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Isotropy index for the connected sum and the direct product of manifolds

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Abstract. A subspace or subgroup is isotropic under a bilinear map if the restriction of the map on it is trivial. We study maximal isotropic subspaces or subgroups under skew-symmetric maps and, in particular, the isotropy index – the maximum dimension of an isotropic subspace or maximum rank of an isotropic subgroup. For a smooth closed orientable manifold M, we describe the geometric meaning of the isotropic subgroups of the first cohomology group with different coefficients under the cup product. We calculate the corresponding isotropy index, as well as the set of ranks of all maximal isotropic subgroups, for the connected sum and the direct product of manifolds. Finally, we study the relationship of the isotropy index with the first Betti number and the co-rank of the fundamental group. We also discuss applications of these results to the topology of foliations.

1. Introduction

Let M be a smooth closed orientable connected n-dimensional manifold. We study *isotropic* subgroups (subspaces) H of its cohomology group (space) $H^1(M;R)$, where R is a field or the ring of integers, under the cup-product

$$\smile : H^1(M;R) \times H^1(M;R) \to H^2(M;R), \tag{1}$$

i.e., $H \subseteq H^1(M;R)$ such that $H \smile H = 0$. To avoid duplication of terminology, such as "rank (dimension)", we will refer to $H^i(M;R)$ as modules over the ring R.

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Specifically, we study the set $\mathcal{H}(M;R)$ of ranks of maximal isotropic submodules of $H^1(M;R)$ and the corresponding *isotropy index*

$$h(M;R) = \max \mathcal{H}(M;R), \tag{2}$$

the maximum rank of an isotropic submodule. We study the structure of (maximal) isotropic submodules for finite connected sums and direct products of manifolds. In particular, we show (Theorems 21 and 27) that

$$\begin{aligned} \mathcal{H}(M_1 \# M_2;R) &= \mathcal{H}(M_1;R) + \mathcal{H}(M_2;R) \quad \text{and} \\ \mathcal{H}(M_1 \times M_2;R) &= \{1\} \cup \mathcal{H}(M_1;R) \cup \mathcal{H}(M_2;R), \end{aligned} \tag{3}$$

and thus

$$\begin{aligned} h(M_1 \# M_2;R) &= h(M_1;R) + h(M_2;R) \quad \text{and} \\ h(M_1 \times M_2;R) &= \max\{h(M_1;R), h(M_2;R)\} \end{aligned} \tag{4}$$

under certain conditions and with certain exceptions described in the corresponding theorems (here the sum of sets is understood element-wise).

Isotropy index bounds the co-rank $b_1'(M)$ of the fundamental group, i.e., the maximum rank of a free homomorphic image of $\pi_1(M)$ [9], and, obviously, is bounded by the first Betti number $b_1(M) = \operatorname{rk} H_1(M;\mathbb{Z})$:

$$b_1'(M) \le h(M;\mathbb{Z}) \le b_1(M). \tag{5}$$

In [21], for a field F, upper and lower bounds on $h(M;F)$ were given in terms of Betti numbers; see Proposition 16. Using (3) and (4), for a given R we describe all possible sets $\mathcal{H}(M;R)$ (Proposition 30) and all possible values of $h(M,R)$ with different M in terms of $b_1(M;R) = \operatorname{rk} H_1(M;R)$ (Theorem 33), as well as extend (5) to fields of characteristic zero and show that in this case these bounds are exact (Proposition 39).

The notion of isotropy has been studied in the context of algebraic geometry. For instance, isotropic subspace theorems by Catanese [4] and Bauer [3] establish relations between isotropic subspaces of $H^1(M;\mathbb{C})$ for a smooth quasi-projective variety M and certain irrational pencils. These theorems have been studied in [5], [7].

The isotropy index has numerous applications to the topological study of manifolds and foliations. As we show, isotropy for manifolds has a clear geometric meaning: (maximal) isotropic subgroups of $H^1(M;\mathbb{Z})$ of rank k correspond to (maximal) systems of k homologically independent, homologically non-intersecting closed orientable codimension-one submanifolds, $h(M;\mathbb{Z})$ being the maximum number of such submanifolds (Theorem 13). While this geometric meaning is defined for $R = \mathbb{Z}$, we show that the relevant aspects of isotropy coincide for $R = \mathbb{Z}$ and $R = \mathbb{Q}$ (Lemma 7), which enables the use of simpler, vector space-based techniques in geometric applications of isotropy.

Consider a foliation defined on M by a Morse form ω, i.e., a closed one-form that is locally the differential of a Morse function. Such foliations have important applications in modern physics, for example, in supergravity [1], [2]. A Morse form foliation defines a decomposition of M into a finite number $m(\omega)$ of minimal components and a finite number $M(\omega)$ of maximal components, i.e., connected components of the union of compact leaves, which are cylinders over a compact leaf. These two numbers are bounded by $h(M;\mathbb{Z})$: $M(\omega) + m(\omega) \leq h(M;\mathbb{Z}) + |\operatorname{Sing}\omega| - 1$, where $\operatorname{Sing}\omega$ is the singular set, which is finite [10]. In homological terms, for the number $c(\omega)$ of homologically independent compact leaves it holds $c(\omega) + m(\omega) \leq h(M;\mathbb{Z})$ [10].

A sufficient condition of existence of a minimal component has been given in [20] in terms of $\operatorname{rk}\omega$, the rank of the group of the periods: if $\operatorname{rk}\omega > h(M;\mathbb{Z})$, then the foliation has a minimal component, i.e., $m(\omega) \geq 1$. Also, in case of strong inequality in the upper bound in (5), i.e., if $h(M;\mathbb{Z}) < b_1(M)$, the foliation of a Morse form in general position has a minimal component [11].

If the subgroup $H_\omega \subseteq H_{n-1}(M)$ generated by the homology classes of all compact leaves of a Morse form foliation is maximal isotropic, then the foliation has no minimal components, i.e., $m(\omega) = 0$ [19]. Subgroups of $H_{n-1}(M)$ are related, by Poincaré duality, with those of $H^1(M;\mathbb{Z})$. In particular, if the homology classes of some compact leaves of a Morse form foliation generate a maximal isotropic subgroup, then $m(\omega) = 0$ [19]. This is the case when the foliation has $h(M;\mathbb{Z})$ homologically independent compact leaves. However, if $M = M_1 \times M_2$, then in some cases our results allow us to conclude that $m(\omega) = 0$ by examining only one leaf (Examples 31 and 32).

Isotropic submodules of L can be defined for arbitrary bilinear map $\varphi : L \times L \to V$, where L, V are finitely generated groups or finite-dimensional vector spaces. This gives the corresponding notions of $\mathcal{H}(\varphi)$ and the isotropy index $h(\varphi)$ as in (2). In order to establish our main results (3) and (4), we study the behavior of isotropic submodules for skew-symmetric maps under operations of extension of scalars (Proposition 8) and direct sum $L_1 \oplus L_2$ (Lemmas 20 and 26).

For $\dim M = 2$, by Poincaré duality, $\mathcal{H}(M;\mathbb{Z})$ and $h(M;\mathbb{Z})$ can be equivalently defined in terms of the intersection map $\circ\colon H_1(M) \times H_1(M) \to \mathbb{Z}$ instead of the cup product (1). For a closed one-form ω on M, the isotropy index $h(\omega)$ is defined by the restriction $\circ|_{\ker[\omega]\times\ker[\omega]}$ of the intersection map to the group $\ker[\omega] \subseteq H_1(M)$, where $[\omega]$ is the integration map. This notion has been extensively used to study the structure of Morse form foliations on closed orientable surfaces M_g^2 of genus g. For example, $c(\omega) \le h(\omega)$; if the foliation has no minimal components, then $h(\omega) = g$ [12]. For so-called weakly generic forms, $m(\omega) \ge g - \frac{1}{2}k(\omega) - h(\omega)$, where $k(\omega)$ is the number of singularities surrounded by a minimal component [14]. Since $h(\omega) \le h(M;\mathbb{Z})$, these inequalities hold also for $h(M;\mathbb{Z})$.

The paper is organized as follows. In Section 2, we give basic facts on isotropy in finite-dimensional vector spaces and finitely generated abelian groups. In Section 3, we introduce the isotropy index for manifolds and consider its properties and geometric meaning. In Section 4, we calculate the isotropy index of the connected sum of two manifolds. In Section 5, we calculate the isotropy index of the direct product of two manifolds and describe the possible sets $\mathcal{H}(M;R)$. In Section 6, we completely characterize the relation between $h(M;R)$ and $b_1(M;R)$. Finally, in Section 7 we consider the relations between $h(M;R)$ and $b_1'(M)$.

2. Isotropy index for vector spaces and abelian groups

In this section, we will define the isotropy index and discuss how it changes from groups to vector fields or between vector fields with different scalars.

We will deal with finite-dimensional vector spaces and finitely generated abelian groups. To avoid duplication of terminology, such as "any subspace or subgroup" or "its dimension or rank", we will use terminology from R-modules, where R will be a field or $\mathbb{Z}$, correspondingly. In particular, *submodule* will stand for subspace or subgroup; *rank* will stand for the dimension of a space or the rank of a group.

2.1. Definitions. Let L, V be finitely generated R-modules and $\varphi\colon L \times L \to V$ a bilinear map; R is a field or $R = \mathbb{Z}$.

Definition 1. A submodule $H \subseteq L$ is called *isotropic* under the map φ if $\varphi|_{H\times H} = 0$, i.e., $\varphi(l_1, l_2) = 0$ for any $l_1, l_2 \in H$.

If R is a field, R-modules L and V are finite-dimensional vector spaces, so we deal with *isotropic subspaces*; if $R = \mathbb{Z}$, then R-modules L, V are finitely generated abelian groups, so we have *isotropic subgroups.*

Since L is Noetherian, every isotropic submodule is contained in some maximal isotropic submodule, not necessarily unique. Denote by $\mathcal{H}(\varphi)$ the set of ranks of maximal isotropic submodules under the map φ:

$$\mathcal{H}(\varphi) = \{\operatorname{rk} H \mid H \text{ is a maximal isotropic submodule of } L\}.$$

Obviously, $\mathcal{H}(\varphi)$ is a finite set of non-negative integers such that

$$0 \notin \mathcal{H}(\varphi) \quad \text{or} \quad \mathcal{H}(\varphi) = \{0\}. \tag{6}$$

Proposition 30 below shows that these are the only restrictions on $\mathcal{H}(\varphi)$.

Definition 2. The *isotropy index* $h(\varphi)$ is the maximum rank of an isotropic submodule of L:

$$h(\varphi) = \max \mathcal{H}(\varphi).$$

Example 3. Consider the skew-symmetric map $\varphi\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, $\varphi(x, y) = [[x, y], l]$, where l is a fixed vector and $[\ ,\ ]$ is the vector product. For any vector $x \not\perp l$, for example $x = l$, the subspace $L_1 = \langle x \rangle$, $\dim L_1 = 1$, is maximal isotropic, and so is $L_2 = l^\perp$, $\dim L_2 = 2$. Thus $\mathcal{H}(\varphi) = \{1, 2\}$, and $h(\varphi) = 2$.

For skew-symmetric maps, usually $h(\varphi) \geq 1$, and thus $0 \notin \mathcal{H}(\varphi)$:

Lemma 4. *Let φ be skew-symmetric. Then $h(\varphi) = 0$, i.e., $\mathcal{H} = \{0\}$, iff either*

- $L = 0$ *or*
- $L = R$, $\operatorname{char} R = 2$, *and* $\varphi \not\equiv 0$.

Proof. Let $h(\varphi) = 0$, then $\varphi(l, l) \neq 0$ for any $0 \neq l \in L$. Unless $L = 0$, for a skew-symmetric map this implies $\operatorname{char} R = 2$. Suppose $\operatorname{rk} L \geq 2$. Consider independent $l_1, l_2 \in L$; $\varphi(l_i, l_i) = 1$. Then, for $l = l_1 + l_2 \neq 0$, we have $\varphi(l, l) = 0$, a contradiction. □

The *kernel* of a bilinear map $\varphi : L \times L \to V$ is

$$\ker \varphi = \{l \in L \mid \varphi(l, l') = 0 \text{ for any } l' \in L\}.$$

Obviously, $\ker \varphi$ is an isotropic submodule; moreover, any maximal isotropic submodule contains $\ker \varphi$, so $h(\varphi) \geq \dim \ker \varphi$.

2.2. Isotropy index for different coefficients. Generally speaking, the isotropy index depends on the coefficients. Namely, let L, V be finitely generated abelian groups and $\varphi\colon L \times L \to V$ a skew-symmetric bilinear map. For a field F, consider the corresponding vector spaces

$$L_F = F \otimes L, \qquad V_F = F \otimes V,$$

and the induced skew-symmetric bilinear map

$$\varphi_F\colon L_F \times L_F \to V_F, \qquad \varphi_F(\alpha_1 \otimes x_1, \alpha_2 \otimes x_2) = \alpha_1\alpha_2 \otimes \varphi(x_1, x_2).$$

The isotropy index depends on the field F, and generally $h(\varphi) \neq h(\varphi_F)$:

Example 5. Consider $\varphi : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}_2 = \mathbb{Z}/2\,\mathbb{Z}$, $\varphi(1,1) = 1$. It has an isotropic subgroup $2\,\mathbb{Z}$, thus $h(\varphi) = 1$. Similarly, for $F = \mathbb{Q}$, we have $\varphi_{\mathbb{Q}} : \mathbb{Z}\times\mathbb{Z} \to 0$, thus again $h(\varphi) = 1$, in accordance with (8) below. However, for $F = \mathbb{Z}_2$, we have $L_F = \mathbb{Z}_2 \otimes \mathbb{Z} = \mathbb{Z}_2$, $V_F = \mathbb{Z}_2 \otimes \mathbb{Z}_2 = \mathbb{Z}_2$, so $\varphi_{\mathbb{Z}_2}\colon \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$, $\varphi(1,1) = 1$; obviously, $\varphi_{\mathbb{Z}_2}^{-1}(0) = 0$, and thus $h(\varphi_{\mathbb{Z}_2}) = 0$:

$$h(\varphi_{\mathbb{Z}_2}) < h(\varphi) = h(\varphi_{\mathbb{Q}}).$$

On the other hand, consider $\varphi : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}$ defined by the matrix $\begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}$, $k \geq 2$. Then $h(\varphi) = h_{\mathbb{Q}}(\varphi) = 1$, but $\varphi_{\mathbb{Z}_p} : \mathbb{Z}_p^2\times\mathbb{Z}_p^2 \to 0$ (thus $h(\varphi_{\mathbb{Z}_p}) = 2$) iff $p \mid k$. So for $p \mid k$ and $q \nmid k$ we have:

$$h(\varphi) = h(\varphi_{\mathbb{Q}}) = h(\varphi_{\mathbb{Z}_q}) < h(\varphi_{\mathbb{Z}_p}).$$

However, extension of scalars for vector spaces does not decrease $h(\varphi_F)$:

Lemma 6. *Let L_F, V_F be finite-dimensional vector spaces over a filed F, and $\varphi_F\colon L_F \times L_F \to V_F$ be a skew-symmetric bilinear map. Let F' be a field, $F \subseteq F'$,*

$$L_{F'} = F' \otimes_F L, \qquad V_{F'} = F' \otimes_F V$$

vector spaces obtained from L_F, V_F by extension of scalars, and $\varphi_{F'}$ the induced map:

$$\varphi_{F'}\colon L_{F'} \times L_{F'} \to V_{F'}, \qquad \varphi_{F'}(\alpha_1' \otimes x_1, \alpha_2' \otimes x_2) = \alpha_1'\alpha_2' \otimes \varphi_F(x_1, x_2). \tag{7}$$

Then

$$h(\varphi_F) \leq h(\varphi_{F'}).$$

Proof. Consider an isotropic subspace $H_F \subseteq L_F$, $\dim H_F = h(\varphi_F) = k$. A basis $\langle e_1, \dots, e_k\rangle = H_F$ can be be extended to a basis for L_F; thus $H_F = F^k$. Extension of scalars from F to F' gives $H_{F'} = F' \otimes_F H_F = F' \otimes_F F^k = (F')^k$, so $\dim H_{F'} = \dim H_F$. By (7), the subspace $H_{F'}$ is isotropic, i.e., $k = \dim H_{F'} \leq h(\varphi_{F'})$. We obtain $h(\varphi_F) \leq h(\varphi_{F'})$. □

A stronger fact holds for groups and $\mathbb{Q}$:

Lemma 7. *Let L, V be finitely generated abelian groups, and $\varphi\colon L\times L\to V$ be a skew-symmetric bilinear map. Denote by*

$$L_{\mathbb{Z}}=L,\qquad L_{\mathbb{Q}}=\mathbb{Q}\otimes L,\qquad V_{\mathbb{Z}}=V,\qquad V_{\mathbb{Q}}=\mathbb{Q}\otimes V$$

the corresponding $\mathbb{Z}$- and $\mathbb{Q}$-modules. Let

$$\varphi_{\mathbb{Q}}\colon L_{\mathbb{Q}}\times L_{\mathbb{Q}}\to V_{\mathbb{Q}},\qquad \varphi_{\mathbb{Q}}(q_1\otimes x_1,q_2\otimes x_2)=q_1q_2\otimes\varphi(x_1,x_2)$$

be the induced skew-symmetric bilinear map. Then

(i) *for every maximal isotropic subgroup $H\subseteq L$, the subspace $H_{\mathbb{Q}}=\mathbb{Q}\otimes H\subseteq L_{\mathbb{Q}}$ is maximal isotropic;*

(ii) *for every maximal isotropic subspace $H_{\mathbb{Q}}\subseteq L_{\mathbb{Q}}$, there is a maximal isotropic subgroup $H\subseteq L$ such that $H_{\mathbb{Q}}=\mathbb{Q}\otimes H$.*

In particular,

$$\mathcal{H}(\varphi)=\mathcal{H}(\varphi_{\mathbb{Q}}),\qquad h(\varphi)=h(\varphi_{\mathbb{Q}}).\tag{8}$$

Proof. (i) Let $H\subseteq L$ be a maximal isotropic subgroup, $H=\langle h_1,\dots,h_n\rangle$. Then $H_{\mathbb{Q}}=\mathbb{Q}\otimes H\subseteq L_{\mathbb{Q}}$ is an isotropic subspace, $\dim H_{\mathbb{Q}}=\operatorname{rk}H$. Consider $0\neq q\otimes x\in L_{\mathbb{Q}}$ such that $\varphi_{\mathbb{Q}}(q\otimes x,H_{\mathbb{Q}})=0$, i.e., all $\varphi(x,h_i)\in V_T$, the torsion subgroup. Then for some $k\neq 0$, we have $\varphi(kx,h_i)=0$. Since H is maximal, $kx\in H$, and thus $q\otimes x\in H_{\mathbb{Q}}$. Therefore, $H_{\mathbb{Q}}$ is maximal.

(ii) Let $H_{\mathbb{Q}}\subseteq L_{\mathbb{Q}}$ be a maximal isotropic subspace, $H_{\mathbb{Q}}=\langle q_1\otimes h_1,\dots,q_n\otimes h_n\rangle$, a basis. Consider $H'=\langle h_1,\dots,h_n\rangle$. Then all $\varphi(h_i,h_j)\in V_T$; thus for some $k\neq 0$, all $\varphi(kh_i,kh_j)=0$. We obtain $H_{\mathbb{Q}}=\mathbb{Q}\otimes H''$ for an isotropic subgroup $H''=kH'=\{kx\mid x\in H'\}$, $k\neq 0$.

Consider a maximal isotropic subgroup $H\supseteq H''$. For any $x\in H$, we have $\varphi(x,H'')=0$, and thus $\varphi_{\mathbb{Q}}(1\otimes x,H_{\mathbb{Q}})=0$. Since $H_{\mathbb{Q}}$ is maximal, $1\otimes x\in H_{\mathbb{Q}}$. We obtain $H_{\mathbb{Q}}=\mathbb{Q}\otimes H$. □

Lemma 7 allows formulating Lemma 6 for fields or $\mathbb{Z}$:

Proposition 8. *Let L_R,V_R be finitely generated R-modules, R being a field or $\mathbb{Z}$, and $\varphi_R\colon L_R\times L_R\to V_R$ be a skew-symmetric bilinear map. Let R' be a field, $R\subseteq R'$,*

$$L_{R'}=R'\otimes_R L,\qquad V_{R'}=R'\otimes_R V$$

modules obtained by extension of scalars, and $\varphi_{R'}\colon L_{R'}\times L_{R'}\to V_{R'}$ the induced map. Then

$$h(\varphi_R)\le h(\varphi_{R'}).\tag{9}$$

In particular, in addition to extension of scalars of vector spaces, (9) holds for a group and a corresponding vector space over F, $\operatorname{char}F=0$, since $\mathbb{Z}\subset\mathbb{Q}\subseteq F$.

3. Isotropy index for manifolds

In this section, we introduce maximal isotropic subgroups (subspaces) of the first cohomology group (space) and the isotropy index for manifolds, and discuss their geometric meaning and properties.

3.1. Definitions. Let M be a smooth closed orientable n-dimensional manifold. Consider the cup product

$$\smile\colon H^1(M;R)\times H^1(M;R)\to H^2(M;R),$$

where $R=\mathbb{Z}$ or R is a field. It is a skew-symmetric bilinear map, and $H^k(M;R)$ are finitely generated R-modules; in case of a field, $H^k(M;R)$ are vector spaces.

Definition 9. A submodule $H\subseteq H^1(M;R)$ is called *isotropic* if it is isotropic under $\smile$ in the sense of Definition 1, i.e., if the restriction of the cup-product to $H\times H$ is zero: $\smile|_{H\times H}=0$.

Accordingly, we denote by $\mathcal{H}(M;R)$ the set of ranks of maximal isotropic submodules:

$$\mathcal{H}(M;R)=\{\operatorname{rk}H\mid H\ \text{ is a maximal isotropic submodule of } H^1(M;R)\};$$

Proposition 30 below shows that (6) is still the only restriction on $\mathcal{H}(M;R)$, i.e., that almost any set of non-negative integers is $\mathcal{H}(M;R)$ for some manifold M.

The *isotropy index*

$$h(M;R)=\max\mathcal{H}(M;R)$$

is the maximum rank of an isotropic submodule of $H^1(M;R)$.

Lemma 7 allows us to work interchangeably with $H^1(M;\mathbb{Z})$ and $H^1(M;\mathbb{Q})$:

Lemma 10. *For a smooth closed orientable manifold M, there exists a maximal isotropic subgroup $H\subseteq H^1(M;\mathbb{Z})$, $\operatorname{rk}H=k$, iff there exists a maximal isotropic subspace $H_Q\subseteq H^1(M;\mathbb{Q})$, $\dim H_Q=k$, i.e.,*

$$\mathcal{H}(M;\mathbb{Z})=\mathcal{H}(M;\mathbb{Q}),\qquad h(M;\mathbb{Z})=h(M;\mathbb{Q}).$$

3.2. Geometric meaning of $h(M;\mathbb{Z})$. The notions of $\mathcal{H}(M;\mathbb{Z})$ and $h(M;\mathbb{Z})$ have a clear geometric meaning, which can be characterized as follows:

Definition 11. An *isotropic system* on a manifold M, $\dim M \geq 2$, is a set of homologically non-intersecting, homologically independent smooth closed orientable connected codimension-one submanifolds $X_1, \ldots, X_k \subset M$, intersecting transversely:

$$[X_i \cap X_j] = 0, \quad i \neq j;\ i, j = 1, \ldots, k. \tag{10}$$

The requirement (10) cannot be simplified to $X_i \cap X_j = \emptyset$, since on some manifolds there exist submanifolds with non-empty, but homologically trivial, intersection:

Example 12. On the Heisenberg 3-nilmanifold, any two homologically independent smooth closed orientable 2-submanifolds have non-empty, but homologically trivial, intersection. This will be shown as Example 40 below; here we only give a graphical illustration.

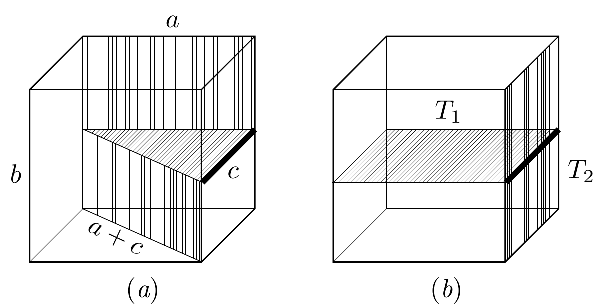

Figure 1. The Heisenberg nilmanifold H^3, represented as a T^2-bundle over the circle S^1. The circle is shown as the vertical line b; T^2 is shown as a horizontal square with the sides a and c, the opposite sides of the square being identified. The top is identified with the bottom with the Dehn twist: $a \sim a + c$, $c \sim c$; all four vertical lines are identified. (a) The curve realizing c is the boundary of a 2-submanifold shown as the hatched rectangle, triangle, and another rectangle; thus $c = 0$. The triangle forms a disk with two holes, which are glued to the cylinder formed by the two rectangles; the resulting figure is a torus with a disk removed, whose boundary realizes c. (b) The two tori T_1, T_2 intersect by a curve realizing c, thus $[T_1 \cap T_2] = 0$. They cannot be made non-intersecting; see Example 40.

The Heisenberg 3-nilmanifold H^3 is a T^2-bundle over the circle S^1, with the monodromy being a Dehn twist $f: T^2 \to T^2$, defined as the quotient space

$$H^3 = \frac{[0,1] \times T^2}{(1,x) \sim (0, f(x))}, \qquad f = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

For the basis cycles a, c of the torus T^2, we have

$$f_*(a) = a + c, \qquad f_*(c) = c;$$

see Figure 1(a). The cycle c is homologically trivial, being realized by the boundary of a 2-submanifold (torus without a disk) shown in Figure 1(a). However, for the two submanifolds $T_i = T^2$ shown in Figure 1(b), we have $[T_1 \cap T_2] = c$.

An algebraic model of H^3 can be given as follows: consider the 3-dimensional Heisenberg group over a ring R,

$$H(R) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in R \right\};$$

then the Heisenberg nilmanifold $H^3 = H(\mathbb{R})/H(\mathbb{Z})$ is the quotient of the real Heisenberg group by the discrete Heisenberg subgroup. It is a compact orientable connected 3-manifold with Nil geometry.

Definition 11 implies the cardinality of an isotropic system $k \le b_1(M)$, the Betti number; thus each isotropic system is contained in a maximal isotropic system.

The following theorem relies on the fact that homology classes $z \in H_{n-1}(M)$ can be realized by smooth closed orientable connected codimension-one submanifolds.

Theorem 13. *Let M be a smooth closed orientable connected manifold, $\dim M \ge 2$, and $D\colon H^1(M;\mathbb{Z}) \to H_{n-1}(M)$ be the Poincaré duality map. Then:*

(i) *Let $\{X_i\}$ be a (maximal) isotropic system. Then $\{D^{-1}[X_i]\}$ form a basis of a (maximal) isotropic subgroup $H \subseteq H^1(M;\mathbb{Z})$.*

(ii) *Let $\{x_i\}$ be a basis of a (maximal) isotropic subgroup $H \subseteq H^1(M;\mathbb{Z})$. Then $\{Dx_i\}$ can be realized by submanifolds X_i that form a (maximal) isotropic system.*

In particular,

- *$\mathcal{H}(M;\mathbb{Z}) = \{k \mid X_1, \dots, X_k \subset M$ is a maximal isotropic system$\}$;*
- *the isotropy index $h(M;\mathbb{Z})$ is the maximum cardinality of an isotropic system of submanifolds of M.*

PROOF. Consider an isotropic subgroup $H \subset H^1(M;\mathbb{Z})$, $\operatorname{rk} H = k$. Since $H^1(M;\mathbb{Z})$ is torsion-free, it has a basis, $H = \langle u_1, \dots, u_k\rangle$. The cup product

$$\smile\colon H^1(M;\mathbb{Z}) \times H^1(M;\mathbb{Z}) \to H^2(M;\mathbb{Z})$$

is dual to the homology classes intersection map

$$\circ\colon H_{n-1}(M) \times H_{n-1}(M) \to H_{n-2}(M);$$

namely,

$$D(u_i \smile u_j) = Du_i \circ Du_j,$$

where D is the Poincaré duality map.

Realize the cycles $Du_i \in H_{n-1}(M)$ by suitable submanifolds $X_i \subset M$, $Du_i = [X_i]$, choosing them to intersect transversely. Then

$$\pm[X_i \cap X_j] = [X_i] \circ [X_j] = Du_i \circ Du_j = D(u_i \smile u_j),$$

where the sign depends on the choice of orientation in X_i and X_j. Since H is isotropic, all $u_i \smile u_j = 0$; thus $[X_i \cap X_j] = 0$ for any $i \neq j$. Since u_i are independent, so are $[X_i]$. If H is maximal, then so is this system, because expanding it would, by duality, expand H.

Similarly, given a (maximal) system of k such submanifolds $X_i \subset M$, the group $H = \langle D^{-1}[X_i]\rangle \subseteq H^1(M;\mathbb{Z})$ is a (maximal) isotropic subgroup, $\operatorname{rk} H = k$. □

Examples 40 and 41 below show that the homological interpretation of the non-intersection requirement 10 is important for Theorem 13: some manifolds have fewer non-intersecting submanifolds, $X_i \cap X_j = \emptyset$, with the properties listed in Definition 11, than homologically non-intersecting such submanifolds, $[X_i \cap X_j] = 0$. For discussion of systems with $X_i \cap X_j = \emptyset$, see Section 7.

3.3. Properties and examples. Recall that the Betti number $b_k(M;R) = \operatorname{rk} H_k(M;R)$; by definition, $b_k(M) = b_k(M;\mathbb{Z})$. By the universal coefficient theorem, if R is a field with $\operatorname{char} R = 0$, then $b_k(M;R) = b_k(M)$. Since $H^k(M;R) = H_k(M;R) = 0$ for $k > \dim M$, the following statements apply to S^1 and a point $*$.

Lemma 14. *Let M be a smooth closed orientable manifold; $R = \mathbb{Z}$ or R be a field. Then*

$$1 \leq h(M;R) \leq b_1(M;R), \tag{11}$$

except that

$$h(M;R) = 0 \tag{12}$$

iff any of the following conditions holds:

- $b_1(M;R) = 0$, *or*
- $b_1(M;R) = 1$, $\operatorname{char} R = 2$, *and the cup product* $\smile \not\equiv 0$.

Theorem 33 below states that this lemma gives the only relation between $h(M;R)$ and $b_1(M;R)$.

Proof. By definition, we have $h(M;R) \leq \operatorname{rk} H^1(M;R)$. For $\mathbb{Z}$, by Poincaré duality, $H^1(M;\mathbb{Z}) \cong H_{n-1}(M)$, so $\operatorname{rk} H^1(M;\mathbb{Z}) = b_{n-1}(M) = b_1(M)$. For a field F, $H^1(M;F) \cong H_1(M;F)$, so $\dim H^1(M;F) = \dim H_1(M;F) = b_1(M;F)$. Since the cup product is skew-symmetric, (12) is given by Lemma 4. $\square$

Example 15. Consider $M = \mathbb{R}P^3$; it is orientable. Its cohomology ring is

$$H^*(\mathbb{R}P^3;\mathbb{Z}_2) \cong \frac{\mathbb{Z}_2[\alpha]}{(\alpha^4)},$$

where $|\alpha| = 1$. Thus each $H^i(\mathbb{R}P^3;\mathbb{Z}_2)$ is a free $\mathbb{Z}_2$-module with generator α^i, i.e., $H^i(\mathbb{R}P^3;\mathbb{Z}_2) = \mathbb{Z}_2$. We have $b_i(\mathbb{R}P^3;\mathbb{Z}_2) = 1$, and for $\alpha \in H^1(\mathbb{R}P^3;\mathbb{Z}_2)$ it holds $\alpha \smile \alpha \neq 0$, i.e., $h(\mathbb{R}P^3;\mathbb{Z}_2) = 0$.

Obviously, $h(M;R) = b_1(M;R)$ iff $\smile \equiv 0$. Since $\ker \smile \subseteq H^1(M;R)$ is an isotropic submodule, $h(M;R) \geq \dim \ker \smile$. There are, though, better estimates:

Proposition 16. *Let M be a smooth closed orientable manifold and $k = \dim \ker \smile$. For $R = \mathbb{Z}$ or R being a field, with the exception specified below, we have:*

(1) *It holds*

$$\frac{b_1(M;R) + k\, b_2(M;R)}{b_2(M;R)+1} \leq h(M;R) \leq \frac{b_1(M;R)\, b_2(M;R) + k}{b_2(M;R)+1}; \tag{13}$$

in particular, if $b_2(M;R) = 1$, then

$$h(M;R) = \frac{1}{2}(b_1(M;R) + k). \tag{14}$$

(2) *If $\smile$ is surjective, then*

$$h(M;R) \leq k + \frac{1}{2} + \sqrt{\left(b_1(M;R) - k - \frac{1}{2}\right)^2 - 2\, b_2(M;R)}. \tag{15}$$

As an exception, if

$$\begin{cases} \operatorname{char} R = 2, \\ b_1(M;R) = 1, \\ k = 0, \end{cases} \tag{16}$$

then $h(M;R) = 0$, and of (13)–(15), *only the upper bound in* (13) *holds.*

Proof. If $h(M;R) \neq 0$, for a field this has been shown in [21]; for $\mathbb{Z}$ it follows from Lemma 10.

If $h(M;R) = 0$, then by Lemma 14, either $b_1(M;R) = 0$ or (16) holds. In the former case, (13)–(15) happen to hold; in (15), we have $b_2(M;R) = 0$. In the latter case, (14) and the lower bound in (13) do not hold, while in (15), we have $b_2(M;R) = 1$ and the square root does not exist. □

The exception is illustrated by Example 15. In fact, (14) and the lower bound in (13) would not need an exception if we took the floor function of the corresponding expressions, which in all other cases except (16) happen to be integer anyway.

Example 17. For a closed orientable surface of genus g, (14) gives $h(M_g^2;R) = g$; for n-torus, $n \geq 2$, (15) gives $h(T^n;R) = 1$. Both cases do not fall under exception (16), since $b_1(M_g^2) = 2g \neq 1$ and $b_1(T^n;R) = n \neq 1$.

Example 18. Thus, $\mathcal{H}(T^n;R) = \{1\}$. Indeed, (6) gives

$$h(M;R) = 1 \quad \text{iff} \quad \mathcal{H}(M;R) = \{1\}.$$

The following example shows a non-singleton $\mathcal{H}(M;R)$:

Example 19. $\mathcal{H}(M_2^2 \times S^1;\mathbb{Z}) = \{1,2\}$. This is seen in Figure 2, but can also be formally proved by Theorem 27 below.

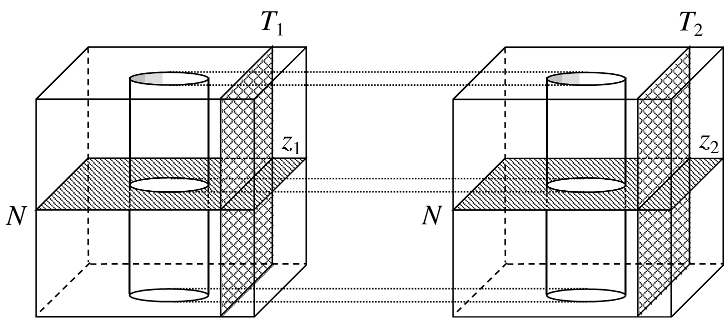

Figure 2. Maximal systems of non-intersecting submanifolds of different cardinality: $\mathcal{H}(M_2^2 \times S^1;\mathbb{Z}) = \{1,2\}$. The double torus $M_2^2 = T^2 \# T^2$, labeled by N, is shown as two horizontal squares, sides of each one being pairwise identified, glued together by a removed circle in the center; S^1 is shown as vertical lines. The two maximal systems of homologically non-intersecting, homologically independent submanifolds are $\{N\}$, a double torus, and $\{T_1, T_2\}$, two tori $z_i \times S^1$, z_1 and z_2 being two homologically independent cycles in the M_2^2.

4. Isotropy index of the connected sum of manifolds

For sets, we denote $A + B = \{a + b \mid a \in A, b \in B\}$.

Lemma 20. *Let L_i, V_i, $i = 1, 2$, be finite-dimensional vector spaces over a field F, and $\varphi_i \colon L_i \times L_i \to V_i$ be bilinear skew-symmetric maps. Denote*

$$L = L_1 \oplus L_2, \qquad V = V_1 \oplus V_2,$$

and let $\varphi \colon L \times L \to V$ be a bilinear skew-symmetric map such that

$$\varphi|_{L_i \times L_i} = \varphi_i, \qquad \varphi|_{L_1 \times L_2} = 0;$$

i.e., φ is defined as component-wise sum of φ_i:

$$\varphi(x, y) = \underbrace{\varphi_1(x_1, y_1)}_{\in V_1} + \underbrace{\varphi_2(x_2, y_2)}_{\in V_2}, \tag{17}$$

where $x_i, y_i \in L_i$ are projections. Then:

(1) *A subspace $H \subseteq L$ is maximal isotropic iff*

$$H = H_1 \oplus H_2, \tag{18}$$

where $H_i \subseteq L_i$ are maximal isotropic under φ_i.

(2) *The set of dimensions of maximal isotropic subspaces*

$$\mathcal{H}(\varphi) = \mathcal{H}(\varphi_1) + \mathcal{H}(\varphi_2).$$

(3) *The isotropy index*

$$h(\varphi) = h(\varphi_1) + h(\varphi_2).$$

Note that these conclusions do not necessarily hold for isotropic subspaces that are not maximal. For example, each 1-dimensional subspace $\langle x \rangle$, $x \in L \setminus (L_1 \cup L_2)$ is isotropic, but (18) does not hold for it.

Proof. By (17), if $H_i \in L_i$ are isotropic, then $H = H_1 \oplus H_2$ is isotropic.

($\Rightarrow$) Let $H \subseteq L$ be a maximal isotropic subspace. Consider $x, y \in H$. By (17), $\varphi(x_1, y_1) = -\varphi(x_2, y_2) \in V_1 \cap V_2 = 0$, i.e., both projections $p_i(H) \subseteq L_i$ are isotropic. Let $H_i \supseteq p_i(H)$ be maximal isotropic subspaces of L_i. Then $H' = H_1 \oplus H_2$ is isotropic, and since $H \subseteq H'$ is maximal, $H = H'$.

($\Leftarrow$) Conversely, let $H_i \subseteq L_i$ be maximal isotropic subspaces; then $H = H_1 \oplus H_2$ is isotropic. Consider $x = x_1 + x_2 \in L \setminus H$, i.e., say, $x_1 \notin H_1$. Since H_1 is maximal isotropic in L_1, there exists $y = y_1 \in H_1 \subseteq H$ such that $\varphi(x_1, y_1) \neq 0$. By (17), $\varphi(x, y) = \varphi(x_1, y_1) + 0 \neq 0$. Thus H is maximal. $\square$

Theorem 21. *Let M_1, M_2 be connected closed orientable manifolds with $\dim M_i \geq 2$, and R be a field or $R = \mathbb{Z}$. Then, for the connected sum $M_1 \# M_2$:*

(1) *A submodule $H \subseteq H^1(M_1 \# M_2; R)$ is maximal isotropic iff*

$$H = H_1 \oplus H_2,$$

where $H_i \subseteq H^1(M_i; R)$ are maximal isotropic submodules.

(2) *The set of ranks of maximal isotropic submodules*

$$\mathcal{H}(M_1 \# M_2; R) = \mathcal{H}(M_1; R) + \mathcal{H}(M_2; R).$$

(3) *The isotropy index of the connected sum*

$$h(M_1 \# M_2; R) = h(M_1; R) + h(M_2; R).$$

Proof. Let R be a field. Denote $L = H^1(M_1 \# M_2; R)$ and $L_i = H^1(M_i; R)$, $i = 1, 2$. Since $\dim M_i \geq 2$, the Mayer–Vietoris sequence gives $L = L_1 \oplus L_2$. The additive structure is given by the induced maps of the inclusions; the cup product translates into component-wise product:

$$x \smile y = (x_1 \smile y_1) + (x_2 \smile y_2),$$

where $x, y \in L$ and $x_i, y_i \in L_i$ are projections. Then, for fields, Lemma 20 gives the result. Now for $R = \mathbb{Z}$, the result follows from Lemma 7. □

Example 22. By Theorem 21 and given Example 18, or by (12) if $g \neq 0$, for a closed orientable surface of genus g it holds

$$\mathcal{H}(M_g^2; R) = \mathcal{H}(\#_{i=1}^{g}\, T^2; R) = \sum_{i=1}^{g} \mathcal{H}(T^2; R) = \{g\}.$$

Example 23. Consider $M = M_2^2 \times S^1$ from Example 19 with $\mathcal{H}(M; \mathbb{Z}) = \{1, 2\}$; see Figure 2. Then

$$\mathcal{H}(M \# M; \mathbb{Z}) = \{2, 3, 4\},$$

and $h(M \# M; \mathbb{Z}) = 4$.

Example 24. Consider $M = M_a^2 \times M_b^2$, surfaces of genus a and b, $1 \leq a \leq b$. Theorem 27 below gives $\mathcal{H}(M; R) = \{1, a, b\}$. Therefore,

$$\mathcal{H}(M \# M; R) = \{2, a + 1, b + 1, a + b, 2\,a, 2\,b\},$$

and $h(M \# M; R) = 2\,b$.

5. Isotropy index of the direct product of manifolds

Lemma 25. *Let L be a vector space and $x, y, u, v \in L$; $x, y \neq 0$. Then $x \otimes v = u \otimes y$ implies $u = ax$, $v = ay$ for some a.*

Proof. Coordinate-wise, we have

$$x_i v_j = u_i y_j \tag{19}$$

for all i, j. For those i, j for which $x_i, y_j \neq 0$, this gives

$$\frac{u_i}{x_i} = \frac{v_j}{y_j} = a_{ij}.$$

Since a_{ij} does not depend on i or j, all $a_{ij} = a$. We obtain $u_i = ax_i$ if $x_i \neq 0$, and $v_j = ay_j$ if $y_j \neq 0$. If $x_i = 0$, (19) gives $0 = u_i y_j$ for all j, thus $u_i = 0$, and similarly, $y_j = 0$ implies $v_j = 0$. □

While in Lemma 20 we had $\varphi|_{L_1 \times L_2} = 0$, now consider $\operatorname{im} \varphi|_{L_1 \times L_2}$ as large as possible:

Lemma 26. *Let L_i, V_i, $i = 1, 2$, be finite-dimensional vector spaces over a field F, and $\varphi_i \colon L_i \times L_i \to V_i$ be bilinear skew-symmetric maps. Denote*

$$L = L_1 \oplus L_2, \qquad V = V_1 \oplus V_2 \oplus V_3,$$

where $V_3 = L_1 \otimes L_2$, and let $\varphi \colon L \times L \to V$ be a bilinear skew-symmetric map such that

$$\varphi|_{L_i \times L_i} = \varphi_i, \qquad \varphi|_{L_1 \times L_2} = \otimes;$$

i.e.,

$$\varphi(x, y) = \underbrace{\varphi_1(x_1, y_1)}_{\in V_1} + \underbrace{\varphi_2(x_2, y_2)}_{\in V_2} + \underbrace{x_1 \otimes y_2 - y_1 \otimes x_2}_{\in V_3}, \tag{20}$$

where $x_i, y_i \in L_i$ are projections. Then:

(1) *A subspace $H \subseteq L$ is isotropic iff*

$$\dim H = 1 \text{ or } H = H_i,$$

where $H_i \subseteq L_i$ is isotropic under φ_i, for $i = 1$ or 2.

(2) *The set of dimensions of maximal isotropic subspaces*

$$\mathcal{H}(\varphi) = \{1\} \cup \mathcal{H}(\varphi_1) \cup \mathcal{H}(\varphi_2),$$

except that $\mathcal{H}(\varphi) = \mathcal{H}(\varphi_i)$ *if* $h(\varphi_j) = 0$, *i.e., if either*

- $L_j = 0$ *or*
- $L_j = F$, $\operatorname{char} F = 2$, *and* $\varphi_j \not\equiv 0$.

(3) *The isotropy index*

$$h(\varphi) = \max\{h(\varphi_1), h(\varphi_2)\}.$$

Note that in contrast to Lemma 20, the first conclusion does not require H to be maximal.

Proof. Let H be an isotropic subspace. We will show that if both projections $p_i(H) \neq 0$, then $\dim H = 1$. Consider $x \in H$ such that both projections $x_i \neq 0$. Let $y \in H$. Since H is isotropic and the three components of (20) are independent, we have

$$\varphi_1(x_1, y_1) = \varphi_2(x_2, y_2) = x_1 \otimes y_2 - y_1 \otimes x_2 = 0.$$

By Lemma 25, $y \in \langle x \rangle$. The conditions for $h(\varphi_j) = 0$ in item (2) are given by Lemma 4. □

Theorem 27. *Let* M_1, M_2 *be connected closed manifolds, and* R *be a field or* $R = \mathbb{Z}$. *Then for the direct product* $M_1 \times M_2$*:*

(1) *A submodule* $H \subseteq H^1(M_1 \times M_2; R)$ *is isotropic iff*

$$\operatorname{rk} H = 1 \quad \textit{or} \quad H = H_i,$$

where $H_i \subseteq H^1(M_i; R)$ *is isotropic for* M_i, $i = 1$ *or* 2.

(2) *The set of ranks of maximal isotropic submodules*

$$\mathcal{H}(M_1 \times M_2; R) = \{1\} \cup \mathcal{H}(M_1; R) \cup \mathcal{H}(M_2; R),$$

except that $\mathcal{H}(M_1 \times M_2; R) = \mathcal{H}(M_i; R)$ *if* $h(M_j; R) = 0$, *i.e., if either*

- $b_1(M_j; R) = 0$, *the Betti number, or*
- $b_1(M_j; R) = 1$, $\operatorname{char} R = 2$, *and* $\smile \not\equiv 0$.

(3) *The isotropy index of the direct product*

$$h(M_1 \times M_2; R) = \max\{h(M_1; R), h(M_2; R)\}.$$

Proof. Let R be a field. Denote

$$L_i = H^1(M_i, R), \quad i = 1, 2, \qquad L = H^1(M_1 \times M_2, R),$$
$$V_i = H^2(M_i, R), \quad i = 1, 2, \qquad V = H^2(M_1 \times M_2, R).$$

By the Künneth formula,

$$L = L_1 \oplus L_2, \qquad V = V_1 \oplus V_2 \oplus V_3,$$

where

$$V_3 = L_1 \otimes L_2.$$

By construction, $L_i \smile L_i \subseteq V_i$ for $i = 1, 2$; $L_1 \smile L_2 \subseteq V_3$, and (20) holds for the cup-products in $M_1 \times M_2$ and M_i, respectively. Lemma 26 gives the result for fields, and Lemma 7 for $R = \mathbb{Z}$. □

Example 12 shows that in Theorem 27, the direct product cannot be replaced by an arbitrary fiber bundle.

Example 28. By Lemma 14, $h(S^1; R) = 1$, so, for a torus $T^n = \times_{i=1}^n S^1$, we have $h(T^n; R) = 1$.

Example 29. By Lemma 14, $h(S^n; R) = 0$, $n \geq 2$, and $h(S^1; R) = 1$, so $h(S^n \times S^1; R) = 1$.

Proposition 30. *For any non-empty finite set $S \subset \mathbb{Z}^*$ of non-negative integers, and for $R = \mathbb{Z}$ or R being a field, $S = \mathcal{H}(M; R)$ for some smooth closed orientable connected manifold M iff $S = \{0\}$ or $0 \notin S$.*

Proof. If $S = \{g\}$, then $S = \mathcal{H}(M_g^2; R)$, a surface of genus g; see Example 22. Let now $S = \{s_1, \dots, s_N\}$, $N \geq 2$. By the condition, $m = \min S \geq 1$. Consider

$$M_1 = M^2_{s_1-m+1} \times \cdots \times M^2_{s_N-m+1}, \qquad \dim M_1 = 2N,$$
$$M_2 = M^2_{m-1} \times S^{2N-2}, \qquad \dim M_2 = 2N.$$

By Theorems 21 and 27, we obtain $\mathcal{H}(M_1 \# M_2; R) = S$. □

Another application of Theorem 27 can be found in the study of the topology of foliations defined by Morse forms. It is known that if the subgroup of $H_{n-1}(M)$ generated by the homology classes of all compact leaves of the foliation is maximal isotropic, then the foliation has no minimal components [19]. This condition obviously holds true when the foliation has $h(M;\mathbb{Z})$ homologically independent compact leaves. However, if $M = M_1 \times M_2$, in some cases Theorem 27 allows to conclude that the foliation has no minimal components by considering only one leaf:

Example 31. As has been mentioned in Example 19, $\mathcal{H}(M_2^2 \times S^1;\mathbb{Z}) = \{1,2\}$; see Figure 2. Even though $h(M_2^2 \times S^1,\mathbb{Z}) = 2$, if a Morse form foliation has the submanifold $N = M_2^2$ as a leaf, then it has no minimal components. In contrast, nothing can be said about a form that has $T_1 = T^2$ as a leaf, because the system $\{T_1\}$ is not maximal.

Example 32. $\mathcal{H}(M_a^2 \times M_b^2;\mathbb{Z}) = \{1,a,b\}$, $a,b \geq 1$. Now, consider a cycle z that winds around the $M_1 = M_a^2$ and also around the $M_2 = M_b^2$, that is, $z = z_1 + z_2$, $0 \neq z_i \in H^1(M_i,\mathbb{Z})$. If a Morse form foliation has a leaf dual to z, then it has no minimal components.

6. Isotropy index and the first Betti number

By definition of the isotropy index,

$$h(M;R) \leq b_1(M;R);$$

for example:

$$h(S^1;R) = 1, \quad b_1(S^1;R) = 1;$$
$$h(M_g^2;R) = g, \quad b_1(M_g^2;R) = 2g;$$
$$h(T^n;R) = 1, \quad b_1(T^n;R) = n.$$

The only relation between $h(M;R)$, $b_1(M;R)$, and R is given by Lemma 14; in particular, any gap between $h(M;R)$ and $b_1(M;R)$ is possible for a given R:

Theorem 33. *Let $h,b \in \mathbb{Z}$, and R be a field or $R = \mathbb{Z}$. There exists a connected smooth closed orientable manifold M with $h(M;R) = h$ and $b_1(M;R) = b$ iff any of the following conditions holds:*

- $1 \leq h \leq b$, *or*
- $h = b = 0$, *or*
- $h = 0$, $b = 1$, *and* $\operatorname{char} R = 2$.

Proof. For $h = b = 0$, consider $M = S^n$. For $h = 0$ and $b = 1$ with $\operatorname{char} R = 2$, consider $M = \mathbb{R}P^3$; see Example 15. Let now $1 \leq h \leq b$. Choose

$m_i \geq 1$ such that

$$\sum_{i=1}^{h} m_i = b. \tag{21}$$

For large enough n such that $n - m_i \geq 2$ for all i, consider an n-manifold

$$M = \mathop{\#}_{i=1}^{h} \left(T^{m_i} \times S^{n-m_i}\right).$$

By Theorem 27, for each summand $M_i = T^{m_i} \times S^{n-m_i}$, we have $h(M_i; R) = 1$, while $b_1(M_i; R) = m_i$. Then by Theorem 21,

$$\begin{aligned} h(M; R) &= \sum_{i=1}^{h} h(M_i; R) \;= \sum_{i=1}^{h} 1 = h, \\ b_1(M; R) &= \sum_{i=1}^{h} b_1(M_i, R) = \sum_{i=1}^{h} m_i = b. \end{aligned} \qquad \square$$

Note that the construction used in the proof requires $n = \dim M \geq 2 + \lceil \frac{b}{h} \rceil$, the ceiling here being the smallest possible value for $\max\{m_i\}$ under (21). This condition is not restrictive when $h = b$, leading to $n \geq 3$; it is not very restrictive when $\frac{b}{2} \leq h < b$, leading to $n \geq 4$, etc.; for $\frac{b}{k+1} \leq h < \frac{b}{k}$, $k = 1, \dots, b-1$, we need $n \geq k + 3$. This requires high dimension when $h \ll b$.

For $\mathbb{Z}_2$, however, $n = 3$ is enough:

Proposition 34. *For $R = \mathbb{Z}_2$, the manifold in Theorem 33 can be chosen with any given* $\dim M \geq 3$.

Proof. Let $R = \mathbb{Z}_2$ and $\dim M = 3$. For $h = b = 0$, consider $M^3 = S^3$. Let now $b \geq 1$. Consider

$$M^3 = \left(\mathop{\#}_{i=1}^{h} \left(S^1 \times S^2\right)\right) \# \left(\mathop{\#}_{i=1}^{b-h} \mathbb{R}P^3\right). \tag{22}$$

Example 29 shows that $h(S^1 \times S^2; \mathbb{Z}_2) = 1$, thus Theorem 21 implies

$$\begin{aligned} h(M^3; \mathbb{Z}_2) &= \sum_{i=1}^{h} h(S^1 \times S^2; \mathbb{Z}_2) + \sum_{i=1}^{b-h} h(\mathbb{R}P^3; \mathbb{Z}_2) = \sum_{i=1}^{h} 1 + \sum_{i=1}^{b-h} 0 = h, \\ b_1(M^3; \mathbb{Z}_2) &= \sum_{i=1}^{h} b_1(S^1 \times S^2; \mathbb{Z}_2) + \sum_{i=1}^{h} b_1(\mathbb{R}P^3; \mathbb{Z}_2) = \sum_{i=1}^{h} 1 + \sum_{i=1}^{b-h} 1 = b. \end{aligned}$$

This trivially generalizes to dim ≥ 5 as

$$M^n = M^3 \times S^{n-3}. \tag{23}$$

Let now $\dim M = 4$. For $1 \leq h < b$, we use (23) with one summand less in (22), namely,

$$M^3 = \left(\mathop{\#}_{i=1}^{h} (S^1 \times S^2) \right) \# \left(\mathop{\#}_{i=1}^{b-h-1} \mathbb{R}P^3 \right).$$

For $h = b$, consider $M^4 = \#_{i=1}^{h}(S^1 \times S^3)$.

Finally, for $h = 0$, $b = 1$, consider an Enriques surface X. Indeed,[1] $X = K3/\sigma$, where σ is an orientation-preserving fixed point-free involution; note that a $K3$ surface is simply connected. Then $H^1(X;\mathbb{Z}_2) = \mathrm{Hom}(\pi_1(X),\mathbb{Z}_2) = \mathbb{Z}_2$; thus $b_1(X;\mathbb{Z}_2) = 1$. For $0 \neq x \in H^1(X;\mathbb{Z})$, $x \smile x$ is a reduction (mod 2) of βx, where $\beta : H^1(X;\mathbb{Z}_2) \to H^2(X;\mathbb{Z})$ is the Bockstein homomorphism. Suppose $x \smile x = 0$, i.e., for some $y \in H^2(X;\mathbb{Z})$, we have $\beta x = 2y$ with $\beta x \neq 0$ because $H^1(X;\mathbb{Z}) = 0$. Since βx is 2-torsion, we obtain that $0 \neq \pi^* y \in H^2(K3;\mathbb{Z})$ is 4-torsion, where π is the quotient map, while the latter group is torsion-free. Thus $h(X;\mathbb{Z}) = 0$. □

7. Isotropy index and the co-rank of the fundamental group

In this section, we give a lower bound on $h(M;R)$ stronger than 1 from (11).

Definition 35. The *co-rank of the fundamental group* of a smooth closed connected manifold M is the maximum rank of a free quotient group of $\pi_1(M)$; we denote it by $b_1'(M)$.

While $h(M;\mathbb{Z})$ is the maximum number of homologically non-intersecting submanifolds $[X_i \cap X_j] = 0$ (Theorem 13), $b_1'(M)$ strengthens the condition to $X_i \cap X_j = \emptyset$:

Theorem 36 ([16, Theorem 2.1])**.** *The co-rank of the fundamental group $b_1'(M)$ is the maximum number of non-intersecting homologically independent smooth closed orientable connected codimension-one submanifolds $X_i \subset M$:*

$$X_i \cap X_j = \emptyset, \quad i \neq j;\ i,j = 1,\ldots,b_1'(M).$$

[1]Example contributed by a colleague who preferred not to be named.

Accordingly, properties of $b_1'(M)$ closely resemble those of $h(M;\mathbb{Z})$. Similarly to (11)–(12), it holds [13]:

$$b_1'(M) = 0 \quad \text{iff} \quad b_1(M) = 0, \tag{24}$$

and otherwise,

$$1 \le b_1'(M) \le b_1(M); \tag{25}$$

in particular, $h(M;\mathbb{Z}) = 0$ iff $b_1'(M) = 0$. Exactly as in Theorems 21 and 27, for the connected sum, $\dim M_i \ge 2$, except for non-orientable surfaces, and the direct product it holds

$$\begin{aligned} b_1'(M_1 \# M_2) &= b_1'(M_1) + b_1'(M_2), && \text{see [15]},\\ b_1'(M_1 \times M_2) &= \max\{b_1'(M_1), b_1'(M_2)\}, && \text{see [13]}. \end{aligned}$$

Example 37. Non-surprisingly, for many manifolds $b_1'(M) = h(M;R)$:

- for the closed orientable surface, $b_1'(M_g^2) = g$ [18] and $h(M_g^2;R) = g$ [21]; see Example 22.
- for n-torus, $b_1'(T^n) = 1$ [8] and $h(T^n;R) = 1$ [21]; see Example 28.
- for manifolds with quasi-Kähler and 1-formal fundamental group, for example, for compact Kähler manifolds, $b_1'(M) = h(M;\mathbb{C})$ [6].
- for $M = \#_{i=1}^h (T^{m_i} \times S^{n-m_i})$ from Theorem 33, it holds $b_1'(M) = h(M;R)$.

A non-trivial theorem from [13] implies that (24)–(25) represent the only relation between $b_1'(M)$ and $b_1(M)$ for any given $\dim M$. The last item in Example 37 shows that the construction from Theorem 33 gives an elementary proof of this fact for large enough $\dim M$:

Theorem 38. *Let $b', b \in \mathbb{Z}$. There exists a connected smooth closed orientable manifold M with $b_1'(M) = b'$ and $b_1(M) = b$ iff either*

$$b' = b = 0, \ \textit{see } (24), \qquad \textit{or} \qquad 1 \le b' \le b, \ \textit{see } (25).$$

Comparing Theorems 13 and 36 gives

$$b_1'(M) \le h(M;\mathbb{Z}); \tag{26}$$

together with (11) this gives a geometric proof of lower and upper bounds on the isotropy index $h(M;\mathbb{Z})$, which have been obtained indirectly in [9]. We extend this to fields of characteristic zero:

Proposition 39. *Let* $R = \mathbb{Z}$ *or* R *be a field,* $\operatorname{char} R = 0$*. For the co-rank of the fundamental group* $b'_1(M)$*, the isotropy index* $h(M;R)$*, and the first Betti number* $b_1(M)$*, it holds*

$$b'_1(M) \le h(M;R) \le b_1(M). \tag{27}$$

Proof. By Proposition 8, for a field F with $\operatorname{char} F = 0$, we have $h(M;\mathbb{Z}) \le h(M;F)$. Equations (26) and (11) complete the proof:

$$b'_1(M) \le h(M;\mathbb{Z}) \le h(M;F) \le b_1(M). \qquad \square$$

Both bounds in (27) are exact (see Example 37 and Theorem 33); in particular, as we have shown, in many cases $b'_1(M)$ is a very strong lower bound for $h(M)$. However, both inequalities can also be strict:

Example 40. Consider the Heisenberg nilmanifold H^3. Its fundamental group $\pi_1(H^3)$ is nilpotent, so $b'_1(H^3) = 1$. Since $H^1(H^3,\mathbb{Z}) = \mathbb{Z}^2$ with zero cup-product [17], we have

$$1 = b'_1(H^3) < h(H^3;\mathbb{Z}) = b_1(H^3) = 2.$$

Example 41. The Kodaira–Thurston nilmanifold $M = H^3 \times S^1$ gives an example of

$$b'_1(M) < h(M;\mathbb{Z}) < b_1(M).$$

Indeed, the fundamental group $\pi_1(M)$ is nilpotent, so $b'_1(M) = 1$; by Theorem 27 and given Example 40, $h(M;\mathbb{Z}) = 2$; and, obviously, $b_1(M) = 3$.

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