Pexiderization of some logarithmic functional equations

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Dedicated to the 60th birthday of Zsolt Páles

Abstract. We study some new logarithmic functional equations and their Pexiderizations on different structures.

1. Introduction

The functional equation

\[ f(xy) = f(x) + f(y) \]  \hspace{1cm} (CL)

with function \( f : \mathbb{R}_+ \to \mathbb{R} \) (or with function \( f : \mathbb{R}_0 \to \mathbb{R} \)) is usually called the Cauchy logarithmic functional equation. Here \( \mathbb{R}_+ \) is the set of positive elements in real numbers \( \mathbb{R} \) and \( \mathbb{R}_0 = \mathbb{R} \setminus \{0\} \).

Several works appeared on functional equations satisfied by logarithmic function, referred to as logarithmic functional equation.

In [8] and [4], for function \( f : \mathbb{R}_+ \to \mathbb{R} \), it is proved that functional equation

\[ f(x + y) - f(x) - f(y) = f \left( \frac{1}{x} + \frac{1}{y} \right) \]  \hspace{1cm} (1)

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and (CL) are equivalent in the sense that each solution of one equation is also solution of the other.

In [9], the authors add the functional equation

$$f(x + y) - f(xy) = f \left( \frac{1}{x} + \frac{1}{y} \right)$$

(2)

to the above list of equivalent equations by proving that (2) and (CL) are equivalent. In addition, the Pexider generalizations of (1) and (2) are considered in [9] in form

$$f(x + y) - g(x) - h(y) = k \left( \frac{1}{x} + \frac{1}{y} \right)$$

(3)

and

$$f(x + y) - g(xy) = h \left( \frac{1}{x} + \frac{1}{y} \right),$$

(4)

respectively for functions $f, g, h, k : \mathbb{R}_+ \to \mathbb{R}$.

In [3], the author gave a simple way to find the general solution of (4).

Then in [5], the equivalence of equations (1) and (2) was proved for function $f : K_0 \to A$, where $K_0 = K \setminus \{0\}$ ($K$ is a field excluding $\mathbb{Z}_2$) and $A$ is an Abelian group which has no 2-torsion.

In [10], the authors complemented the works [4], [8] and [9] mentioned above by solving a few other logarithmic functional equations in Pexider form.

Here we study two new logarithmic functional equations and their Pexiderizations for functions mapping $\mathbb{R}_+$ or $\mathbb{T}_+$ (where $\mathbb{T}_+$ is the set of positive elements in an ordered field $\mathbb{T}$) into $\mathbb{R}$ or into a uniquely 2-divisible Abelian group $A$.

### 2. The first new logarithmic equation

It is easy to see that any solution of (CL) is a solution of the functional equation

$$f(x + y) + f \left( \frac{x + y}{xy} \right) = f \left( \frac{(x + y)^2}{xy} \right)$$

(5)

for function $f : \mathbb{R}_+ \to \mathbb{R}$. We will prove the equivalence of equations (5) and (CL).

First, we present the general solution of the Pexiderized version

$$f(x + y) + g \left( \frac{x + y}{xy} \right) = h \left( \frac{(x + y)^2}{xy} \right)$$

(6)

of (5) for $x, y \in \mathbb{R}_+$.
Theorem 2.1. The functions \( f, g : \mathbb{R}_+ \to \mathbb{R} \) and \( h : D = \{ t \in \mathbb{R}_+ | t \geq 4 \} \to \mathbb{R} \) satisfy functional equation (6) for all \( x, y \in \mathbb{R}_+ \) if and only if they have the form

\[
\begin{align*}
  f(x) &= l(x) + a \quad (x \in \mathbb{R}_+), \\
  g(x) &= l(x) + b \quad (x \in \mathbb{R}_+), \\
  h(x) &= l(x) + a + b \quad (x \in D),
\end{align*}
\]

(7)

where \( l : \mathbb{R}_+ \to \mathbb{R} \) is a logarithmic function (i.e. satisfies (CL) for all \( x, y \in \mathbb{R}_+ \)) and \( a, b \in \mathbb{R} \) are arbitrary constants.

Proof. Assume that the functions \( f, g, h \) satisfy equation (6) for all \( x, y \in \mathbb{R}_+ \). Set \( x + y = t, \ (x + y)/(xy) = s \) in (6) to get

\[
f(t) + g(s) = h(ts) \quad (t, s \in \mathbb{R}_+, t \cdot s \geq 4). \tag{8}
\]

Let \( t, s \in \mathbb{R}_+ \) be arbitrary, then there exists \( u \in \mathbb{R}_+ \) such that \( uts \geq 4 \). Then we have, by (8),

\[
h(uts) = f(ut) + g(s) \quad \text{and} \quad h(uts) = f(u) + g(ts),
\]

so we get

\[
g(ts) - g(s) = f(ut) - f(u) : = \alpha(t) \quad (t, s \in \mathbb{R}_+).
\]

Hence \( g(ts) = \alpha(t) + g(s) \) for all \( t, s \in \mathbb{R}_+ \). Thus (see [1]) there exists a logarithmic function \( l : \mathbb{R}_+ \to \mathbb{R} \) such that

\[
g(t) = l(t) + b \quad \text{and} \quad \alpha(t) = l(t) \quad (t \in \mathbb{R}_+), \tag{9}
\]

where \( b \in \mathbb{R} \) is a constant.

Putting \( x = y = t/2 \) in (6), we obtain

\[
f(t) = -g \left( \frac{4}{t} \right) + h(4) = -l \left( \frac{4}{t} \right) - b + h(4) = l(t) + a \quad (t \in \mathbb{R}_+), \tag{10}
\]

where \( a \in \mathbb{R} \) is a constant.

Finally, we get from (6), (10) and (9) that

\[
h \left( \frac{(x + y)^2}{xy} \right) = f(x + y) + g \left( \frac{x + y}{xy} \right)
\]

\[
= l(x + y) + a + l \left( \frac{x + y}{xy} \right) + b = l \left( \frac{(x + y)^2}{xy} \right) + a + b \quad (x, y \in \mathbb{R}_+),
\]

so we have \( h(t) = l(t) + a + b \) for all \( t \geq 4 \), since \( t = (x + y)^2/(xy) = x/y + y/x + 2 \geq 4 \).

The converse can be easily obtained by a simple calculation. \( \Box \)
Corollary 2.1. Functional equation (5) and (CL) are equivalent.

Proof. In case $f = g = h$, Theorem 2.1 implies that $a = b = 0$, thus the only solution of (5) is the logarithmic function $f(x) = l(x)$ for all $x \in \mathbb{R}_+$. The converse is easy to check. \qed

To generalize Theorem 2.1, we need the following

Lemma 2.1 (see [6]). If $S$ is a non-empty set and $f, g : T_+ \to S$ are functions such that
\[
f(x + y) = g(xy) \quad (x, y \in T_+),
\]
then $f$ and $g$ are constant.

Proof. Let $\mu \in T_+ (\mu > 1)$ be arbitrary. Replacing $x$ by $\mu x$ and $y$ by $(1/\mu)y$ in (11), we find that
\[
f \left( \frac{\mu x + 1}{\mu} \right) = g(xy) = f(x + y) \quad (x, y \in T_+). \tag{12}
\]
Let $u \in T_+$ be arbitrary and choose $x, y \in T_+$ and $\mu > 1$ such that $\mu x + (1/\mu)y = 1$ and $x + y = u$. This system of equations is satisfied if and only if
\[
x = \frac{\mu - u}{\mu^2 - 1} \quad \text{and} \quad y = \frac{u^2u - \mu}{\mu^2 - 1}.
\]
Let $\mu = u + 1/u$, then $x = u/(u^4 + u^2 + 1)$, $y = (u^5 + u^3)/(u^4 + u^2 + 1)$. Putting these in (12), we have
\[
f(u) = f(1) \quad (u \in T_+).
\]
Thus $f$ is constant on $T_+$, and hence so is $g$. \qed

Remark 2.1. In the case $T_+ = \mathbb{R}_+$ this lemma was proved in [2].

Lemma 2.2. Let $A$ be an Abelian group. The functions $f, g, K : T_+ \to A$ satisfy functional equation
\[
f(x + y) + g \left( \frac{x + y}{xy} \right) = K \left( \frac{x}{y} \right) \quad (x, y \in T_+) \tag{13}
\]
if and only if
\[
f(x) = l(x) + a \quad (x \in T_+),
\]
\[
g(x) = l(x) + b \quad (x \in T_+),
\]
\[
K(x) = l \left( x + \frac{1}{x} + 2 \right) + a + b \quad (x \in T_+) \tag{14}
\]
where the function $l : T_+ \to A$ satisfies the Cauchy logarithmic equation (CL) for all $x, y \in T_+$, and $a, b \in A$ are arbitrary constants.
Proof. Assume that $f, g, K$ satisfy (13) for all $x, y \in \mathbb{T}_+$. Replacing $x$ and $y$ by $x/2$ in (13), we have that

$$f(x) + g\left(\frac{4}{x}\right) = K(1) \quad (x \in \mathbb{T}_+),$$

which gives that

$$g\left(\frac{x + y}{xy}\right) = -f\left(\frac{4xy}{x + y}\right) + K(1) \quad (x, y \in \mathbb{T}_+).$$

Applying (16) in (6), we get

$$f(x + y) - f\left(\frac{4xy}{x + y}\right) + K(1) = K\left(\frac{x}{y}\right) \quad (x, y \in \mathbb{T}_+).$$

Now let $\lambda \in \mathbb{T}_+$ be arbitrary. The equation (17) shows that

$$\Delta_\lambda f(x + y) = \Delta_\lambda f\left(\frac{4xy}{x + y}\right) \quad (x, y \in \mathbb{T}_+),$$

where the function $\Delta_\lambda f : \mathbb{T}_+ \to A$ is defined by

$$\Delta_\lambda f(x) = f(\lambda x) - f(x) \quad (x \in \mathbb{T}_+).$$

Replace $x$ by $x(x + y)/4$ and $y$ by $y(x + y)/4$ in (18), we obtain that

$$F_\lambda(x + y) = \Delta_\lambda f(xy) \quad (x, y \in \mathbb{T}_+)$$

with function $F_\lambda : \mathbb{T}_+ \to A$ defined by

$$F_\lambda(x) = \Delta_\lambda f\left(\left(\frac{x}{2}\right)^2\right) \quad (x \in \mathbb{T}_+).$$

From (19), by Lemma 2.1, we can infer that the function $\Delta_\lambda f$ is constant, that is for all $\lambda \in \mathbb{T}_+$ there exists a constant $c(\lambda) \in A$ such that $\Delta_\lambda f(x) = c(\lambda)$ for all $x \in \mathbb{T}_+$. This equality and the definition of $\Delta_\lambda f$ imply that

$$f(\lambda x) - f(x) = c(\lambda) \quad (\lambda, x \in \mathbb{T}_+).$$

From (20), by the substitution $x = 1$, we obtain that $c(\lambda) = f(\lambda) - f(1)$ for all $\lambda \in \mathbb{T}_+$. This equation and (20) give that $f(\lambda x) = f(x) + f(\lambda) - f(1)$ for all
$x, \lambda \in \mathbb{T}_+$, which shows that there exists a logarithmic function $l : \mathbb{T}_+ \to A$, such that

$$f(x) = l(x) + a \quad (x \in \mathbb{T}_+),$$

(21)

where $a = f(1) \in A$ is a constant.

Now (15) implies that

$$g(x) = -f\left(\frac{4}{x}\right) + K(1) = -l\left(\frac{4}{x}\right) - a + K(1) = l(x) + b \quad (x \in \mathbb{T}_+),$$

(22)

with constant $b = K(1) - a - l(4) \in A$, thus $f$ and $g$ is of the form as in (14).

Finally, setting $y = 1$ in (13), from (21) and (22) we get that

$$K(x) = f(x + 1) + g\left(\frac{x + 1}{x}\right) = l(x + 1) + a + l\left(\frac{x + 1}{x}\right) + b$$

$$= l\left(\frac{(x + 1)^2}{x}\right) + a + b = l\left(x + \frac{1}{x} + 2\right) + a + b \quad (x, y \in \mathbb{T}_+),$$

which completes the proof. The converse is easy to check.

**Theorem 2.2.** Let $D = \{ t \in \mathbb{T}_+ | \exists u \in \mathbb{T}_+ : t = u + \frac{1}{u} + 2 \}$ and $A$ be an Abelian group. The functions $f, g : \mathbb{T}_+ \to A$, $h : D \to A$ satisfy functional equation (6) for all $x, y \in \mathbb{T}_+$ if and only if

$$f(x) = l(x) + a \quad (x \in \mathbb{T}_+),$$

$$g(x) = l(x) + b \quad (x \in \mathbb{T}_+),$$

$$h(x) = l(x) + a + b \quad (x \in D),$$

(23)

where the function $l : \mathbb{T}_+ \to A$ satisfies the Cauchy logarithmic equation (CL) for all $x, y \in \mathbb{T}_+$, and $a, b \in A$ are arbitrary constants.

**Proof.** Assume that functions $f, g, h$ satisfy (6) for all $x, y \in \mathbb{T}_+$. Then one can easily see that functions $f, g$ and function $K : D \to A$ defined by

$$K(x) = h\left(x + \frac{1}{x} + 2\right) \quad (x \in \mathbb{T}_+)$$

satisfy the functional equation (13). It follows from Lemma 2.2 that $f, g, K$ are of the form (14). Finally, (14) and the definition of $K$ imply (23) for functions $f, g, h$. Conversely, functions in (23) indeed satisfy (6).

**Corollary 2.2.** Functional equations (5) and (CL) are equivalent for function $f : \mathbb{T}_+ \to A$, too.

**Proof.** See the proof of Corollary 2.1.
3. The second new logarithmic equation

One can easily see that any solution of (CL) is a solution of the functional equation

\[ f(x(y+1)) + f(y(x+1)) = f(x(x+1)) + f(y(y+1)) \quad (x, y \in \mathbb{T}_+) \] (24)

for function \( f : \mathbb{T}_+ \to A \). We will prove the equivalence of equations (24) and (CL).

To do this, consider the functional equation

\[ f(y(x+1)) + g(x(y+1)) = h(x) + h(y) \quad (x, y \in \mathbb{T}_+) \] (25)

for functions \( f, g, h : \mathbb{T}_+ \to A \).

**Theorem 3.1.** Let \( A \) be a uniquely 2-divisible Abelian group. The functions \( f, g, h : \mathbb{T}_+ \to A \) satisfy the functional equation (25) if and only if

\[
\begin{align*}
  f(x) &= l(x) + a & (x \in \mathbb{T}_+), \\
  g(x) &= l(x) + b & (x \in \mathbb{T}_+), \\
  h(x) &= l(x(x+1)) + \frac{a+b}{2} & (x \in \mathbb{T}_+),
\end{align*}
\] (26)

where \( l : \mathbb{T}_+ \to A \) is a logarithmic function (that is \( l \) satisfies (CL)), and \( a, b \in A \) are arbitrary constants.

**Proof.** Suppose that \( f, g, h : \mathbb{T}_+ \to A \) satisfy (25) for all \( x, y \in \mathbb{T}_+ \). Replace \( y \) by \( 1/y \) in (25), we get

\[ f \left( \frac{x+1}{y} \right) + g \left( x \left( \frac{1}{y} + 1 \right) \right) = h(x) + h \left( \frac{1}{y} \right) \quad (x, y \in \mathbb{T}_+). \] (27)

Interchange \( x \) and \( y \) in (27) and deduce that

\[ f \left( \frac{y+1}{x} \right) + g \left( y \left( \frac{1}{x} + 1 \right) \right) = h \left( \frac{1}{x} \right) + h(y) \quad (x, y \in \mathbb{T}_+). \] (28)

Adding equations (27) and (28), we get that

\[
\begin{align*}
  f \left( \frac{x+1}{y} \right) + f \left( \frac{y+1}{x} \right) + g \left( y \left( \frac{1}{x} + 1 \right) \right) + g \left( x \left( \frac{1}{y} + 1 \right) \right) \\
  &= h(x) + h \left( \frac{1}{x} \right) + h(y) + h \left( \frac{1}{y} \right) \quad (x, y \in \mathbb{T}_+). \end{align*}
\] (29)
Replace $x$ by $x/y$ and $y$ by $1/y$ in (29), we obtain that
\[
f(x + y) + f\left(\frac{y + 1}{x}\right) + g\left(x\frac{1}{y} + 1\right) + g\left(x + y\frac{1}{xy}\right) = h\left(\frac{x}{y}\right) + h\left(\frac{y}{x}\right) + h(y) + h\left(\frac{1}{y}\right) \quad (x, y \in \mathbb{T}_+). \quad (30)
\]

Interchange here $x$ and $y$ and add the resulting equation to equation (30) to get
\[
2f(x + y) + f\left(\frac{x + 1}{y}\right) + f\left(\frac{y + 1}{x}\right) + g\left(x\frac{1}{y} + 1\right) + g\left(y\frac{1}{x} + 1\right) + 2g\left(x + y\frac{1}{xy}\right) = 2h\left(\frac{x}{y}\right) + 2h\left(\frac{y}{x}\right) + h(x) + h\left(\frac{1}{x}\right) + h(y) + h\left(\frac{1}{y}\right) \quad (x, y \in \mathbb{T}_+). \quad (31)
\]

Comparing equations (29) and (31) and using the uniquely 2-divisibility of $A$, we see that functions $f, g, h$ satisfy the functional equation
\[
f(x + y) + g\left(\frac{x + y}{xy}\right) = h\left(\frac{x}{y}\right) + h\left(\frac{y}{x}\right) \quad (x, y \in \mathbb{T}_+). \quad (32)
\]

It follows that the functions $f, g$ and the function $K : \mathbb{T}_+ \to A$ defined by
\[
K(x) = h(x) + h\left(\frac{1}{x}\right) \quad (x \in \mathbb{T}_+)
\]
satisfy functional equation (13) in Lemma 2.2. Then Lemma 2.2 shows that $f$ and $g$ are of the form (26).

Finally, from (25), with substitution $x = y$, using the uniquely 2-divisibility of $A$ and the form of $f, g$, we get that
\[
h(x) = \frac{1}{2} [f(x(x + 1)) + g(x(x + 1))] = \frac{1}{2} [l(x(x + 1)) + a + l(x(x + 1)) + b] = l(x(x + 1)) + \frac{a + b}{2} \quad (x, y \in \mathbb{T}_+),
\]
which gives (26) for $h$, too. The converse is evident again.

**Corollary 3.1** (see [6]). Let $A$ be a uniquely 2-divisible Abelian group. The function $f : \mathbb{T}_+ \to A$ satisfies (24) for all $x, y \in \mathbb{T}_+$ if and only if $f(x) = l(x) + a$ for all $x \in \mathbb{T}_+$, where $l : \mathbb{T}_+ \to A$ satisfies (CL) for all $x, y \in \mathbb{T}_+$, and $a \in A$ is an arbitrary constant. Furthermore, (24) with condition $f(1) = 0$ and (CL) are equivalent for function $f : \mathbb{T}_+ \to A$. 

Pexiderization of some logarithmic functional equations

Proof. $f$ satisfies (25) with $g = f$ and $h(x) = f(x(x + 1))$. Thus Theorem 3.1 gives that $f(x) = l(x) + a$ for all $x \in T_+$, where $l : T_+ \to A$ satisfies (CL) for all $x, y \in T_+$, and $a \in A$ is an arbitrary constant. This proves the first part of our Corollary. If $f(1) = 0$, then $a = 0$, thus we have that $f(x) = l(x)$ for all $x \in T_+$, that is, $f$ satisfies (CL). The converse is easy to see. □

Remark 3.1. In case $T_+ = \mathbb{R}_+, A = \mathbb{R}$, Theorem 3.1 and Corollary 3.1 imply that (24) with condition $f(1) = 0$ and (CL) are equivalent for function $f : \mathbb{R}_+ \to \mathbb{R}$.

4. A third new logarithmic functional equation

As a counterpart of equation (24), we recall our former result [7, Theorem 2]:

Theorem 4.1. Let $A$ be a uniquely 2-divisible Abelian group. The function $\gamma : T_+ \to A$ satisfies the functional equation

$$
\gamma \left( \frac{x + 1}{y} \right) + \gamma \left( \frac{y + 1}{x} \right) = \gamma \left( \frac{x + 1}{x} \right) + \gamma \left( \frac{y + 1}{y} \right)
$$

(33)

if and only if it is of the form

$$
\gamma(x) = l(x) + c \quad (x \in T_+)
$$

where $l : T_+ \to A$ satisfies (CL) for all $x, y \in T_+$, and $c \in A$ is an arbitrary constant.

Now, one can easily derive from Theorem 4.1 the following

Corollary 4.1. (33), with condition $\gamma(1) = 0$, and (CL) are equivalent for function $f : T_+ \to A$ (or for function $f : \mathbb{R}_+ \to \mathbb{R}$).

References


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