Completely continuous commutator of Marcinkiewicz integral

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Abstract. Let $M_{\Omega}$ be the higher-dimensional Marcinkiewicz integral introduced by Stein. In this paper, by Fourier transform estimates, approximation and a sufficient condition for strongly pre-compact set in $L^p(L^2[1, 2], l^2; \mathbb{R}^n)$, the authors proved that if $b \in \text{CMO}(\mathbb{R}^n)$ and $\Omega \in L((\ln L)^\frac{1}{2}(S^{n-1}))$, then for $p \in (1, \infty)$, the commutator generated by $b$ and $M_{\Omega}$ is a completely continuous operator on $L^p(\mathbb{R}^n)$.

1. Introduction

As an analogy to the classical Littlewood–Paley $g$-function, MARCINKIEWICZ [20] introduced the operator defined by

$$
M(f)(x) = \left( \int_0^{\infty} \frac{|F(x+t) - F(x-t) - 2F(x)|^2}{t^3} \, dt \right)^{\frac{1}{2}},
$$

where $F(x) = \int_0^x f(t) \, dt$. This operator is now called the Marcinkiewicz integral. ZYGMUND [26] proved that $M$ is bounded on $L^p([0, 2\pi])$ for $p \in (1, \infty)$. STEIN [21] generalized the Marcinkiewicz operator to the case of higher dimension. Let $\Omega$ be homogeneous of degree zero, integrable and have mean value zero on the unit sphere $S^{n-1}$. Define the Marcinkiewicz integral operator $M_{\Omega}$ by

$$
M_{\Omega}(f)(x) = \left( \int_0^{\infty} |F_{\Omega,t}f(x)|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}},
$$

(1.1)

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where
\[ F_{\Omega,t}f(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy \]
for \( f \in S(\mathbb{R}^n) \). This operator has been studied by many authors (see [1], [6], [12], [13], and the related references therein). STEIN [21] proved that if \( \Omega \in \text{Lip}_\alpha(S^{n-1}) \) with \( \alpha \in (0, 1] \), then \( M_\Omega \) is bounded on \( L^p(\mathbb{R}^n) \) for \( p \in (1, 2] \). Benedek, Calderón and Panzon showed that the \( L^p(\mathbb{R}^n) \) \( (p \in (1, \infty)) \) boundedness of \( M_\Omega \) holds true under the condition that \( \Omega \in C^4(S^{n-1}) \). Using the one-dimensional result and Riesz transforms similarly as in the case of singular integrals (see [4]) and interpolation, WALSH [24] proved that for \( p \in (1, \infty) \), \( \Omega \in L(\ln L)^{1/3} (\ln \ln L)^{2(1-2/r^*)} (S^{n-1}) \) is a sufficient condition such that \( M_\Omega \) is bounded on \( L^p(\mathbb{R}^n) \), where \( r = \min \{p, p'\} \) and \( p' = p/(p-1) \). DING, FAN and PAN [12] proved that if \( \Omega \in H^1(S^{n-1}) \) (the Hardy space on \( S^{n-1} \)), then \( M_\Omega \) is bounded on \( L^p(\mathbb{R}^n) \) for all \( p \in (1, \infty) \); AL-SALMAN et al. [1] proved that \( \Omega \in L(\ln L)^{1/2} (S^{n-1}) \) is a sufficient condition such that \( M_\Omega \) is bounded on \( L^p(\mathbb{R}^n) \) for all \( p \in (1, \infty) \).

The commutator of \( M_\Omega \) is also of interest and has been considered by many authors. Let \( b \in \text{BMO}(\mathbb{R}^n) \), the commutator generated by \( M_\Omega \) and \( b \) is defined by
\[ [b, M_{\Omega,b}]f(x) = \left( \int_0^\infty \left| \int_{|x-y|\leq t} (b(x) - b(y)) \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy \right|^2 \frac{dt}{t^2} \right)^{1/2}, \quad \text{(1.2)} \]
TORCHINSKY and WANG [22] showed that if \( \Omega \in \text{Lip}_\alpha(S^{n-1}) \) \( (\alpha \in (0, 1]) \), then \( M_{\Omega,b} \) is bounded on \( L^p(\mathbb{R}^n) \) with bound \( C\|b\|_{\text{BMO}(\mathbb{R}^n)} \) for all \( p \in (1, \infty) \). HU and YAN [19] proved that \( \Omega \in L(\ln L)^{1/2} (S^{n-1}) \) is a sufficient condition such that \( M_{\Omega,b} \) is bounded on \( L^2(\mathbb{R}^n) \). CHEN and LU [5] improved the result in [19] and showed that if \( \Omega \in L(\ln L)^{1/2} (S^{n-1}) \), then \( M_{\Omega,b} \) is bounded on \( L^p(\mathbb{R}^n) \) with bound \( C\|b\|_{\text{BMO}(\mathbb{R}^n)} \) for all \( p \in (1, \infty) \).

Let \( \text{CMO}(\mathbb{R}^n) \) be the closure of \( C^\infty_0(\mathbb{R}^n) \) in the \( \text{BMO}(\mathbb{R}^n) \) topology, which coincides with \( \text{VMO}(\mathbb{R}^n) \), the space of functions of vanishing mean oscillation introduced by COIFMAN and WEISS [11], see also [3]. UCHIYAMA [23] proved that if \( S \) is a Calderón–Zygmund operator, and \( b \in \text{BMO}(\mathbb{R}^n) \), then the commutator of \( S \) defined by
\[ [b, S]f(x) = b(x)Sf(x) - S(bf)(x) \]
is a compact operator on \( L^p(\mathbb{R}^n) \) \( (p \in (1, \infty)) \) if and only if \( b \in \text{CMO}(\mathbb{R}^n) \). CHEN, DING and WANG [8] considered the compactness of \( M_{\Omega,b} \) on \( L^p(\mathbb{R}^n) \), and proved that if \( \Omega \) satisfies certain regularity condition of Dini type, then for \( p \in (1, \infty) \),
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$M_{\Omega, b}$ is compact on $L^p(\mathbb{R}^n)$ if and only if $b \in \text{CMO}(\mathbb{R}^n)$. The purpose of this paper is to prove that, in order to guarantee the compactness of $M_{\Omega, b}$ on $L^p(\mathbb{R}^n)$, the regularity condition of $\Omega$ is superfluous. To formulate our main result, we first recall some definitions.

**Definition 1.1.** Let $X$ be a normed linear space and $X^*$ be its dual space, \( \{x_k\} \subset X \) and $x \in X$. If for all $f \in X^*$,

$$\lim_{k \to \infty} |f(x_k) - f(x)| = 0,$$

then $\{x_k\}$ is said to converge to $x$ weakly, or $x_k \rightharpoonup x$.

**Definition 1.2.** Let $X$, $Y$ be two Banach spaces and $S$ be a bounded operator from $X$ to $Y$.

(i) If for each bounded set $G \subset X$, $SG = \{Sx : x \in G\}$ is a strongly pre-compact set in $Y$, then $S$ is called a compact operator from $X$ to $Y$;

(ii) if for $\{x_k\} \subset X$ and $x \in X$,

$$x_k \to x \text{ in } X \Rightarrow \|Sx_k - Sx\|_Y \to 0,$$

then $S$ is said to be a completely continuous operator.

It is well known that if $X$ is a reflexive space and $S$ is completely continuous from $X$ to $Y$, then $S$ is also compact from $X$ to $Y$. On the other hand, if $S$ is a linear compact operator from $X$ to $Y$, then $S$ is also a completely continuous operator. However, if $S$ is not linear, then the compactness of $S$ does not imply that $S$ is completely continuous. For example, the operator

$$Sx = \|x\|_2$$

is compact from $l^2$ to $\mathbb{R}$, but not completely continuous.

The main result in this paper can be stated as follows.

**Theorem 1.1.** Let $\Omega$ be homogeneous of degree zero and have mean value zero on $S^{n-1}$. Suppose that $\Omega \in L(\ln L)^{3/2}(S^{n-1})$. Then for $b \in \text{CMO}(\mathbb{R}^n)$ and $p \in (1, \infty)$, $M_{\Omega, b}$ is completely continuous on $L^p(\mathbb{R}^n)$.

**Remark 1.1.** Recently, CHEN and HU [7] considered the compactness of the commutator of homogeneous singular integral operators defined by

$$T_{\Omega}f(x) = \text{p. v.} \int_{\mathbb{R}^n} \frac{\Omega(x - y)}{|x - y|^n} f(y) dy,$$
here Ω is homogeneous of degree zero, integrable on $S^{n-1}$ and has mean value zero. Using the idea of approximating $T_Ω$ by a sequence of operators with smooth kernels, Chen and Hu considered the compactness of the commutator of $T_Ω$ when $Ω$ satisfies

$$\sup_{ζ \in S^{n-1}} \int_{S^{n-1}} |Ω(η)| \left( \ln \frac{1}{|η \cdot ζ|} \right)^{θ} dη < \infty$$

for some $θ > 2$. It should be pointed out that this idea comes from Watson’s paper [25]. In this paper, we will also employ the idea of Watson. However, the operators $M$ and $M_Ω, b$ are not linear, the proof of Theorem 1.1 involves much more technical problems, such as an appropriate sufficient condition of strongly pre-compact sets in space $L^p(L^2[1, 2], l^2; R^n)$ (see Lemma 3.4 below), and the argument in this paper is more complicated.

We make some conventions. In what follows, $C$ always denotes a positive constant that is independent of the main parameters involved, but whose value may differ from line to line. We use the symbol $A \lesssim B$ to denote that there exists a positive constant $C$ such that $A \leq CB$. For a set $E \subset R^n$, $χ_E$ denotes its characteristic function. Let $M$ be the Hardy–Littlewood maximal operator. For $r \in (0, \infty)$, we use $M_r$ to denote the operator $M_r f(x) = (M(|f|^r)(x))^{1/r}$.

2. Approximation

Let $Ω$ be homogeneous of degree zero, integrable on $S^{n-1}$. For $t \in [1, 2]$ and $j \in Z$, set

$$K_j^t(x) = \frac{1}{2j} \frac{Ω(x)}{|x|^{n-1}} χ_{\{2^{j-1}t < |x| \leq 2^jt\}}(x).$$

(2.1)

As it was proved in [15], there exists a constant $α \in (0, 1)$ such that for $t \in [1, 2]$ and $ξ \in R^n \\{0\}$,

$$|\hat{K}_j^t(ξ)| \lesssim ||Ω||_{L^∞(S^{n-1})} \min \{1, |2^jξ|^{-α}\}.$$  

(2.2)

Moreover, if $\int_{S^{n-1}} Ω(x')dx' = 0$, then

$$|\hat{K}_j^t(ξ)| \lesssim ||Ω||_{L^1(S^{n-1})} \min \{1, |2^jξ|\}.$$  

(2.3)

Let

$$\tilde{M}_Ω f(x) = \left( \int_1^2 \sum_{j \in Z} |F_j f(x, t)|^2 dt \right)^{\frac{1}{2}},$$

where
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with

$$F_j f(x, t) = \int_{\mathbb{R}^n} K^j_t(x - y) f(y) dy.$$  

For $b \in \text{BMO}(\mathbb{R}^n)$, let $\tilde{M}_{\Omega, b}$ be the commutator of $\tilde{M}_{\Omega}$ defined by

$$\tilde{M}_{\Omega, b} f(x) = \left( \int_1^2 \sum_{j \in \mathbb{Z}} |F_{j, b} f(x, t)|^2 dt \right)^{1 \over 2},$$

with

$$F_{j, b} f(x, t) = \int_{\mathbb{R}^n} (b(x) - b(y)) K^j_t(x - y) f(y) dy.$$  

A trivial computation leads to that

$$M_{\Omega} f(x) \approx \tilde{M}_{\Omega} f(x), \quad M_{\Omega, b} f(x) \approx \tilde{M}_{\Omega, b} f(x).$$  

(2.4)

Let $\phi \in C^\infty_0(\mathbb{R}^n)$ be a nonnegative function such that $\int_{\mathbb{R}^n} \phi(x) dx = 1$, $\text{supp} \phi \subset \{x : |x| \leq 1/4\}$. For $l \in \mathbb{Z}$, let $\phi_l(y) = 2^{-nl} \phi(2^{-l} y)$. It is easy to verify that for any $\beta \in (0, 1)$,

$$|\hat{\phi}_l(\xi) - 1| \lesssim \min\{1, |2^l \xi|^\beta\}.  \quad (2.5)$$

Let

$$F_{l, j} f(x, t) = \int_{\mathbb{R}^n} K^j_t \ast \phi_{j - l}(x - y) f(y) dy.$$  

Define the operator $\tilde{M}_{\Omega, l}$ by

$$\tilde{M}_{\Omega, l} f(x) = \left( \int_1^2 \sum_{j \in \mathbb{Z}} |F_{l, j} f(x, t)|^2 dt \right)^{1 \over 2}.  \quad (2.6)$$

For $b \in \text{BMO}(\mathbb{R}^n)$, let $\tilde{M}_{\Omega, l, b}$ be the commutator of $\tilde{M}_{\Omega, l}$, that is,

$$\tilde{M}_{\Omega, l, b} f(x) = \left( \int_1^2 \sum_{j \in \mathbb{Z}} |F_{l, j, b} f(x, t)|^2 dt \right)^{1 \over 2},  \quad (2.7)$$

with

$$F_{l, j, b} f(x, t) = \int_{\mathbb{R}^n} (b(x) - b(y)) K^j_t \ast \phi_{j - l}(x - y) f(y) dy.$$  

For $j \in \mathbb{Z}$ and $l \in \mathbb{N}$, let

$$U_{l, j, l}(y) = K^j_l \ast \phi_{l - j}(y) - K^j_l(y).$$
Let $E_0 = \{x' \in S^{n-1} : |\Omega(x')| \leq 2\}$ and $E_d = \{x' \in S^{n-1} : 2^d < |\Omega(x')| \leq 2^{d+1}\}$ for $d \in \mathbb{N}$. Denote by $\Omega_d$ the restriction of $\Omega$ to $E_d$, namely, $\Omega_d(x') = \Omega(x') \chi_{E_d}(x')$. Set
\[
U_{l,j,d,t}(y) = K_{d,t}^j * \phi_{l-j}(y) - K_{d,t}^j(y),
\]
with
\[
K_{d,t}^j(y) = \frac{1}{2^j} \frac{\Omega_d(x)}{|x|^{n-1}} \chi_{\{2^{-j+1} < |x| \leq 2^j\}}(x).
\]

**Lemma 2.1.** Let $\Omega$ be homogeneous of degree zero and $\Omega \in L^1(S^{n-1})$. Then for $p \in (1, \infty)$,
\[
\left\| \left( \sum_{l \in \mathbb{Z}} |U_{l,j,d,t} * f_l|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|\Omega\|_{L^1(S^{n-1})} \left\| \left( \sum_{l \in \mathbb{Z}} |f_l|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}.
\]

For the proof of Lemma 2.1, see [19].

**Lemma 2.2.** Let $\Omega$ be homogeneous of degree zero and have mean value zero, $\tilde{M}_l^j$ be the operator defined by (2.6). Suppose that $\Omega \in L \ln L(S^{n-1})$, then for $l \in \mathbb{N}$ and $p \in (1, \infty)$,
\[
\| \tilde{M}_l^j f - \tilde{M}_l^j f \|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.
\]

**Proof.** It is obvious that
\[
|\tilde{M}_l^j f(x) - \tilde{M}_l^j f(x)| \leq \left( \int_1^2 \sum_j |U_{l,j,t} * f(x)|^2 \, dt \right)^{\frac{1}{2}}.
\]

Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be a radial function such that $\text{supp} \psi \subset \{1/4 \leq |\xi| \leq 4\}$ and
\[
\sum_{i \in \mathbb{Z}} \psi(2^{-i} \xi) = 1, \quad |\xi| \neq 0.
\]
Define the multiplier operator $S_i$ by
\[
\tilde{S}_i f(\xi) = \psi(2^{-i} \xi) \hat{f}(\xi).
\]
Let
\[
D_1 f(x) = \sum_{m=-\infty}^0 \left( \int_1^2 \sum_j |U_{l,j,t} * (S_{m-j} f)(x)|^2 \, dt \right)^{\frac{1}{2}},
\]
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\[ D_2 f(x) = \sum_{d=1}^{\infty} \sum_{m=N_d}^{\infty} \left( \int_1^2 \sum_j |U_{i,j,d,t} \ast (S_{m-j}f)(x)|^2 \, dt \right)^{\frac{1}{2}} \]

and

\[ D_3 f(x) = \sum_{d=1}^{\infty} \sum_{m=1}^{N_d} \left( \int_1^2 \sum_{j \in \mathbb{Z}} |U_{i,j,d,t} \ast (S_{m-j}f)(x)|^2 \, dt \right)^{\frac{1}{2}}. \]

It then follows that for \( f \in \mathcal{S}(\mathbb{R}^n) \),

\[
\left\| \left( \int_1^2 \sum_j \left| U_{i,j,t} \ast f(x) \right|^2 \, dt \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \leq \sum_{i=1}^{3} \| D_i f \|_{L^p(\mathbb{R}^n)}.
\]

We now estimate the term \( D_1 \). By Fourier transform estimate, we know that

\[
\left\| \left( \int_1^2 \sum_j \left| U_{i,j,t} \ast (S_{m-j}f)(x) \right|^2 \, dt \right)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^n)}^2
\]

\[
= \int_1^2 \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} \left| U_{i,j,t} \ast (S_{m-j}f)(x) \right|^2 \, dx \, dt
\]

\[
= \int_1^2 \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} \left| \hat{K}^j(\xi) \hat{\phi}_{j-l}(\xi) - 1 \right|^2 |\psi(2^{-m+j} \xi)|^2 |\hat{f}(\xi)|^2 \, d\xi \, dt
\]

\[
\lesssim \| \Omega \|_{L^1(S^{n-1})} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} |2^{j-l} \xi|^2 |\psi(2^{-m+j} \xi)|^2 |\hat{f}(\xi)|^2 \, d\xi
\]

\[
\lesssim 2^{2m-2l} \| \Omega \|_{L^1(S^{n-1})}^2 \| f \|_{L^p(\mathbb{R}^n)}. \quad (2.9)
\]

On the other hand, for \( p \in (2, \infty) \), applying the Minkowski inequality and Lemma 2.1, we have that

\[
\left\| \left( \int_1^2 \sum_j \left| U_{i,j,t} \ast (S_{m-j}f)(x) \right|^2 \, dt \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}^2
\]

\[
\leq \int_1^2 \left( \int_{\mathbb{R}^n} \left( \sum_{j \in \mathbb{Z}} \left| U_{i,j,t} \ast (S_{m-j}f)(x) \right|^2 \right)^{\frac{p}{2}} \, dx \right)^{\frac{2}{p}} \, dt
\]

\[
\leq \| \Omega \|_{L^1(S^{n-1})}^2 \| f \|_{L^p(\mathbb{R}^n)}^2. \quad (2.10)
\]

To estimate

\[
\left\| \left( \int_1^2 \sum_j \left| U_{i,j,t} \ast (S_{m-j}f)(x) \right|^2 \, dt \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}
\]
for $p \in (1, 2)$, we consider the mapping $\mathcal{F}$ defined by

$$
\mathcal{F} : \{h_j(x)\}_{j \in \mathbb{Z}} \longrightarrow \{U_{t,j} * h_j(x)\}.
$$

Note that for any $p \in (1, 2)$,

$$
|U_{t,j} * h_j(x)| \lesssim M_M h_j(x) + M_M h_j(x),
$$

with $M_M$ the maximal operator defined by

$$
M_M h(x) = \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |\Omega(x - y)||f(y)|dy.
$$

It is well known that $M_M$ is bounded on $L^p(\mathbb{R}^n)$ with bound $C\|\Omega\|_{L^1(S^{n-1})}$ for all $p \in (1, \infty)$. A straightforward computation then tells us that for $p_0 \in (1, \infty)$

$$
\int_{\mathbb{R}^n} \left( \int_1^2 \sum_{j \in \mathbb{Z}} |U_{t,j} * h_j(x)|^{p_0} \, dt \right) \, dx \lesssim \|\Omega\|_{L^1(S^{n-1})} \int_{\mathbb{R}^n} \left( \sum_{j \in \mathbb{Z}} |h_j(x)|^{p_0} \right) \, dx. \quad (2.11)
$$

Also, we have that

$$
\sup_{j \in \mathbb{Z}} \sup_{t \in \{1, 2\}} |U_{t,j} * h_j(x)| \lesssim \|\Omega\|_{L^1(S^{n-1})} \sup_{j \in \mathbb{Z}} |h_j(x)|,
$$

which implies that for $p_1 \in (1, \infty)$,

$$
\left\| \sup_{j \in \mathbb{Z}} \sup_{t \in \{1, 2\}} |U_{t,j} * h_j| \right\|_{L^{p_1}(\mathbb{R}^n)} \lesssim \|\Omega\|_{L^1(S^{n-1})} \left\| \sup_{j \in \mathbb{Z}} |h_j| \right\|_{L^{p_1}(\mathbb{R}^n)}. \quad (2.12)
$$

For $p \in (1, 2)$, interpolating the inequalities (2.11) and (2.12) (with $p_0 \in (1, 2)$, $p_1 \in (2, \infty)$ and $1/p = 1/2 + (2 - p_0)/(2p_1)$) leads to that

$$
\left\| \left( \int_1^2 \sum_{j \in \mathbb{Z}} |U_{t,j} * h_j|^{2} \, dt \right)^{\frac{p}{2}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|\Omega\|_{L^1(S^{n-1})} \left( \sum_{j \in \mathbb{Z}} |h_j|^{2} \right)^{\frac{p}{2}} \|f\|_{L^p(\mathbb{R}^n)},
$$

and so

$$
\left\| \left( \int_1^2 \sum_{j} |U_{t,j} * (S_{m-j} f)(x)|^{2} \, dt \right)^{\frac{p}{2}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|\Omega\|_{L^1(S^{n-1})} \left( \sum_{j \in \mathbb{Z}} |S_{m-j} f|^{2} \right)^{\frac{p}{2}} \|f\|_{L^p(\mathbb{R}^n)}.
$$
As in the inequality (2.13), we have that for \( p \in (1, \infty) \),
\[
\left\| \left( \int_1^2 \sum_{j \in \mathbb{Z}} |U_{l,j,t} \ast (S_{m-j}f)(x)|^2 \, dt \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|\Omega\|_{L^1(S^{n-1})} \|f\|_{L^p(\mathbb{R}^n)}. \tag{2.13}
\]
Interpolating the inequalities (2.9) and (2.13) gives us that for \( p \in (1, \infty) \),
\[
\left\| \left( \int_1^2 \sum_{j \in \mathbb{Z}} |U_{l,j,t} \ast (S_{m-j}f)(x)|^2 \, dt \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \lesssim 2^{\nu m} \|\Omega\|_{L^1(S^{n-1})} \|f\|_{L^p(\mathbb{R}^n)},
\]
with \( t_p \in (0, 1) \) a constant depending only on \( p \). Therefore,
\[
\|D_1f\|_{L^p(\mathbb{R}^n)} \lesssim \|\Omega\|_{L^1(S^{n-1})} \|f\|_{L^p(\mathbb{R}^n)}.
\]

We turn our attention to the term \( D_2 \). Again by the Plancherel theorem and the Fourier transform estimates (2.2) and (2.5), we have that
\[
\left\| \left( \int_1^2 \sum_{j \in \mathbb{Z}} |U_{l,j,d,t} \ast (S_{m-j}f)(x)|^2 \, dt \right)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^n)} \lesssim \|\Omega\|_{L^\infty(S^{n-1})} \left( \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} |2^j \xi|^{-2\alpha} |2^j \xi|^\alpha \psi(2^{m+j} \xi)|^2 |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}
\]
\[
\lesssim \|\Omega\|_{L^\infty(S^{n-1})} 2^{-\alpha m} \|f\|_{L^2(\mathbb{R}^n)}. \tag{2.14}
\]
As in the inequality (2.13), we have that for \( p \in (1, \infty) \),
\[
\left\| \left( \int_1^2 \sum_{j \in \mathbb{Z}} |U_{l,j,d,t} \ast (S_{m-j}f)(x)|^2 \, dt \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|\Omega\|_{L^1(S^{n-1})} \|f\|_{L^p(\mathbb{R}^n)}. \tag{2.15}
\]
Interpolating the inequalities (2.14) and (2.15) then gives that
\[
\left\| \left( \int_1^2 \sum_{j \in \mathbb{Z}} |U_{l,j,d,t} \ast (S_{m-j}f)(x)|^2 \, dt \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|\Omega\|_{L^\infty(S^{n-1})} 2^{-m\delta_p} \|f\|_{L^p(\mathbb{R}^n)}.
\]
This in turn implies that
\[
\|D_2f\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{d=1}^\infty 2^d \sum_{m=Nd}^{\infty} 2^{-m\delta_p} \|f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)},
\]
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if we choose \( N \in \mathbb{N} \) such that \( N > 2\delta_p \).

It remains to consider the term \( D_3 \). Again as (2.13), we have that for \( p \in (1, \infty) \),
\[
\left\| \left( \int_1^2 \sum_j \left| U_{t,j,d,t} \ast (S_{m-j}f)(x) \right|^2 dt \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \| \Omega_d \|_{L^1(\mathbb{R}^n)} \| f \|_{L^p(\mathbb{R}^n)}.
\]
This, in turn implies that
\[
\| D_3 f \|_{L^p(\mathbb{R}^n)} \lesssim \sum_{d=1}^\infty 2^d \sum_{m=1}^N \| \Omega_d \|_{L^1(S^{n-1})} \| f \|_{L^p(\mathbb{R}^n)} \lesssim \| f \|_{L^p(\mathbb{R}^n)}.
\]
Combining the estimates for \( D_1, D_2 \) and \( D_3 \) leads to (2.8).

The following result shows that \( \{ \tilde{M}_\Omega f \}_{t \in \mathbb{N}} \) approximate to \( M_\Omega \) properly, and will be useful in the proof of Theorem 1.1.

**Theorem 2.1.** Let \( \Omega \) be homogeneous of degree zero and have mean value zero. Suppose that \( \Omega \in L(\ln L)^\gamma(S^{n-1}) \) for some \( \gamma \in (1, \infty) \), then for \( l \in \mathbb{N} \) and \( p \in (1, \infty) \),
\[
\| \tilde{M}_\Omega f - \tilde{M}_{\Omega_l} f \|_{L^p(\mathbb{R}^n)} \lesssim l^{-\delta_p} \| f \|_{L^p(\mathbb{R}^n)},
\]
with \( \delta_p \) a constant depending only on \( p \) and \( n \).

**Proof.** By the estimates in [17], we know that if \( \Omega \) satisfies (1.3) for some \( \theta \in (0, \infty) \), then, for \( \xi \in \mathbb{R}^n \setminus \{0\}, j \in \mathbb{Z} \) and \( t \in [1, 2] \),
\[
|\hat{K}_j^l(\xi)| \lesssim \ln^{-\theta}(\|2^j \xi\|).
\]

For each \( \xi \in \mathbb{R}^n \setminus \{0\} \) and \( l \in \mathbb{N} \), let \( j_0 \) be the integer such that \( 2^{j-1} \xi \leq 2^j \xi \). A trivial computation involving the Fourier transform estimates (2.1)–(2.3) leads to that
\[
\sum_{j \in \mathbb{Z}} |\hat{K}_j^l(\xi) \hat{\phi}(2^{j-1} \xi) - \hat{K}_j^l(\xi)|^2 \lesssim \sum_{j \in \mathbb{Z}, j \leq j_0} |2^{j-1} \xi|^2 + \sum_{j \in \mathbb{Z}, j > j_0} \ln^{-2\gamma}(2^j \xi) \lesssim l^{-2\theta+1}.
\]
This, via the Plancherel theorem, leads to
\[
\| \tilde{M}_\Omega f - \tilde{M}_{\Omega_l} f \|_{L^2(\mathbb{R}^n)} \lesssim l^{-\theta+\frac{1}{2}} \| f \|_{L^2(\mathbb{R}^n)}.
\]
On the other hand, it was pointed out in [18] that, if \( \Omega \in L(\ln L)^\gamma(S^{n-1}) \) for \( \gamma \in (1, \infty) \), then \( \Omega \) satisfies (1.3) for \( \theta \in (1, \gamma) \). Therefore, by interpolating the inequalities (2.8) and (2.16), we know that under the hypothesis of Theorem 2.1,
\[
\| \tilde{M}_\Omega f - \tilde{M}_{\Omega_l} f \|_{L^p(\mathbb{R}^n)} \lesssim l^{-\gamma+1/2+\epsilon} \| f \|_{L^p(\mathbb{R}^n)},
\]
with \( \epsilon \in (0, \gamma - 1/2) \). This completes the proof of Theorem 2.1. \( \Box \)
3. Proof of Theorem 1.1

To prove Theorem 1.1, we will use some lemmas.

**Lemma 3.1.** Let \( \Omega \) be homogeneous of degree zero and belong to \( L^1(S^{n-1}) \), \( K^j_t \) be defined as in (2.1). Then for \( l \in \mathbb{N} \), \( s \in (1, \infty) \), \( j_0 \in \mathbb{Z} \) and \( y \in \mathbb{R}^n \) with \( |y| < 2^{j_0-4} \),

\[
\sum_{j>j_0} \sum_{k \in \mathbb{Z}} 2^{kn/s} \left( \int_{2^k < |x| \leq 2^{k+1}} |K^j_t \ast \phi_{j-l}(x+y) - K^j_t \ast \phi_{j-l}(x)|^{s'} \, dx \right)^{\frac{1}{s'}} \lesssim 2^{l(n+1)} 2^{-j_0} |y|.
\]

**Proof.** We follow the argument used in [25] (see also [7]), with suitable modification. Observe that \( \text{supp} K^j_t \ast \phi_{j-l} \subset \{ x : 2^{j_0-2} \leq |x| \leq 2^{j_0+2} \} \), and

\[
\| \phi_{j-l}(\cdot + y) - \phi_{j-l}(\cdot) \|_{L^{s'}}(\mathbb{R}^n) \lesssim 2^{(j-l)n/s} 2^{-j_0} |y|.
\]

Thus, for all \( k \in \mathbb{N} \),

\[
\sum_{j \in \mathbb{Z}} 2^{kn/s} \left( \int_{2^k < |x| \leq 2^{k+1}} |K^j_t \ast \phi_{j-l}(x+y) - K^j_t \ast \phi_{j-l}(x)|^{s'} \, dx \right)^{\frac{1}{s'}} \lesssim 2^{kn} \| K^j_t \|_{L^1(\mathbb{R}^n)} \| \phi_{j-l}(\cdot + y) - \phi_{j-l}(\cdot) \|_{L^{s'}(\mathbb{R}^n)} \lesssim 2^{l(n+1)} |y|.
\]

This, in turn, leads to that

\[
\sum_{j>j_0} \sum_{k \in \mathbb{Z}} 2^{kn/s} \left( \int_{2^k < |x| \leq 2^{k+1}} |K^j_t \ast \phi_{j-l}(x+y) - K^j_t \ast \phi_{j-l}(x)|^{s'} \, dx \right)^{\frac{1}{s'}} = \sum_{j_0+3 \leq j \in \mathbb{Z}} 2^{kn} \left( \int_{2^k < |x| \leq 2^{k+1}} |K^j_{\Omega} \ast \phi_{j-l}(x+y) - K^j_{\Omega} \ast \phi_{j-l}(x)|^{s'} \, dx \right)^{\frac{1}{s'}} \lesssim 2^{l(n+1)} 2^{-j_0} |y|,
\]

and completes the proof of Lemma 3.1. \( \square \)

For \( l \in [1, 2] \) and \( j \in \mathbb{Z} \), let \( K^j_t \) be defined as in (2.1), \( \phi \) and \( \phi_l \) (with \( l \in \mathbb{Z} \)) be as in Section 2. By Lemma 2.2 and the \( L^p(\mathbb{R}^n) \) boundedness of \( M_{\Omega} \), we see that if \( \Omega \) is homogeneous of degree zero, has mean value zero and \( \Omega \in L \ln L(S^{n-1}) \),
then for $p \in (1, \infty)$, $\widetilde{M}^{l,j_0}_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ with bound independent of $l$. For $j_0 \in \mathbb{Z}$, define the operator $\widetilde{M}^{l,j_0}_\Omega$ by

$$\widetilde{M}^{l,j_0}_\Omega f(x) = \left( \int_1^2 \sum_{j \in \mathbb{Z}, j > j_0} \left| \int_{\mathbb{R}^n} K^j_t * \phi_{j-1}(x - y)f(y)dy \right|^2 dt \right)^{\frac{1}{2}},$$

and the commutator $\tilde{M}^{l,j_0,b}_\Omega$ by

$$\tilde{M}^{l,j_0,b}_\Omega f(x) = \left( \int_1^2 \sum_{j \in \mathbb{Z}, j > j_0} \left| \int_{\mathbb{R}^n} (b(x) - b(y))K^j_t * \phi_{j-1}(x - y)f(y)dy \right|^2 dt \right)^{\frac{1}{2}},$$

with $b \in \text{BMO}(\mathbb{R}^n)$.

**Lemma 3.2.** Let $\Omega$ be homogeneous of degree zero and have mean value zero. Suppose that $\Omega \in L((\ln L)^\gamma(S^{n-1}))$ for some $\gamma \in (1, \infty)$, then for $p \in (1, \infty)$, $l \in \mathbb{N}$ and $j_0 \in \mathbb{Z}$, $\tilde{M}^{l,j_0}_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ with bound independent of $l$ and $j_0$.

**Proof.** Let $p \in (1, \infty)$ and $l \in \mathbb{Z}$. By Theorem 2.1, it follows that $\tilde{M}^{l}_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ with bound independent of $l$. Observe that

$$\tilde{M}^{l,j_0}_\Omega f(x) \lesssim \tilde{M}^{l}_\Omega f(x) + \Lambda^{l,j_0}_\Omega f(x),$$

with

$$\Lambda^{l,j_0}_\Omega f(x) = \left( \int_1^2 \sum_{j \leq j_0} \left| F^j_f(x, t) \right|^2 dt \right)^{\frac{1}{2}}.$$

Thus, it suffices to prove that $\Lambda^{l,j_0}_\Omega$ is bounded on $L^p$ with bound independent of $j_0$ and $l$. To this aim, we first note that if $\text{supp} f \subset Q$ for a cube $Q$ having side length $2^{n}$, then $\text{supp} \Lambda^{l,j_0}_\Omega f \subset 20\sqrt{n}Q$. On the other hand, if $\{Q_k\}_k$ is a sequence of cubes with disjoint interiors and having side length $2^{n}$, then the cubes $\{20\sqrt{n}Q_k\}$ have bounded overlaps. Thus, we may assume that $\text{supp} f \subset Q$, with $Q$ a cube centered at $h \in \mathbb{R}^n$ and having side length $2^{j_0}$. For such a $f \in L^p(\mathbb{R}^n)$, we see that if $x \in 20\sqrt{n}Q$, then

$$\int_1^2 \sum_{j > j_0 + 20n} |F^j_f(x, t)|^2 dt = 0.$$

Therefore, for $x \in 20\sqrt{n}Q$,

$$\Lambda^{l,j_0}_\Omega f(x) \leq \tilde{M}^{l}_\Omega f(x) + \left( \int_1^2 \sum_{j_0 < j \leq j_0 + 20n} \left| F^j_f(x, t) \right|^2 dt \right)^{\frac{1}{2}} \lesssim \tilde{M}^{l}_\Omega f(x) + M_{\Omega}M f(x).$$

The desired $L^p(\mathbb{R}^n)$ boundedness of $\tilde{M}^{l,j_0}_\Omega$ then follows directly. $\square$
Lemma 3.3. Let $\Omega$ be homogeneous of degree zero and integrable on $S^{n-1}$. Then for $b \in C_c^\infty(\mathbb{R}^n)$, $l \in \mathbb{N}$, $j_0 \in \mathbb{Z}^n$ and $p \in (1, \infty)$,

$$
\|\mathcal{M}_{\Omega, b}^{l,j_0} f - \mathcal{M}_{\Omega}^{l} f\|_{L^p(\mathbb{R}^n)} \lesssim 2^{hn} \|f\|_{L^p(\mathbb{R}^n)}.
$$

**Proof.** Let $b \in C_c^\infty(\mathbb{R}^n)$ with $\|\nabla b\|_{L^\infty(\mathbb{R}^n)} = 1$. By the fact that $\text{supp} \ 2^{j} \in \{x : 2^{j-2} \leq |x| \leq 2^{j+2}\}$, it is easy to verify that

$$
\sum_{j \leq j_0} \int_{\mathbb{R}^n} |K_j^l \ast \phi_{j-1}(x-y)||x-y||f(y)|dy
\lesssim \sum_{j \leq j_0} \sum_{k \in \mathbb{Z}} 2^k \int_{2^k < |x-y| \leq 2^{k+1}} |K_j^l \ast \phi_{j-1}(x-y)||f(y)|dy
\lesssim \sum_{j \leq j_0} \sum_{k-j \leq 3} 2^k \int_{2^k < |x-y| \leq 2^{k+1}} |K_j^l \ast \phi_{j-1}(x-y)||f(y)|dy
\lesssim 2^hn \Omega M f(x).
$$

Thus,

$$
\left|\mathcal{M}_{\Omega, b}^{l,j_0} f(x) - \mathcal{M}_{\Omega}^{l} f(x)\right|^2
\leq \sum_{j < j_0} \int_1^2 \left( \int_{\mathbb{R}^n} (b(x) - b(y))K_j^l \ast \phi_{j-1}(x-y)f(y) \right)^2 dt
\lesssim \int_1^2 \left( \sum_{j \leq j_0} \int_{\mathbb{R}^n} |x-y||K_j^l \ast \phi_{j-1}(x-y)f(y)|dy \right)^2 dt
\lesssim \{2^hn \Omega M f(x)\}^2.
$$

The desired conclusion now follows immediately. □

Let $p, r \in [1, \infty)$, $q \in [1, \infty]$, $L^p(L^q([1, 2]), l^r; \mathbb{R}^n)$ be the space of sequences of functions defined by

$$
L^p(L^q([1, 2]), l^r; \mathbb{R}^n) = \{ \tilde{f} = \{f_k\}_{k \in \mathbb{Z}} : \|\tilde{f}\|_{L^p(L^q([1, 2]), l^r; \mathbb{R}^n)} < \infty \},
$$

with

$$
\|f\|_{L^p(L^q([1, 2]), l^r; \mathbb{R}^n)} = \left\| \left( \int_1^2 \left( \sum_{k \in \mathbb{Z}} |f_k(x, t)|^r \right)^{\frac{q}{r}} dt \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)}.
$$

With usual addition and scalar multiplication, $L^p(L^q([1, 2]), l^r; \mathbb{R}^n)$ is a Banach space.

**Lemma 3.4.** Let $p \in (1, \infty)$, $\mathcal{G} \subset L^p(L^2([1, 2]), l^2; \mathbb{R}^n)$. Suppose that $\mathcal{G}$ satisfies the following five conditions:
(a) $G$ is bounded, that is, there exists a constant $C$ such that for all $\vec{f} \in G$, 
$$
\|\vec{f}\|_{L^p(L^2([1, 2]), l^2; \mathbb{R}^n)} \leq C;
$$
(b) for each fixed $\epsilon > 0$, there exists a constant $A > 0$, such that for all $\vec{f} = \{f_k\}_{k \in \mathbb{Z}} \in G$, 
$$
\left\| \left( \int_1^2 \sum_{k \in \mathbb{Z}} |f_k(\cdot, t)|^2 dt \right)^{\frac{1}{2}} \chi_{\{|\cdot| > A\}}(\cdot) \right\|_{L^p(\mathbb{R}^n)} < \epsilon;
$$
(c) for each fixed $\epsilon > 0$ and $N \in \mathbb{N}$, there exists a constant $\varrho > 0$, such that for all $\vec{f} = \{f_k\}_{k \in \mathbb{Z}} \in G$, 
$$
\left\| \sup_{|h| \leq \delta} \left( \int_1^2 \sum_{|k| \leq N} |f_k(x, t) - f_k(x+h, t)|^2 dt \right)^{\frac{1}{2}} \chi_{B(0, D)} \right\|_{L^p(\mathbb{R}^n)} < \epsilon;
$$
(d) for each fixed $\epsilon > 0$ and $N \in \mathbb{N}$, there exists a constant $\sigma \in (0, 1/2)$ such that for all $\vec{f} = \{f_k\}_{k \in \mathbb{Z}} \in G$, 
$$
\left\| \sup_{|s| \leq \rho} \left( \int_1^2 \sum_{|k| \leq N} |f_k(\cdot, t+s) - f_k(\cdot, t)|^2 dt \right)^{\frac{1}{2}} \chi_{B(0, D)} \right\|_{L^p(\mathbb{R}^n)} < \epsilon,
$$
(e) for each fixed $D > 0$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $\vec{f} = \{f_k\}_{k \in \mathbb{Z}} \in G$, 
$$
\left\| \left( \int_{|k| > N} \sum_{|k|} |f_k(\cdot, t)|^2 dt \right)^{\frac{1}{2}} \chi_{B(0, D)} \right\|_{L^p(\mathbb{R}^n)} < \epsilon.
$$

Then $G$ is a strongly pre-compact set in $L^p(L^2([1, 2]), l^2; \mathbb{R}^n)$.

**Proof.** We employ the argument used in the proof of [9, Theorem 5], with some refined modifications. We claim that for each fixed $\epsilon > 0$, there exists a $\delta = \delta_\epsilon > 0$, and a mapping $\Phi_\epsilon$ on $L^p(L^2([1, 2]), l^2; \mathbb{R}^n)$, such that $\Phi_\epsilon(G) = \{\Phi_\epsilon(\vec{f}) : \vec{f} \in G\}$ is a strong pre-compact set in $L^p(L^2([1, 2]), l^2; \mathbb{R}^n)$, and for any $\vec{f}, \vec{g} \in G$, 
$$
\|\Phi_\epsilon(\vec{f}) - \Phi_\epsilon(\vec{g})\|_{L^p(L^2([1, 2]), l^2; \mathbb{R}^n)} < \delta \Rightarrow \|\vec{f} - \vec{g}\|_{L^p(L^2([1, 2]), l^2; \mathbb{R}^n)} < 9\epsilon.
$$

If we can prove this, then by Lemma 6 in [9], we see that $G$ is a strongly pre-compact set in $L^p(L^2[1, 2], l^2; \mathbb{R}^n)$. 
Now let $\epsilon > 0$. We choose $A > 1$ large enough as in assumption (b), $N \in \mathbb{N}$ such that for all $\{f_k\}_{k \in \mathbb{Z}} \in \mathcal{G}$,

$$
\left\| \left( \int_0^2 \sum_{|k| > N} |f_k(\cdot, t)|^2 dt \right)^{\frac{1}{2}} \chi_{B(0, 2A)} \right\|_{L^p(\mathbb{R}^n)} < \epsilon.
$$

Let $\varrho \in (0, 1/2)$ be small enough as in assumption (c), and $\sigma \in (0, 1/2)$ be small enough such that (d) holds true. Let $Q$ be the largest cube centered at the origin such that $2Q \subset B(0, \varrho)$, $Q_1, \ldots, Q_J$ be $J$ copies of $Q$ such that they are non-overlapping, and $B(0, A) \subset \bigcup_{j=1}^J Q_j \subset B(0, 2A)$, $I_1, \ldots, I_L \subset [1, 2]$ be non-overlapping intervals with the same length $|I|$, such that $|s - t| \leq \sigma$ for all $s, t \in I_j$ ($j = 1, \ldots, L$) and $\bigcup_{j=1}^J I_j = [1, 2]$. Define the mapping $\Phi_\epsilon$ on $L^p(L^2([1, 2]), I^2; \mathbb{R}^n)$ by

$$
\Phi_\epsilon(\vec{f}) (x, t) = \left\{ \ldots, 0, \ldots, 0, \sum_{j=1}^J \sum_{m=1}^L m_{Q_j \times I_j} (f_{-N}) \chi_{Q_j \times I_j}(x, t), \ldots, \sum_{j=1}^J \sum_{m=1}^L m_{Q_j \times I_j} (f_N) \chi_{Q_j \times I_j}(x, t), 0, \ldots, \right\},
$$

where, and in the following,

$$
m_{Q_j \times I_j} (f_k) = \frac{1}{|Q_j|} \frac{1}{|I_j|} \int_{Q_j \times I_j} f_k(x, t) dx dt.
$$

Note that

$$
|m_{Q_j \times I_j} (f_k)| \leq \left( \frac{1}{|Q_j| |I_j|} \int_{I_j} \int_{Q_j} |f_k(y, t)|^2 dy dt \right)^{\frac{1}{2}}.
$$

For $\vec{f} = \{f_k\}_{k \in \mathbb{Z}}$ and $p \in [2, \infty)$, we have that by the Hölder inequality,

$$
\|\Phi_\epsilon(\vec{f})\|_{L^p(L^2([1, 2]), I^2; \mathbb{R}^n)} = |Q|^1 - \frac{1}{2} |I|^{1 - \frac{N}{2}} \sum_{i=1}^J \sum_{j=1}^L \left( \int_{I_j} \int_{Q_i} \sum_{k \in \mathbb{Z}} |f_k(y, t)|^2 dy dt \right)^{\frac{1}{2}}
$$

$$
\leq \sum_{i=1}^J \sum_{j=1}^L \int_{I_j} \int_{Q_i} \left( \sum_{k \in \mathbb{Z}} |f_k(y, t)|^2 \right)^{\frac{1}{2}} dy dt \leq \|\vec{f}\|_{L^p(L^2([1, 2]), I^2; \mathbb{R}^n)}.
$$

On the other hand, we have that

$$
\sup_{-N < k < N} \sup_{t \in [1, 2]} \left| \sum_{i=1}^J \sum_{j=1}^L m_{Q_j \times I_j}(f_k) \chi_{Q_j \times I_j}(x, t) \right| \leq \sup_{k \in \mathbb{Z}} \sup_{t \in [1, 2]} |f_k(x, t)|.
$$
which implies that for $p_1 \in (1, \infty)$,
\[
\|\Phi_{\tau}(\vec{f})\|_{L^{p_1}(L^\infty([1, 2]), L^\infty; \mathbb{R}^n)} \lesssim \|\vec{f}\|_{L^{p_1}(L^\infty([1, 2]), L^\infty; \mathbb{R}^n)}. \tag{3.1}
\]
We also have that for $p_0 \in (1, \infty)$,
\[
|m_{Q_i \times I_j}(f_k)| \leq \left( \frac{1}{|Q_i||I_j|} \int_{I_j} \int_{Q_i} |f_k(y, t)|^{p_0} \, dy \, dt \right)^{\frac{1}{p_0}},
\]
and so
\[
\|\Phi_{\tau}(\vec{f})\|_{L^{p_0}(L^{p_0}([1, 2]), L^{p_0}; \mathbb{R}^n)} \lesssim \|\vec{f}\|_{L^{p_0}(L^{p_0}([1, 2]), L^{p_0}; \mathbb{R}^n)}. \tag{3.2}
\]
By interpolation, we deduce from (3.1) and (3.2) that for any $p \in (1, 2)$,
\[
\|\Phi_{\tau}(\vec{f})\|_{L^p(L^2([1, 2]), L^2; \mathbb{R}^n)} \lesssim \|\vec{f}\|_{L^p(L^2([1, 2]), L^2; \mathbb{R}^n)}.
\]
Thus, $\Phi_{\tau}(\mathcal{G}) = \{\Phi_{\tau}(\vec{f}) : \vec{f} \in \mathcal{G}\}$ is a strongly pre-compact set in $L^p(L^2([1, 2]), L^2; \mathbb{R}^n)$. Denote $\mathcal{D} = \bigcup_{j=1}^J Q_i$. Write
\[
\left\| \vec{f}_D - \Phi_{\tau}(\vec{f}) \right\|_{L^p(L^2([1, 2]), L^2; \mathbb{R}^n)}
\leq \left\| \left( \int_1^2 \sum_{|k| \leq N} |f_k(\cdot, t)\chi_D - \sum_{i=1}^L \sum_{j=1}^J m_{Q_i \times I_j}(f_k)\chi_{Q_i \times I_j}(\cdot, t)|^2 \, dt \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}
+ \left\| \left( \int_1^2 \sum_{|k| > N} |f_k(\cdot, t)|^2 \right)^{\frac{1}{2}} \chi_{B(0, 2A)} \right\|_{L^p(\mathbb{R}^n)}.
\]
Let
\[
E = \left\| \sup_{|h| \leq \sigma} \left( \int_1^2 \sum_{|k| \leq N} |f_k(\cdot, t) - f_k(\cdot + h, t)|^2 \, dt \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)},
\]
\[
F = \left\| \sup_{|s| \leq \sigma} \left( \int_1^2 \sum_{|k| \leq N} |f_k(\cdot, t) - f_k(\cdot, t + s)|^2 \, dt \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}.
\]
Noting that for $x \in Q_i$ with $1 \leq i \leq J$,
\[
\left\{ \int_1^2 \sum_{|k| \leq N} |f_k(x, t)\chi_D - \sum_{i=1}^L \sum_{j=1}^J m_{Q_i \times I_j}(f_k)\chi_{Q_i \times I_j}(x, t)|^2 \, dt \right\}^{\frac{1}{2}}
\lesssim |Q|^{-\frac{1}{2}} |I|^{-\frac{1}{2}} \left\{ \sum_{j=1}^J \int_{I_j} \int_{Q_i} \int_{|k| \leq N} |f_k(x, t) - f_k(y, s)|^2 \, dy \, ds \, dt \right\}^{\frac{1}{2}}
\]}
Marcinkiewicz integral

\[
\lesssim |Q|^{-\frac{1}{2}} \left\{ \int_{2|Q} \int_1^2 \sum_{|k| \leq N} |f_k(x, s) - f_k(x + h, s)|^2 ds dh \right\}^{\frac{1}{2}}
+ |I|^{-\frac{1}{2}} \left\{ \sum_{j=1}^L \int_{I_j} \int_{I_j} \sum_{|k| \leq N} |f_k(x, t) - f_k(x, s)|^2 dt ds \right\}^{\frac{1}{2}},
\]

we then get that

\[
\sum_{i=1}^J \int_{Q_i} \left\{ \int_1^2 \left( \sum_{|k| \leq N} \left| f_k(x, t) - \sum_{l=1}^J m_{Q_i} f_k \chi_{Q_i} \right|^2 \right) dt \right\}^{\frac{1}{2}} \approx E + F.
\]

It then follows from the assumption (b) that for all \( \vec{f} \in G \),

\[
\| \vec{f} - \Phi_\epsilon(\vec{f}) \|_{L^p(L^2([1,2]),I^2;\mathbb{R}^n)} \lesssim \left( \int_1^2 \left( \sum_{k \in \mathbb{Z}} \left| f_k(\cdot, t) - \sum_{l=1}^J m_{Q_i} f_k \chi_{Q_i} \right|^2 \right) dt \right)^{\frac{1}{2}} \chi_{\{|\cdot| > A\}} \left\| \vec{f} \right\|_{L^p(\mathbb{R}^n)} < 3\epsilon.
\]

Note that

\[
\| \vec{f} - \vec{g} \|_{L^p(L^2([1,2]),I^2;\mathbb{R}^n)} \lesssim \left( \int_1^2 \left( \sum_{k \in \mathbb{Z}} \left| f_k(\cdot, t) - \sum_{l=1}^J m_{Q_i} f_k \chi_{Q_i} \right|^2 \right) dt \right)^{\frac{1}{2}} \chi_{\{|\cdot| > A\}} \left\| \vec{f} \right\|_{L^p(\mathbb{R}^n)} + \left\| \Phi_\epsilon(\vec{f}) - \Phi_\epsilon(\vec{g}) \right\|_{L^p(L^2([1,2]),I^2;\mathbb{R}^n)}.
\]

Our claim then follows directly. This completes the proof of Lemma 3.4.

For \( b \in \text{BMO}(\mathbb{R}^n) \), set

\[
F^i_{j,b} f(x, t) = \int_{\mathbb{R}^n} (b(x) - b(y)) K^i_{j} \ast \phi_{j-1}(x - y) f(y) \, dy.
\]

**Proof of Theorem 1.1.** Let \( j_0 \in \mathbb{Z}, \ b \in C_0^\infty(\mathbb{R}^n) \) with \( \text{supp} \ b \subset B(0, R) \), \( p \in (1, \infty) \) and \( \delta \in (0, 1) \). Without loss of generality, we may assume that \( \|b\|_{L^\infty(\mathbb{R}^n)} + \|\nabla b\|_{L^\infty(\mathbb{R}^n)} = 1 \). We claim that

(i) for each fixed \( \epsilon > 0 \), there exists a constant \( A > 0 \) such that

\[
\left\| \left( \int_1^2 \sum_{j \in \mathbb{Z}} |F^i_{j,b} f(x, t)|^2 dt \right)^{\frac{1}{2}} \chi_{\{|\cdot| > A\}} (\cdot) \right\|_{L^p(\mathbb{R}^n)} < \epsilon \|f\|_{L^p(\mathbb{R}^n)};
\]
(ii) for $s \in (1, \infty)$,
\[
\left( \int_1^2 \sum_{j > j_0} |F_{j,b} f(x, t) - F_{j,b} f(x+h, t)|^2 dt \right)^{\frac{1}{2}} \lesssim 2^{-j_0} |h| \left( \mathcal{M}^{1, j_0}_\Omega f(x) + 2^{l(n+1)} M_s f(x) \right);
\]

(iii) for each $\epsilon > 0$ and $N \in \mathbb{N}$, there exists a constant $\sigma \in (0, 1/2)$ such that
\[
\left\| \sup_{|s| \leq \sigma} \left( \int_1^2 \sum_{|j| \leq N} |F_{j,b} f(x, s + t) - F_{j,b} f(x, t)|^2 dt \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} < \epsilon \|f\|_{L^p(\mathbb{R}^n)};
\]

(iv) for each fixed $D > 0$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that
\[
\left\| \left( \int_1^2 \sum_{j > N} |F_{j,b} f(\cdot, t)|^2 dt \right)^{\frac{1}{2}} \chi_{B(0,D)} \right\|_{L^p(\mathbb{R}^n)} < \epsilon \|f\|_{L^p(\mathbb{R}^n)}.
\]

We now prove claim (i). Let $t \in [1, 2]$. For each fixed $x \in \mathbb{R}^n$ with $|x| > 4R$, observe that $\text{supp} \ K_1^j * \phi_{j-l} \subset \{2^{j-2} \leq |x| \leq 2^{j+2}\}$, and $\int_{|z| < R} |K_1^j * \phi_{j-l}(x - z)| d\mu \neq 0$ only if $2^j \approx |x|$. A trivial computation leads to that
\[
\int_{|z| < R} |K_1^j * \phi_{j-l}(x - z)| d\mu \lesssim \left( \int_{|z| < 2|x|} |K_1^j * \phi_{j-l}(z)|^2 d\mu \right)^{\frac{1}{2}} R^2 \lesssim \|K_1^j \|_{L^1(S^{n-1})} \|\phi_{j-l}\|_{L^2(\mathbb{R}^n)} R^2 \lesssim 2^{n/2} |x|^{-\frac{n+1}{2}} R^2.
\]

On the other hand, we have that
\[
\sum_{j \in \mathbb{Z}} \left( \int_{|y| < R} |K_1^j * \phi_{j-l}(x - y)||f(y)|^s dy \right)^{\frac{1}{s}} = \sum_{j \in \mathbb{Z}, 2^j \approx |x|} \left( \int_{|y|/2 \leq |y-x| \leq 2|x|} |K_1^j * \phi_{j-l}(x - y)||f(y)|^s dy \right)^{\frac{1}{s}} \lesssim \left( M_{\Omega} M(|f|^s)(x) \right)^{\frac{1}{s}}.
\]

Another application of the Hölder inequality then yields
\[
\sum_{j \in \mathbb{Z}} |F_{j,b} f(x, t)|^2 \lesssim
\]
By the Young inequality, it is obvious that for 
$p$

\[ \sum_{j \in \mathbb{Z}} \left( \int_{|y| < R} |K_j^f \ast \phi_{j-1}(x-y)||f(y)|^p dy \right)^{\frac{2}{p}} \times \left( \int_{|y| < R} |K_j^f \ast \phi_{j-1}(x-y)|dy \right)^{\frac{2}{p}} \]

\[ \lesssim 2^{\frac{2j}{p}} |x|^{-\frac{2}{p}} R^{\frac{2}{p}} \left( M_\Omega M(|f|^*) (x) \right)^{\frac{2}{p}}. \]

This, in turn implies our claim (i).

We turn our attention to claim (ii). Write

\[ |F_{j, h}^l (x, t) - F_{j, h}^l (x + h, t)| \leq |b(x) - b(x + h)||F_{j}^l f(x, t)| + J_j^l f(x, t), \]

with

\[ J_j^l f(x, t) = \left| \int_{\mathbb{R}^n} (K_j^f \ast \phi_{j-1}(x-y) - K_j^f \ast \phi_{j-1}(x+h-y))(b(x+h) - b(y)) f(y) dy \right|. \]

It follows from Lemma 3.1 that

\[ \left( \sum_{j > j_0} |J_j^l f(x, t)|^2 \right)^{\frac{1}{2}} \lesssim \sum_{j > j_0} \int_{\mathbb{R}^n} |K_j^f \ast \phi_{j-1}(x-y) - K_j^f \ast \phi_{j-1}(x+h-y)||f(y)| dy \]

\[ \lesssim 2^{(n+1)} |h| 2^{-j_0} M_s f(x). \]

Therefore,

\[ \left( \int_{t} \sum_{j > j_0} |F_{j, h}^l f(x, t) - F_{j, h}^l f(x + h, t)|^2 dt \right)^{\frac{1}{2}} \]

\[ \lesssim |h| M_\Omega^{l, j_0} f(x) + 2^{(n+1)} 2^{-j_0} |h| M_s f(x). \] \hspace{1cm} (3.3)

The claim (ii) now follows from the (3.3) and Lemma 3.2.

We now verify claim (iii). For each fixed \( \sigma \in (0, 1/2) \) and \( t \in [1, 2] \), let

\[ U_{l, \sigma}^j (z) = \frac{1}{2^j} \frac{\Omega(z)}{|z|^{n-1}} \chi_{2^{j} (t-\sigma) \leq |z| \leq 2^{j+1}} + \frac{1}{2^j} \frac{\Omega(z)}{|z|^{n-1}} \chi_{2^{j+1} \leq |z| \leq 2^{j+1} (t+\sigma)} , \]

and

\[ G_{l, t, \sigma}^j f(x) = \int_{\mathbb{R}^n} (U_{l, \sigma}^j \ast |\phi_{l-j}|)(x-y)|f(y)| dy. \]

Note that

\[ \| U_{l, \sigma}^j \ast |\phi_{l-j}| \|_{L^1(\mathbb{R}^n)} \lesssim \sigma. \]

By the Young inequality, it is obvious that for \( p_1 \in (1, \infty) \),

\[ \left\| \sup_{|j| \leq N} \sup_{t \in [1, 2]} |G_{l, t, \sigma}^j f| \right\|_{L^{p_1}(\mathbb{R}^n)} \lesssim \sigma \| f \|_{L^{p_1}(\mathbb{R}^n)}, \] \hspace{1cm} (3.4)
and for \( p_0 \in (1, \infty) \),
\[
\int_{\mathbb{R}^n} \int_1^2 \sum_{|j| \leq N} |G_{j,t,x}^j f(x)|^{p_0} \, dt \, dx \lesssim N \sigma^{p_0} \|f\|_{L^{p_0} (\mathbb{R}^n)}^{p_0}.
\] (3.5)

We get from (3.4) and (3.5) that for \( p \in (1, 2) \),
\[
\left\| \left( \int_1^2 \sum_{|j| \leq N} |G_{j,t,x}^j f(x)|^2 \, dt \right)^{\frac{1}{2}} \right\|_{L^p (\mathbb{R}^n)} \lesssim N \sigma \|f\|_{L^p (\mathbb{R}^n)}.
\] (3.6)

On the other hand, for \( p \in [2, \infty) \), we obtain from the Minkowski inequality and the Young inequality that
\[
\left\| \left( \int_1^2 \sum_{|j| \leq N} |G_{j,t,x}^j f(x)|^2 \, dt \right)^{\frac{1}{2}} \chi_{B(0,D)} \right\|_{L^p (\mathbb{R}^n)} \lesssim (2N\sigma)^2 \|f\|_{L^p (\mathbb{R}^n)}.
\] (3.7)

Since
\[
\sup_{|s| \leq \sigma} |F_{j,t,b}^j f(x) - F_{j,t,b}^j f(x, t + s)| \leq G_{j,t,x}^j f(x),
\]
our claim (iii) now follows from (3.6) and (3.7) immediately if we choose \( \sigma = \epsilon/(2N) \).

It remains to prove (iv). Let \( D > 0 \) and \( N \in \mathbb{N} \) such that \( 2^{N-2} > D \). Then for \( j > N \) and \( x \in \mathbb{R}^n \) with \( |x| \leq D \),
\[
\int_{\mathbb{R}^n} |K_{t,x} f(x - y) f(y)| \, dy = \int_{\mathbb{R}^n} |K_{t,x} f(x - y) f(y)| \chi_{|y| \leq 2^{j+3}}(y) \, dy
\]
\[
\lesssim \int_{|y| \leq 2^{j+3}} |f(y)| \|K_{t,x} f(x)\|_{L^1 (\mathbb{R}^n)} \|f\|_{L^\infty (\mathbb{R}^n)} \lesssim 2^{Nl} 2^{-\frac{n}{2}} \|f\|_{L^p (\mathbb{R}^n)}.
\]

Therefore,
\[
\left\| \left( \int_1^2 \sum_{j > N} |F_{j,t,b} f(\cdot, t)|^2 \, dt \right)^{\frac{1}{2}} \chi_{B(0,D)} \right\|_{L^p (\mathbb{R}^n)} \lesssim 2^{Nl} \left( \frac{D}{2N} \right)^{\frac{p}{2}} \|f\|_{L^p (\mathbb{R}^n)}.
\]
We can now conclude the proof of Theorem 1.1. Let $p \in (1, \infty)$. Our claims (i)–(iv), via Lemma 3.2 and Lemma 3.4, prove that for $b \in C_0^\infty(\mathbb{R}^n)$, $l \in \mathbb{N}$ and $j_0 \in \mathbb{Z}_-$, the operator $F_{j_0}^l$ defined by

$$F_{j_0}^l : f(x) \rightarrow \{ \ldots, 0, \ldots, F_{j_0}^l b f(x, t), F_{j_0+1}^l b f(x, t), \ldots \}$$

is compact, and completely continuous from $L^p(\mathbb{R}^n)$ to $L^p(L^2([1, 2], L^2; \mathbb{R}^n))$. Thus, $\widetilde{M}_{\Omega,b}^{j_0,l}$ is completely continuous on $L^p(\mathbb{R}^n)$. This, via Lemma 3.3 and Theorem 2.1, shows that for $b \in C_0^\infty(\mathbb{R}^n)$, $\mathcal{M}_{\Omega,b}$ is completely continuous on $L^p(\mathbb{R}^n)$. Note that

$$|\mathcal{M}_{\Omega,b} f_k(x) - \mathcal{M}_{\Omega,b} f(x)| \lesssim \mathcal{M}_{\Omega,b}(f_k - f)(x) \lesssim \widetilde{M}_{\Omega,b}(f_k - f)(x).$$

Thus, for $b \in C_0^\infty(\mathbb{R}^n)$, $\mathcal{M}_{\Omega,b}$ is completely continuous on $L^p(\mathbb{R}^n)$. Recalling that when $\Omega \in L(1)(S_n^{-1})$, $\mathcal{M}_{\Omega,b}$ is bounded on $L^p(\mathbb{R}^n)$ with bound $C\|b\|_{\text{BMO}(\mathbb{R}^n)}$ (see [5]), we finally obtain that for $b \in \text{CMO}(\mathbb{R}^n)$, $\mathcal{M}_{\Omega,b}$ is completely continuous on $L^p(\mathbb{R}^n)$.

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