

$$G/(G')^{\text{p}} = A/(G')^{\text{p}} \cdot B/(G')^{\text{p}}, \quad (G/(G')^{\text{p}})' = G'/(G')^{\text{p}};$$

$G/(G')^{\text{p}}$ is supersoluble by Lemma 5(1). Therefore, $G^{\text{id}} \leq (G')^{\text{ind}} \leq G''$. Thus, $G^{\text{id}} = (G')^{\text{p}} \leq G''$.

The subgroups $A'$ and $B'$ are normal in $G$ and nilpotent, it implies that $A'B' \leq F(G)$. The subgroups $A(A'B')/(A'B')$ and $B(A'B')/(A'B')$ of the quotient group $G/(A'B')$ are abelian and normal, so $G/(A'B')$ is nilpotent and $G^{\text{p}} \leq A'B' < G'$. Since $G^{\text{p}} \neq G'$, $G/G^{\text{p}}$ is non-abelian. \hfill $\square$
Statements (2) and (3.1) of Lemma 6 were first obtained in [11, Theorem 1]. We give the more concise proof.

**Theorem 2.** If every proper normal subgroup of a soluble non-supersoluble group $G$ is supersoluble, then the following statements hold:

1. If the quotient group $G/G^{\pi}$ is non-cyclic, then
   1.1) $G = [G^{\pi}]g_p$ for some $p \in \pi(G)$;
   1.2) $F(G) = G^{\pi} \times O_p(G)$;
   1.3) all proper subgroups of $G_p/O_p(G)$ are abelian;
   1.4) the subgroup $[G^{\pi}]P$ is supersoluble for all $P < G_p$;

2. If the quotient group $G/G^{\pi}$ is cyclic, then
   2.1) $\pi_{ind}(G) = \{p\}$ for some $p \in \pi(G)$ and $G = G^{\pi} \langle x \rangle$, where $\langle x \rangle$ is a minimal supplement to $G^{\pi}$ in $G$ and $x \in G_p$;
   2.2) $G^{\pi} = G'$ and $G^{\pi} \langle x^p \rangle$ is supersoluble.

Conversely, if Statements (1.1) and (1.4) or (2.1) and (2.2) hold for $G$, then every proper normal subgroup of $G$ is supersoluble.

**Proof.** Let $G$ be a soluble non-supersoluble group all whose proper normal subgroups are supersoluble. In view of Lemma 6(1), the quotient group $G/G^{\pi}$ is a $p$-group for some $p \in \pi(G)$. Since $G^{\pi} \leq G'$, $G/G'$ is an abelian $p$-group and $G = G'G_p$.

**Case 1.** The quotient group $G/G^{\pi}$ is non-cyclic.

Let $A/G^{\pi}$ and $B/G^{\pi}$ be different subgroups of index $p$. Hence $G = AB$ and the subgroups $A$ and $B$ are normal in $G$ and supersoluble. If $G'$ is nilpotent, then $G$ is supersoluble by Lemma 5(1). This contradiction implies that $G'$ is not nilpotent. Since the subgroups $A'$ and $B'$ are normal in $G$ and nilpotent, it follows that $A'B' \leq F(G)$. The subgroups $A(A'B')/(A'B')$ and $B(A'B')/(A'B')$ are abelian and normal in $G/(A'B')$. Then $G/(A'B')$ is nilpotent and $G^{\pi} \leq A'B'$. Therefore $G^{\pi}$ is a nilpotent Hall $p'$-subgroup of $G$ and $F(G) = G^{\pi} \times O_p(G)$. Since $G'$ is not nilpotent, $G^{\pi} < G'$ and $G_p$ is non-abelian.

Let $X$ be a maximal subgroup of $G_p$ such that $O_p(G) \subseteq X$. Hence $H = [G^{\pi}]X$ is normal in $G$ and supersoluble. This implies that $H'$ is normal in $G$ and nilpotent, therefore $H' \leq F(G) = G^{\pi} \times O_p(G)$. Consequently, $X/O_p(G)$ is abelian and all proper subgroups of $G_p/O_p(G)$ are abelian.

Let $P < G_p$ and $P_1$ be a maximal subgroup of $G_p$ such that $O_p(G) \subseteq P_1$. Then $[G^{\pi}]P_1$ is normal in $G$ and supersoluble. Hence $[G^{\pi}]P$ is supersoluble.

**Case 2.** The quotient group $G/G^{\pi}$ is cyclic.
Since \( G/G^{p_1} \) is cyclic and \( G^{p_1} \leq G' \), we obtain \( G^{p_1} = G' \). Suppose that \( X \) is a subgroup of the minimal order in \( G_p \) such that \( G = G'X \). Then \( G' \cap X \leq \Phi(X) \) [15, 3.21]. Hence \( G/G' \simeq X/G' \cap X \) is a cyclic \( p \)-group. It follows that \( X = \langle x \rangle \) is also a cyclic \( p \)-group [15, 3.20]. Consequently, \( G = G'(x) \), where \( x \in G_p \), and \( G'(x^p) \) is a supersoluble normal subgroup in \( G \) of index \( p \).

Conversely, let \( G \) be a a soluble non-supersoluble group satisfying (1.1) and (1.4) or (2.1) and (2.2). In view of Lemma 3, every proper normal subgroup of \( G \) is supersoluble.

\[ \square \]

**Corollary 5.** If \( G \) is a non-supersoluble group such that the commutator subgroup \( G' \) is nilpotent and all proper normal subgroups of \( G \) are supersoluble, then

1. \( G = [G',\langle x \rangle] \), where \( \langle x \rangle \) is a Sylow \( p \)-subgroup of \( G \) for some \( p \in \pi(G) \);
2. \( [G',\langle x^p \rangle] \) is supersoluble.

Conversely, if \( G \) is a group such that the commutator subgroup \( G' \) is nilpotent and Statements (1)–(2) hold, then all proper normal subgroups of \( G \) are supersoluble [11, Theorem 2].

**Proof.** Since \( G' \) is nilpotent and \( G^{p_1} \leq G' \), \( G^{p_1} \) is a Hall \( p' \)-subgroup of \( G \) by Lemma 6(1). In view of Lemma 6(3.1), the quotient group \( G/G^{p_1} \) is cyclic. The conclusion follows from Theorem 2(2). \[ \square \]

**References**


VICTOR S. MONAKHOV
DEPARTMENT OF MATHEMATICS
FRANCISK SKORINA GOMEL STATE UNIVERSITY
SOVETSKAYA STR., 104
GOMEL 246019
BELARUS

E-mail: victor.monakhov@gmail.com

IRINA L. SOKHOR
DEPARTMENT OF MATHEMATICS
FRANCISK SKORINA GOMEL STATE UNIVERSITY
SOVETSKAYA STR., 104
GOMEL 246019
BELARUS

E-mail: irina.sokhor@gmail.com

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