An elementary proof for the time-monotonicity of the solutions of linear parabolic equations

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In Banach spaces, POLÁČIK [12] has recently investigated the monotonicity properties with respect to the time variable of solutions of semilinear parabolic problems of form

$$\begin{cases} u' + Au = f(u) \\ u(0) = u_0, \end{cases}$$

where $A$ is a sectorial operator, $f$ is smooth enough and the domain of the fractional power $A^\alpha$ is strongly ordered for some $\alpha$. Later MIERCZYŃSKI [8] generalized Poláčik’s result for $C^1$ strongly monotone semiflows. They proved that under certain conditions the set of points near the equilibrium point having not eventually strongly monotone trajectories lie on a manifold of co-dimension one.

Both the above mentioned papers include the case of the present paper as certain linear parabolic equations are treated here using the technique of [11] to obtain a new elementary proof.

Let $n \in \mathbb{N}^+, \, \Omega \subset \mathbb{R}^n$ be a bounded domain, $\partial \Omega$ belonging to the Hölder class $C^{2+\alpha}$ for some positive $\alpha$, and $L$ the following symmetric second order linear differential operator

$$Lu := \sum_{i,j=1}^{n} \partial_i (a_{ij} \partial_j u) + du,$$

where $a_{ij} \in C^{1+\alpha}(\bar{\Omega})$, $a_{ij} = a_{ji}$, $i, j = 1, \ldots, n$; $d \in C^{\alpha}(\bar{\Omega})$, $d \leq 0$ and suppose that $L$ is uniformly elliptic in $\Omega$, i.e. there exists a positive number

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κ such that for every \( \zeta \in \mathbb{R}^n \)

\[
\kappa |\zeta|^2 \leq \sum_{i,j=1}^{n} a_{ij} \zeta_i \zeta_j.
\]

Let \( Q := (0, +\infty) \times \Omega, \Gamma := [0, +\infty) \times \partial \Omega \) and \( \Omega_0 := \{0\} \times \Omega \).

It is well-known [7] that there exists a sequence of solutions of the classical eigenvalue problem

\[
\begin{align*}
L w + \lambda w &= 0 \quad \text{in } \Omega \\
w &= 0 \quad \text{on } \partial \Omega \\
w &\in C^{2+\alpha}(\bar{\Omega})
\end{align*}
\]

Denote the sequence of eigenvalues by \( \lambda_k, k \in \mathbb{N}^+ \) (let them form a monotone nondecreasing sequence) and the corresponding eigenfunctions normed in \( L^2(\Omega) \) by \( w_k \).

Let \( \varphi \in L^2(\Omega) \) be a given function. We examine the generalized solution of the initial-boundary value problem

\[
\begin{align*}
\partial_0 u - Lu &= 0 \quad \text{in } Q \\
u |_{\Gamma} &= 0 \\
u |_{\Omega_0} &= \tilde{\varphi} \\
u &\in H^{0,1}(Q)
\end{align*}
\]

where \( \tilde{\varphi}(0, x) := \varphi(x) \) for \( x \in \Omega \). For the definition of \( H^{0,1}(Q) \) see e.g. [14]. Let \( \xi_k := \int_{\Omega} \varphi w_k \), \( k \in \mathbb{N}^+ \).

We recall that there exists a unique weak solution of (2),

\[
u(t, x) = \sum_{k=1}^{\infty} \xi_k e^{-\lambda_k t} w_k(x), \quad (t, x) \in Q
\]

(convergence is understood in the norm of \( H^{0,1}(Q) \)) and it is smooth in \( Q \setminus \Omega_0 \) (see e.g. [14]). If \( \varphi \in C^{2+\alpha}(\Omega) \) then \( u \in C^{1+\alpha/2,2+\alpha}(Q) \) [3].

Results of Narasimhan [10] and Friedman [3] claim a solution of the classical initial-boundary value problem corresponding to (2) with \( \varphi \in C(\Omega) \) tends to zero uniformly in \( \Omega \) as \( t \) tends to infinity.

Under weaker conditions on the coefficients of \( L \) and \( \partial \Omega \) we have proved [11] for any \( \varphi \in L^2(\Omega) \) and fixed \( x \in \Omega \) the monotonicity of the function \( t \mapsto u(t, x) \) for \( t \) large enough. Moreover, we have shown that for any compact subset \( K \) of \( \Omega \) there exists a positive number \( T \) such that for
every \( x \in K \) the function \( t \mapsto u(t, x) \) is monotone in \([T, +\infty)\) provided the first Fourier coefficient of \( \varphi \) is not equal to zero.

Now under the given stronger conditions which ensure the existence of eigenfunctions in the classical sense we prove the same result instead of a compact subset for the whole \( \Omega \).

Due to the theorem of Krein and Rutman ([2], [6]) the principal eigenvalue \( \lambda_1 \) of \( L \) is simple and the corresponding eigenfunction \( w_1 \) does not vanish in \( \Omega \), thus it can be chosen a positive function in \( \Omega \).

**Theorem 1.** Let \( u \) be the (unique) weak solution of the initial-boundary value problem (2) with the conditions given previously. Suppose that the first Fourier coefficient \( \xi_1 \) of \( \varphi \) is not equal to zero. Then there exists a positive number \( T \) such that for every \( x \in \Omega \) the function \( t \mapsto u(t, x), t > T \) is strictly decreasing if \( \xi_1 > 0 \), and strictly increasing if \( \xi_1 < 0 \).

**Proof.** Theorem 3 in [11] gives the following estimate for the maximum of the absolute value of \( w_k \):

\[
\max_{\bar{\Omega}} |w_k| \leq M^* \lambda_k^*, \quad k \in \mathbb{N}^+,
\]

where \( M^* \) and \( s^* \) are appropriate positive constants independent of \( k \).

Therefore we have

\[
\partial_0 u(t, x) = -e^{-\lambda_1 t}w_1(x) \left( \xi_1 \lambda_1 + e^{-(\lambda_2 - \lambda_1)t} \sum_{k=2}^\infty \xi_k \lambda_k e^{-(\lambda_k - \lambda_2)t} \cdot \frac{w_k(x)}{w_1(x)} \right),
\]

where \( (t, x) \in Q \).

First, we examine term

\[
\frac{w_k(x)}{w_1(x)}.
\]

Under our assumptions the outward normal derivative of \( w_1 \) does not vanish on \( \partial \Omega \) (see e.g. [4] or [13]), thus there exists a positive \( \varepsilon \) such that

\[
\partial_\nu w_1 \leq -\varepsilon \quad \text{on} \quad \partial \Omega.
\]

For every \( y \in \partial \Omega \) let us take an open, convex neighbourhood \( U_y \subset \mathbb{R}^n \) such that in a system of coordinates chosen appropriately \( U_y \cap \partial \Omega \) is the graph of a function belonging to the \( C^{2+\alpha} \) class. We can take \( U_y \) such that \( \partial_\nu w_1 \leq -\varepsilon/2 \) is valid in \( U_y \cap \Omega \) since \( w_1 \in C^1(\Omega) \). In addition we may assume for every \( x \in U_y \cap \Omega \) the existence of a point \( \beta_x \in \partial \Omega \) such
that the direction $\beta_x - x$ coincides with the outward normal direction at $\beta_x$ (e.g. let $\min\{|\beta - x| : \beta \in \partial \Omega\}$ be attained at $\beta_x$). From the open cover

$$\partial \Omega \subset \bigcup_{y \in \partial \Omega} U_y$$

we can select a finite cover $\{U_{y_1}, \ldots, U_{y_N}\}$. Let $K$ be the following compact set:

$$K := \Omega \setminus \bigcup_{i=1}^{N} U_{y_i}.$$ 

With $\delta := \min\{w_1(x) : x \in K\}$ we have

$$(7) \quad \frac{|w_k(x)|}{w_1(x)} \leq \frac{M^*}{\delta} \lambda_k^* \quad \text{for } x \in K.$$ 

Now we will examine term (5) near the boundary. Due to the homogeneous Dirichlet boundary condition we can write

$$\frac{|w_k(x)|}{w_1(x)} = \frac{|w_k(x) - w_k(\beta_x)|}{w_1(x) - w_1(\beta_x)} = \left| \frac{\partial_x w_k(\eta_x)}{\partial_x w_1(\eta_x)} \right| \quad \text{for } x \in \Omega \cap \left( \bigcup_{i=1}^{N} U_{y_i} \right),$$

where $\beta_x \in \partial \Omega$, the direction $\beta_x - x$ coincides with the outward normal direction $\nu$, and $\eta_x$ is an appropriate point in the segment $(\beta_x, x)$. Therefore, by using (6) we have

$$(8) \quad \frac{|w_k(x)|}{w_1(x)} \leq \frac{2}{\varepsilon} \max_{\Omega} |\nabla w_k| \leq \frac{2}{\varepsilon} \|w_k\|_{C^1(\bar{\Omega})} \quad \text{in } \Omega \cap \left( \bigcup_{i=1}^{N} U_{y_i} \right).$$

Ladyženskaja and Ural’ceva [7] proved boundedness in $C^1(\bar{\Omega})$-norm for the solution of the generalized elliptic boundary value problem under certain conditions. By using their proof we have obtained a bound in $C^1(\bar{\Omega})$-norm for the solution $w_k$ of the eigenvalue problem (1) depending on the eigenvalue $\lambda_k$. In the Appendix we have shown the existence of positive numbers $N^*$ and $r^*$ such that

$$(9) \quad \|w_k\|_{C^1(\bar{\Omega})} \leq N^* \lambda_k^*^{r^*}, \quad k \in \mathbb{N}^+.$$ 

(For the details see Theorem 2.)

By using estimates (7), (8) and (9) we obtain

$$\left| \frac{w_k(x)}{w_1(x)} \right| \leq C \lambda_k^g, \quad k \in \mathbb{N}^+, \quad x \in \Omega$$

for
where $C := \max \left\{ \frac{2N^*}{\varepsilon}, \frac{M^*}{\delta} \right\}$ and $\sigma := \max \{r^*, s^*\}$.

Finally we examine the series in (4) as it was done in [11].

\begin{equation}
\left| \sum_{k=2}^{\infty} \xi_k \lambda_k e^{-(\lambda_k - \lambda_2)t} \cdot \frac{w_k(x)}{w_1(x)} \right| \leq C \sum_{k=2}^{\infty} |\xi_k| |\lambda_k|^\sigma e^{-(\lambda_k - \lambda_2)t}
\end{equation}

for $(t, x) \in Q$. The series on the right-hand side of (10) admits a finite sum for every $t \in \mathbb{R}^+$ due to the following estimate for the eigenvalues $\lambda_k$:

\begin{equation}
C_1 k^{2/n} \leq \lambda_k \leq C_2 k^{2/n}, \quad k \in \mathbb{N}^+
\end{equation}

($C_1$ and $C_2$ are appropriate positive constants, see e.g. [9], [14]). Moreover, it is easy to see that both series in (10) have an upper bound independent of $t$ (see [11]), thus the function

$$t \mapsto e^{-(\lambda_2 - \lambda_1)t} \sum_{k=2}^{\infty} \xi_k \lambda_k e^{-(\lambda_k - \lambda_2)t} \cdot \frac{w_k(x)}{w_1(x)}$$

tends to zero uniformly in $\Omega$ as $t \to +\infty$. For this reason there exists a positive number $T$ such that

$$\text{sign}\{\partial_0 u(t, x)\} = \text{sign}\{-\xi_1 \lambda_1\} \quad \text{for} \ (t, x) \in (T, +\infty) \times \Omega.$$ 

Theorem 1 is proved.

**Appendix**

Here we prove formula (9), i.e. we give an upper bound for the $C^1(\Omega)$-norm of the eigenfunctions $w_k$ of (1) depending on the eigenvalue $\lambda_k$. The proof was obtained by complementing the proof of Theorem 15.1 in [7].

**Theorem 2.** There exist positive numbers $N^*, r^* \in \mathbb{R}^+$ such that for the eigenfunctions $w_k$ of (1) normed in $L^2(\Omega)$

\begin{equation}
\|w_k\|_{C^1(\Omega)} \leq N^* \lambda_k^{r^*}, \quad k \in \mathbb{N}^+
\end{equation}

holds (or, equivalently $\|w_k\|_{C^1(\Omega)} \leq N k^r$ for some $N, r \in \mathbb{R}^+$).

**Proof.** Let $p > n$. According to Ladyženskaja and Ural’ceva there exists a positive constant $K_1$ such that

\begin{equation}
\|v\|_{W^{2,p}(\Omega)} \leq K_1 (\|Lv\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)})
\end{equation}

for arbitrary $v \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ (see [7], formula (11.8) in part III). Applying this a priori estimate to $w_k$ we obtain

\begin{equation}
\|w_k\|_{W^{2,p}(\Omega)} \leq K_1 (\lambda_k + 1)\|w_k\|_{L^p(\Omega)} \leq K_2 \lambda_k \|w_k\|_{L^p(\Omega)}, \quad k \in \mathbb{N}^+
\end{equation}

for arbitrary $v \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ (see [7], formula (11.8) in part III).
for an appropriate positive number $K_2$.

According to the Sobolev imbedding theorem (see e.g. [1]) $W^{2,p}(\Omega) \subset C^1(\Omega)$ for $p > n$, and there exists a positive number $K_3$ such that for every $k \in \mathbb{N}^+$

\[(15) \quad \|w_k\|_{C^1(\bar{\Omega})} \leq K_3\|w_k\|_{W^{2,p}(\Omega)}.\]

From (14) and (15) we obtain the following inequality with some positive constant $K_4$:

\[\|w_k\|_{C^1(\bar{\Omega})} \leq K_4 \lambda_k \|w_k\|_{L^p(\Omega)}, \quad k \in \mathbb{N}^+.\]

The $L^p(\Omega)$-norm of $w_k$ can trivially be estimated by using the maximum norm of $w_k$:

\[\|w_k\|_{L^p(\Omega)} \leq \text{mes}(\Omega)^{1/p} \max_{\bar{\Omega}} |w_k|, \quad k \in \mathbb{N}^+.\]

Finally we use (3), i.e. the estimate for the maximum norm of $w_k$ to get

\[\|w_k\|_{L^p(\Omega)} \leq K_5 \lambda_k^{s^*}, \quad k \in \mathbb{N}^+\]

with appropriate positive constants $K_5$ and $s^*$, which leads to

\[\|w_k\|_{C^1(\bar{\Omega})} \leq K_4 K_5 \lambda_k^{s^*+1}, \quad k \in \mathbb{N}^+.\]

Applying estimate (11) we find a bound depending on $k$ for some $N$, $r \in \mathbb{R}^+$:

\[\|w_k\|_{C^1(\bar{\Omega})} \leq Nk^r, \quad k \in \mathbb{N}^+.\]

Theorem 2 is proved.

Remark 1. Supposing some more smoothness on $\partial \Omega$, results of Koshelev [5] could have been used instead of (13). As a special case, his paper gives conditions for the existence in $W^{2,p}(\Omega)$ of the solution of (1), and gives a bound for the $W^{2,p}(\Omega)$-norm of the solution depending on its $L^p(\Omega)$-norm.

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References

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