Joining means

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Dedicated to Professor Zsolt Páles on the occasion of his 60th birthday

Abstract. Modifying and generalizing some ideas from [1], we come to the notion of a marginal joint of two arbitrary means given on adjacent intervals. The construction of the joints makes use of the notion of a set-valued joiner. Also, the converse is proved: any mean can be obtained as a marginal joint of its two restrictions, produced with the use of a so-called reconstructing joiner having the smallest values in a sense. We conclude the paper by answering the question when the reconstructing joiner of the mean is a single-valued function.

1. Introduction

Let $I$ be an interval of reals. A function $M : I \times I \to I$ is called a mean on $I$ if

$$\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$$

for all $x, y \in I$.

Take any interior point $\xi$ of $I$, and put

$$I_\xi := \{x \in I : x \leq \xi\}, \quad \xi I := \{x \in I : \xi \leq x\}$$

(1)

and

$$I^\circ_\xi := \{x \in I : x < \xi\}, \quad \xi I^\circ := \{x \in I : \xi < x\}.$$  

(2)

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Our main question is as follows.

**Problem 1.** Given two arbitrary means $M$ and $N$ on the intervals $I_\xi$ and $\xi I$, respectively, find a mean, say $M \oplus N$, on the interval $I$ such that

$$M \oplus N|_{I_\xi \times I_\xi} = M \quad \text{and} \quad M \oplus N|_{\xi I \times \xi I} = N.$$  

Any such mean $M \oplus N$ will be called a joint of $M$ and $N$.

Observe that, given any mean $K$ on $I$, the formula

$$(M \oplus N)(x,y) = \begin{cases} 
M(x,y), & \text{if } (x,y) \in I_\xi \times I_\xi \\
K(x,y), & \text{if } (x,y) \in I_\xi^\circ \cup \xi I^\circ \times I_\xi^\circ \\
N(x,y), & \text{if } (x,y) \in \xi I \times \xi I
\end{cases}$$

defines a joint of $M$ and $N$. However, its values taken in the set $I_\xi^\circ \times \xi I^\circ \cup \xi I^\circ \times I_\xi^\circ$ need not be connected with $M$ and $N$ at all. Such trivial joints will not be of interest for us. In the sequel, we will focus on joints carrying information on the means $M$ and $N$.

In the paper [1], Z. Daróczy and the authors solved Problem 1, assuming that the marginal functions $h_1, h_2 : I \to I$, given by

$$h_1(x) = \begin{cases} 
M(x,\xi), & \text{if } x \in I_\xi \\
N(x,\xi), & \text{if } x \in \xi I
\end{cases} \quad (3)$$

and

$$h_2(y) = \begin{cases} 
M(\xi,y), & \text{if } y \in I_\xi \\
N(\xi,y), & \text{if } y \in \xi I
\end{cases} \quad (4)$$

are continuous and strictly increasing. Next, we solve Problem 1 for arbitrary means.
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2. Joiners

The idea is to modify and generalize some ideas from [1]. The main tool used there to produce joints $M \oplus N$ is the notion of the so-called joining function. Now, we replace it by its set-valued analogue, called by us a joiner. Using it we construct a set-valued joint with the selections being joints of the means $M$ and $N$.

In what follows, we assume that:

- $\xi$ is an interior point of an interval $I$;
- the intervals $I_{\xi}, \xi I$ and $\xi I^o$ are defined by (1) and (2), respectively;
- $M$ and $N$ are means on $I_{\xi}$ and $\xi I$, respectively;
- $h_1, h_2 : I \to I$ are the marginal functions given by (3) and (4), respectively.

Given any functions $f : I_{\xi} \to I_{\xi}$ and $g : \xi I \to \xi I$ satisfying $f(\xi) = g(\xi)$, we use the notation $f \cup g$ for the function mapping $I$ into itself, defined by

$$(f \cup g)(x) := \begin{cases} f(x), & \text{if } x \in I_{\xi}, \\ g(x), & \text{if } x \in \xi I. \end{cases}$$

Moreover, we define the product $h_1 \times h_2 : I \times I \to I \times I$ by the equality

$$(h_1 \times h_2)(x, y) = (h_1(x), h_2(y)).$$
Definition 1. A multifunction
\[ K : (h_1 \times h_2) (I^\circ_\xi \times I^\circ_\xi \times I^\circ_\xi) \rightarrow 2^I \setminus \{\emptyset\} \]
is called a joiner of the pair \((M, N)\) if
\[
(h_1|_{\xi^\circ} \cup h_2|_{\xi^\circ})^{-1} (K (h_1(x), h_2(y))) \cap [x, y] \neq \emptyset \quad \text{for} \quad (x, y) \in I^\circ_\xi \times I^\circ_\xi, \quad (5)
\]
and
\[
(h_2|_{\xi^\circ} \cup h_1|_{\xi^\circ})^{-1} (K (h_1(x), h_2(y))) \cap [y, x] \neq \emptyset \quad \text{for} \quad (x, y) \in I^\circ_\xi \times I^\circ_\xi, \quad (6)
\]
or, equivalently, if for every \((x, y) \in I^\circ_\xi \times I^\circ_\xi \cup I^\circ_\xi \times I^\circ_\xi\) there is a \(\kappa(x, y) \in I\) such that
\[
\min\{x, y\} \leq \kappa(x, y) \leq \max\{x, y\},
\]
and the function \((h_1|_{\xi^\circ} \cup h_2|_{\xi^\circ}) \circ \kappa\) is a selection of \(K \circ (h_1 \times h_2)\).

Observe that if \(K_1\) is a joiner of the pair \((M, N)\), then so is any \(K_2\) such that
\[
K_1(x, y) \subset K_2(x, y)
\]
for each \((x, y) \in (h_1 \times h_2) (I^\circ_\xi \times I^\circ_\xi \cup I^\circ_\xi \times I^\circ_\xi)\).

The most trivial example of a joiner is the multifunction \(K\) given by \(K(u, v) = I\) : for any point \((x, y) \in I^\circ_\xi \times I^\circ_\xi\) we have
\[
(h_1|_{\xi^\circ} \cup h_2|_{\xi^\circ})^{-1} (K (h_1(x), h_2(y))) = (h_1|_{\xi^\circ} \cup h_2|_{\xi^\circ})^{-1} (I)
\]
\[
= h_1^{-1} (I_\xi) \cup h_2^{-1} (I_\xi) = I_\xi \cup I_\xi = I,
\]
so condition (5) holds. Similarly, (6) follows directly.

Clearly, such a joiner is not of interest. It is evident that the main point is to consider joiners and set-valued joints with relatively small values.

The following examples of joiners originated in fact in the paper [1].

Example 1. If the marginal functions \(h_1, h_2 : I \rightarrow I\) are continuous and strictly increasing, and
\[
K(x, y) = h_1(x) + h_2(y) - \xi, \quad (x, y) \in (h_1 \times h_2) (I^\circ_\xi \times I^\circ_\xi \cup I^\circ_\xi \times I^\circ_\xi),
\]
then the formula
\[
K(x, y) := \{K(x, y)\}
\]
defines a single-valued joiner of the pair \((M, N)\) (cf. [1, Ex. 2.1]).
Example 2. Let \( \varphi : \mathbb{I}_x \to \mathbb{R} \) and \( \psi : \mathbb{I}_y \to \mathbb{R} \) be continuous strictly monotonic functions vanishing at \( \xi \), and let \( p, q \in (0, 1) \). Consider the quasi-arithmetic means \( M = A_p^\varphi \) and \( N = A_q^\psi \) on the intervals \( \mathbb{I}_x \) and \( \mathbb{I}_y \), respectively:

\[
M(x, y) = A_p^\varphi(x, y) = \varphi^{-1}((1 - p)\varphi(x) + p\varphi(y)), \quad x, y \in \mathbb{I}_x,
\]

\[
N(x, y) = A_q^\psi(x, y) = \psi^{-1}((1 - q)\psi(x) + q\psi(y)), \quad x, y \in \mathbb{I}_y.
\]

For any \((x, y) \in (h_1 \times h_2) (\mathbb{I}_x \times \mathbb{I}_y)\), put

\[
K(x, y) = \begin{cases}
\varphi^{-1}(p(\varphi(x) + \psi(y))), & \text{if } \varphi(x) + \psi(y) < 0, \\
\psi^{-1}((1 - q)(\varphi(x) + \psi(y))), & \text{if } \varphi(x) + \psi(y) \geq 0.
\end{cases}
\]

Similarly, having \((x, y) \in (h_1 \times h_2) (\mathbb{I}_y \times \mathbb{I}_x)\), we put

\[
K(x, y) = \begin{cases}
\varphi^{-1}((1 - p)(\psi(x) + \varphi(y))), & \text{if } \psi(x) + \varphi(y) < 0, \\
\psi^{-1}(q(\psi(x) + \varphi(y))), & \text{if } \psi(x) + \varphi(y) \geq 0.
\end{cases}
\]

Then \( K \), defined by (7), is a joiner of the pair \((A_p^\varphi, A_q^\psi)\) (cf. [1, Ex. 2.2]).

More generally, one can easily check (see [1, p. 224]) that if

\[
K : (h_1 \times h_2) (\mathbb{I}_x \times \mathbb{I}_y \cup \mathbb{I}_y \times \mathbb{I}_x) \to \mathbb{I}
\]

is any joining function for the pair \((M, N)\) in the sense of the paper [1], and the marginal functions \(h_1, h_2 : \mathbb{I} \to \mathbb{I}\) are continuous and strictly increasing, then the single-valued multifunction \( K \), given on \((h_1 \times h_2) (\mathbb{I}_x \times \mathbb{I}_y \cup \mathbb{I}_y \times \mathbb{I}_x)\) by (7), is a joiner of the pair \((M, N)\).

3. Marginal joints of means

The following result yields a pretty general procedure of joining two means.

**Theorem 1.** Let \( K \) be any joiner of the pair \((M, N)\). Then the values of the multifunction \( M \oplus K N \) defined as

\[
\begin{align*}
\{M(x, y)\}, & \quad \text{if } (x, y) \in \mathbb{I}_x \times \mathbb{I}_y, \\
(h_11_{\mathbb{I}_x} \cup h_2|_{\mathbb{I}_y})^{-1}(K(h_1(x), h_2(y))) \cap [x, y], & \quad \text{if } (x, y) \in \mathbb{I}_x \times \mathbb{I}_y, \\
(h_21_{\mathbb{I}_y} \cup h_1|_{\mathbb{I}_x})^{-1}(K(h_1(x), h_2(y))) \cap [y, x], & \quad \text{if } (x, y) \in \mathbb{I}_y \times \mathbb{I}_x, \\
\{N(x, y)\}, & \quad \text{if } (x, y) \in \mathbb{I}_y \times \mathbb{I}_y,
\end{align*}
\]

are non-empty, and every its selection is a mean on the interval \( \mathbb{I} \), extending both the means \( M \) and \( N \).
Proof. It is enough to follow Definition 1. □

Definition 2. Any mean $M \oplus_K N$ constructed above is called a **marginal $K$-joint of the means** $M$ and $N$.

4. The converse problem

Now, we would like to answer the following question.

*Can any mean $L$ on the interval $I$ be reconstructed as a $K$-joint $M \oplus_K N$ of the restricted means $M := L|_{I \times I}$ and $N := L|_{I \times I}$, with a suitable joiner $K$?*

However, that question is not well-posed since it can be answered in the following quite trivial way. Namely, if $K$ is defined by $K(x,y) = I$, then the joint $M \oplus_K N$ is given by

$$(M \oplus_K N)(x,y) = \begin{cases} \{L(x,y)\}, & \text{if } (x,y) \in I \times I, \\ [x,y], & \text{if } (x,y) \in I^2 \times I, \\ [y,x], & \text{if } (x,y) \in I \times I. \end{cases}$$

and every its selection is a mean on $I$. Note that $L$ is one of those selections. The reason of that phenomenon is completely clear: the values of the used joiner are too big. So, we will try to answer the following modified question.

**Problem 2.** *Can any mean $L$ on the interval $I$ be reconstructed as a $K$-joint $M \oplus_K N$ of the restricted means $M := L|_{I \times I}$ and $N := L|_{I \times I}$, with a suitable joiner $K$ with relatively small values?*

Fix a mean $L$ on the interval $I$, and put

$$M := L|_{I \times I} \quad \text{and} \quad N := L|_{I \times I}.$$ Define marginal functions $h_1, h_2 : I \to I$ by the formulas

$$h_1(x) = L(x,\xi) \quad \text{and} \quad h_2(y) = L(\xi,y),$$

respectively. Then

$$h_1(x) = \begin{cases} M(x,\xi), & \text{if } x \in I \xi, \\ N(x,\xi), & \text{if } x \in \xi I, \end{cases}$$

and

$$h_2(x) = \begin{cases} M(\xi,y), & \text{if } x \in I \xi, \\ N(\xi,y), & \text{if } x \in \xi I. \end{cases}$$
The formula
\[(u, v) \sim (x, y) :\iff h_1(u) = h_1(x) \quad \text{and} \quad h_2(v) = h_2(y)\]
defines an equivalence relation in the set \(I^2_\xi \times I^0 \cup \xi I^0 \times I^2_\xi\). Denoting by \((x_0, y_0)\) the equivalence class of the \(\xi\) defines an equivalence relation in the set \(I^2_\xi \times I^0 \cup \xi I^0 \times I^2_\xi\), we have
\[
\{(x, y) \in I^2_\xi \times I^0 \cup \xi I^0 \times I^2_\xi : h_1(x) = h_1(x_0) \quad \text{and} \quad h_2(y) = h_2(y_0)\}
\]
\[
= \{(x, y) \in I^2_\xi \times I^0 \cup \xi I^0 \times I^2_\xi : (h_1 \times h_2)(x, y) = (h_1(x_0), h_2(y_0))\}
\]
\[
= (h_1 \times h_2)^{-1} \{\{h_1(x_0), h_2(y_0)\}\} \cap (I^2_\xi \times I^0 \cup \xi I^0 \times I^2_\xi)\].

This means that the equivalence class \((x_0, y_0)\) is the level set of the point \((h_1(x_0), h_2(y_0))\) under the product \(h_1 \times h_2\) restricted to \(I^2_\xi \times I^0 \cup \xi I^0 \times I^2_\xi\).

The next result gives a positive answer to the question posed in this section.

**Theorem 2.** Let \(\mathbb{K}_0 : (h_1 \times h_2) (I^2_\xi \times I^0 \cup \xi I^0 \times I^2_\xi) \rightarrow 2^I \setminus \{\emptyset\}\) be given by
\[
\mathbb{K}_0(h_1(x), h_2(y)) = (h_1|_{I^2_\xi} \cup h_2_{|I^2_\xi}) (L((x_0, y_0)))\), \quad (x, y) \in I^2_\xi \times I^0\, (8)
\]
and
\[
\mathbb{K}_0(h_1(x), h_2(y)) = (h_2|_{I^2_\xi} \cup h_1_{|I^2_\xi}) (L((x_0, y_0)))\), \quad (x, y) \in \xi I^0 \times I^2_\xi\, (9)
\]

Then
(i) \(\mathbb{K}_0\) is a joiner of the pair \((L|_{I^2_\xi \times I^2_\xi}, L_{|\xi I^0 \times I^2_\xi})\) and satisfies the condition
\[
L((x_0, y_0)) \subset (L|_{I^2_\xi \times I^2_\xi} \oplus \mathbb{K}_0 L_{|\xi I^0 \times I^2_\xi}) (x, y), \quad (x, y) \in I^2_\xi \times I^0 \cup \xi I^0 \times I^2_\xi\]
and
(ii) if \(\mathbb{K}\) is a joiner of the pair \((L|_{I^2_\xi \times I^2_\xi}, L_{|I^0 \times I^2_\xi})\) and satisfies
\[
L((x_0, y_0)) \subset (L|_{I^2_\xi \times I^2_\xi} \oplus \mathbb{K} L_{|\xi I^0 \times I^2_\xi}) (x, y), \quad (x, y) \in I^2_\xi \times I^0 \cup \xi I^0 \times I^2_\xi\]
then \(\mathbb{K}_0 \subseteq \mathbb{K}\), that is,
(i) the mean \(L\) can be reconstructed as a selection for the marginal \(\mathbb{K}_0\)-joint of
its restrictions \(L|_{I^2_\xi \times I^2_\xi}\) and \(L_{|\xi I^0 \times I^2_\xi}\),
and
(ii) \(\mathbb{K}_0\) is the smallest (in the sense of inclusion) joiner of the pair \((L|_{I^2_\xi \times I^2_\xi}, \ L_{|\xi I^0 \times I^2_\xi})\) satisfying condition (10).
Proof. (i) For every \((x, y) \in I_\xi^x \times I_\xi^y\), we have
\[ (h_1|_{I_\xi^x} \cup h_2|_{I_\xi^y})^{-1} \left[ \left( h_1|_{I_\xi^x} \cup h_2|_{I_\xi^y} \right) \left( L((x_0, y_0)) \right) \right] \cap [x, y] \supset L((x_0, y_0)) \cap [x, y] \ni L(x, y). \]
Similarly, if \((x, y) \in \xi I_\xi^y \times I_\xi^y\), then
\[ (h_2|_{I_\xi^y} \cup h_1|_{I_\xi^x})^{-1} \left[ \left( h_2|_{I_\xi^y} \cup h_1|_{I_\xi^x} \right) \left( L((x_0, y_0)) \right) \right] \cap [y, x] \supset L((x_0, y_0)) \cap [y, x] \ni L(x, y). \]
Thus \(K_0\) is a joiner of the pair \((L_{I_\xi^x \times I_\xi^y}, L_{I_\xi^x \times I_\xi^y})\) and
\[ (L_{I_\xi^x \times I_\xi^y} \oplus K_0 L_{I_\xi^x \times I_\xi^y}) \supset L((x_0, y_0)), \]
for each \((x, y) \in I_\xi^x \times I_\xi^y \cup \xi I_\xi^x \times I_\xi^y\).

(ii) Take any joiner \(K\) of the pair \((L_{I_\xi^x \times I_\xi^y}, L_{I_\xi^x \times I_\xi^y})\) satisfying condition (10). Then, for any \((x, y) \in I_\xi^x \times I_\xi^y\), we have
\[ K_0 (h_1(x), h_2(y)) = (h_1|_{I_\xi^x} \cup h_2|_{I_\xi^y}) \left( L((x_0, y_0)) \right) \]
\[ \subset (h_1|_{I_\xi^x} \cup h_2|_{I_\xi^y}) \left( L_{I_\xi^x \times I_\xi^y} \oplus K L_{I_\xi^x \times I_\xi^y} \right) (x, y) \]
\[ = (h_1|_{I_\xi^x} \cup h_2|_{I_\xi^y}) \left[ (h_1|_{I_\xi^x} \cup h_2|_{I_\xi^y})^{-1} \left( K (h_1(x), h_2(y)) \right) \cap [x, y] \right] \]
\[ \subset K (h_1(x), h_2(y)). \]
Repeating the calculation for an arbitrary point \((x, y) \in \xi I_\xi^x \times I_\xi^y\), we come to the assertion. \[\square\]

Definition 3. The multifunction \(K_0\), introduced in Theorem 2, is called \(\xi\)-reconstructing joiner for the mean \(L\).

5. Reconstructing joiner with singleton values

The following question arises naturally.

Problem 3. Is it possible that the reconstructing joiner is in fact a single-valued function?

Below we give a full answer to this question, providing a simple characterization of means with single-valued reconstructing joiners.
We say that a mean $L$ preserves its $\xi$-margins if the equalities $L(u_1, \xi) = L(u_2, \xi)$ and $L(\xi, v_1) = L(\xi, v_2)$ imply that
\[
L(\min\{L(u_1, v_1), \xi\}, \max\{L(u_1, v_1), \xi\}) = L(\min\{L(u_2, v_2), \xi\}, \max\{L(u_2, v_2), \xi\})
\]
for all $(u_1, v_1), (u_2, v_2) \in I_\xi^0 \times \xi I^0$, and
\[
L(\max\{L(u_1, v_1), \xi\}, \min\{L(u_1, v_1), \xi\}) = L(\max\{L(u_2, v_2), \xi\}, \min\{L(u_2, v_2), \xi\})
\]
for all $(u_1, v_1), (u_2, v_2) \in \xi I^0 \times I_\xi^0$.

Remark 1. Observe that for symmetric means the above defining condition becomes much simpler: the equalities $L(u_1, \xi) = L(u_2, \xi)$ and $L(v_1, \xi) = L(v_2, \xi)$ imply that
\[
L(L(u_1, v_1), \xi) = L(L(u_2, v_2), \xi)
\]
for all $(u_1, v_1), (u_2, v_2) \in I_\xi^0 \times \xi I^0$ and $(u_1, v_1), (u_2, v_2) \in \xi I^0 \times I_\xi^0$.

Remark 2. If the marginal functions $L(\cdot, \xi)$ and $L(\xi, \cdot)$ are one-to-one, then the mean $L$ preserves its $\xi$-margins.

Example 3. Take $I = \mathbb{R}$ and $\xi = 0$, and put
\[
L(x, y) = \max\{x, y\}, \quad x, y \in I.
\]
Of course, the marginal function $L(\cdot, 0)$ is not one-to-one. Nevertheless, $L$ preserves its 0-margins. To see this, take any $(u_1, v_1), (u_2, v_2) \in I_0^0 \times 0 I^0$ satisfying $L(u_1, 0) = L(u_2, 0)$ and $L(v_1, 0) = L(v_2, 0)$. Then $u_1 < 0 < v_1$ and $u_2 < 0 < v_2$, whence also $L(u_1, v_1) = v_1$ and $L(u_2, v_2) = v_2$. Therefore,
\[
L(L(u_1, v_1), 0) = L(v_1, 0) = L(v_2, 0) = L(L(u_2, v_2), 0).
\]
A similar condition holds whenever $(u_1, v_1), (u_2, v_2) \in 0 I^0 \times I_0^0$. Thus, by Remark 1, the mean $L$ preserves its 0-margins.

Example 4. Of course, not every mean preserves its margins. To see this, take $\xi = 1$ and define $L$ as the contraharmonic mean on the interval $(0, +\infty)$ by the equality
\[
L(x, y) = \frac{x^2 + y^2}{x + y}.
\]
Suppose that $L$ preserves its 1-margins. As it is symmetric, we may use Remark 1. Take an arbitrary $v \in (1, +\infty)$ and put $u_1 = \frac{1}{5}, u_2 = \frac{2}{3}, v_1 = v_2 = v$. Then $(u_1, v_1), (u_2, v_2) \in I_1^\xi \times I_1^\xi$,

$$L (u_1, 1) = L \left( \frac{1}{5}, 1 \right) = \frac{13}{15} = L \left( \frac{2}{3}, 1 \right) = L (u_2, 1),$$

and, of course, $L (v_1, 1) = L (v_2, 1) = L (v, 1)$. Thus, by Remark 1,

$$L (L (u_1, v), 1) = L (L (u_2, v), 1). \quad (11)$$

Since

$$\lim_{v \to \infty} L (u, v) = \lim_{v \to \infty} \frac{u^2 + v^2}{u + v} = +\infty, \quad u \in (0, +\infty),$$

we may choose $v$ in such a way that $L (u_1, v) > 1$ and $L (u_2, v) > 1$. Then, as the restriction of $L (\cdot, 1)$ to $(1, +\infty)$ is one-to-one, we have $L (u_1, v) = L (u_2, v)$. Taking into account that $u_1 = \frac{1}{5}$ and $u_2 = \frac{2}{3}$, we see that

$$\frac{25v^2 + 1}{25v + 5} = \frac{9v^2 + 4}{9v + 6},$$

whence

$$15v^2 - 13v - 2 = 0,$$

that is, $v \in \{-\frac{2}{15}, 1\}$, a contradiction.

The main result of this section reads as follows.

**Theorem 3.** The $\xi$-reconstructing joiner of the mean $L$ has only singletons among the values if and only if $L$ preserves its $\xi$-margins.

**Proof.** Assume that the reconstructing joiner $K_0$ given by (8) and (9) is single-valued. Take any points $(u_1, v_1), (u_2, v_2) \in I_1^\xi \times I_1^\xi$ satisfying $L (u_1, \xi) = L (u_2, \xi)$ and $L (\xi, v_1) = L (\xi, v_2)$. Consider the following four possible cases:

(a) $L (u_1, u_2) \leq \xi$ and $L (v_1, v_2) \leq \xi$;
(b) $L (u_1, u_2) \leq \xi < L (v_1, v_2)$;
(c) $L (v_1, v_2) \leq \xi < L (u_1, u_2)$;
(d) $\xi < L (u_1, u_2)$ and $\xi < L (v_1, v_2)$.
According to (8), we have
\[ \mathcal{K}_0 (h_1(u_1), h_2(v_1)) = h_1 (L((u_1, v_1)) \cap I_{\xi}) \cup h_2 (L((u_1, v_1)) \cap I_{\xi}^\circ). \]

Therefore, as the above set is a singleton, we conclude that:
\[ h_1 (L(u_1, v_1)) = h_1 (L(u_2, v_2)), \]
i.e. \( L(L(u_1, v_1), \xi) = L(L(u_2, v_2), \xi) \) in case (a),
\[ h_1 (L(u_1, v_1)) = h_2 (L(u_2, v_2)), \]
i.e. \( L(L(u_1, v_1), \xi) = L(\xi, L(u_2, v_2)) \) in case (b),
\[ h_2 (L(u_1, v_1)) = h_1 (L(u_2, v_2)), \]
i.e. \( L(\xi, L(u_1, v_1)) = L(L(u_2, v_2), \xi) \) in case (c),
\[ h_2 (L(u_1, v_1)) = h_2 (L(u_2, v_2)), \]
i.e. \( L(\xi, L(u_1, v_1)) = L(\xi, L(u_2, v_2)) \) in case (d). In all cases (a)–(d), the obtained equalities mean that
\[
L(\min\{L(u_1, v_1), \xi\}, \max\{L(u_1, v_1), \xi\}) = L(\min\{L(u_2, v_2), \xi\}, \max\{L(u_2, v_2), \xi\}).
\]

A similar reasoning gives the second desired equality in the case when \((u_1, v_1), (u_2, v_2) \in \xi I^\circ \times I_{\xi}^\circ\). So \(L\) preserves its \(\xi\)-margins.

A careful analysis of the above proof shows that also the converse implication holds true.

Finally, we notice the following immediate consequence of Theorem 3 and Remark 2.

**Corollary 1.** If the marginal functions \(L(\cdot, \xi)\) and \(L(\xi, \cdot)\) are one-to-one, then the \(\xi\)-reconstructing joiner of \(L\) is single-valued.
References


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