On the factors of Stern polynomials II. 
Proof of a conjecture of M. Gawron

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Abstract. Let $B_n(x)$ be the $n$-th Stern polynomial in the sense of Klavžar et al. [2]. Gawron’s conjecture [1] about the natural density of indices $n$ such that $B_n(t) = 0$, where $t = -1/2, -1/3$, is proved and generalized. Similar questions are treated.

Klavžar, Milutinović and Petr [2] defined Stern polynomials $B_n(x)$ by the conditions $B_0(x) = 0$, $B_1(x) = 1$, $B_{2n}(x) = xB_n(x)$, $B_{2n+1}(x) = B_n(x) + B_{n+1}(x)$. Gawron [1] proved that the only rational zeros of $B_k(x)$ are $0, -1/2, -1/3$ and proved that for $t = -1/2, t = -1/3$,

$$d_m(t) = \frac{|\{0 \leq k < m : B_k(t) = 0\}|}{m},$$

we have $\liminf_{m \to \infty} d_m(t) = 0$. He conjectured ([1, Conjecture 2.7]) that

$$\lim_{m \to \infty} d_m(t) = 0.$$  (2)

We shall consider a more general problem: how often an irreducible (over $\mathbb{Q}$) polynomial $f$ with integral coefficients divides $B_n$. Denoting a zero of $f$ by $t$, we introduce $d_m(t)$ by formula (1). Since $B_{2n+1}(0) = 1$, if $t \neq 0$, $t^{-1} = \tau$ is an algebraic integer and we set $F = \mathbb{Q}(t)$. $N_{F/\mathbb{Q}}$ is the norm from $F$ to $\mathbb{Q}$.

Theorem 1. For every algebraic integer $\tau = t^{-1}$, different from $0$ and roots of unity, (2) holds.

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Corollary 1. For \( t = -\frac{1}{2}, -\frac{1}{3} \), (2) holds.

Corollary 2. For every prime \( p > 2 \), the upper density of indices \( m \) such that \( p \mid B_m(1) \) does not exceed \( 2/p \).

As to \( t \) being a root of unity, we have only partial results.

Theorem 2. The density of indices \( n \) such that \( (x + 1)^2 \mid B_n(x) \) is zero.

Theorem 3. If \( t \) is a primitive root of unity of order \( e > 2 \), then, for every positive integer \( m \),

\[
d_m(t) \leq \frac{1}{m} + \frac{1}{m} \left\lfloor \frac{m-1}{\Phi_e(2)} \right\rfloor ,
\]

where \( \Phi_e \) is the cyclotomic polynomial of order \( e \). Moreover, if \( e = 2^a > 2 \), or \( e = 2 \cdot 3^a > 2 \), then, for every positive integer \( m \),

\[
d_m(t) \leq \frac{1}{m} + \frac{1}{m} \left\lfloor \frac{m-1}{3\Phi_e(2)} \right\rfloor .
\]

As to the other conjecture in [1, Conjecture 4.3], we have only a much weaker result.

Theorem 4. The density of indices \( n \) such that \( B_n \) is reciprocal is zero.

[1, Conjecture 4.3] asserts that the number of \( n \leq x \) such that \( B_n \) is reciprocal is \( O((\log x)^k) \) for a certain \( k \).

Notation. For a prime ideal \( \mathfrak{p} \nmid \tau \) of \( F \), let \( q = N_{F/\mathbb{Q}}\mathfrak{p} \), and let \( W_p(t) \) be the set of all pairs \( (\alpha, \beta) \in \mathbb{F}_q^2 \) obtainable from \((1,0)\) by repeated use of the transformations \( T_0(\alpha, \beta) = (t\alpha + \beta, \beta) \) and \( T_1(\alpha, \beta) = (\alpha, t\beta + \alpha) \), where \( t \) is to be interpreted as an element of \( \mathbb{F}_q \). For an integer \( m \neq 0 \), \( P(m) \) is the greatest prime factor of \( m \). For an algebraic integer \( \tau \), \( M(\tau) \) is the Mahler measure of \( \tau \), i.e.,

\[
M(\tau) = \prod_{|\tau^{(i)}| > 1} |\tau^{(i)}|,
\]

where \( \tau^{(i)} \) are all conjugates of \( \tau \).

Lemma 1. There exist infinitely many prime ideals \( \mathfrak{p} \) of \( F \) such that there is \( (\alpha, \beta) \in W_p(t) \) satisfying

\[
T_0(\alpha, \beta) = (\alpha, \beta).
\]
PROOF. Let us consider the sequence $u_n = N_{F/Q}((2\tau - 1)\tau^n - 1)$, where $u_n$ is a linear recurrence defined over $\mathbb{Q}$. Let $\omega_1, \ldots, \omega_s$ be the characteristic roots of the sequence $u_n$ (the distinct zeros of the companion polynomial), and let $l$ be the least common multiple of the finite orders of the ratios $\omega_i/\omega_j$ in the multiplicative group $\mathbb{C}^\ast$. No two characteristic roots of the sequence $u_{lm}$ ($m = 0, 1, \ldots$) have the ratio of finite order. Hence, by the theorem of Pólya [4], $\limsup P(u_{lm}) = \infty$, unless $u_{lm} = A(m)a^m$, where $A \in \mathbb{Q}[x]$ and $a \in \mathbb{Q}^\ast$. Now, $\limsup P(A(m)) = \infty$, unless $A$ is constant and

$$u_{lm} = Aa^m.$$

However, since if an algebraic integer $\tau \neq 0$ is not a root of unity, by a theorem of Kronecker, some of its conjugates $\tau^{(i)}$ lies outside the unit circle. Hence $M(\tau) > 1$. For a large $n$ suitably chosen (see [7]), we have for all $i$, hence for infinitely many $m$,

$$|u_{mi}| = (1 + o(1)) \prod_{|\tau^{(i)}| > 1} |2\tau^{(i)} - 1| M(\tau)^{|mi|} \prod_{|\tau^{(i)}| = 1} |2\tau^{(i)} - 2|,$$

$$|u_{2mi}| = (1 + o(1)) \prod_{|\tau^{(i)}| > 1} |2\tau^{(i)} - 1| M(\tau)^{2|mi|} \prod_{|\tau^{(i)}| = 1} |2\tau^{(i)} - 2|,$$

and, since by (6), $u_{2mi}u_0 = u_{mi}^2$, we obtain

$$(1 + o(1)) \prod_{|\tau^{(i)}| \neq 1} |2\tau^{(i)} - 1| = (1 + o(1)) \prod_{|\tau^{(i)}| > 1} |2\tau^{(i)} - 1|^{-1}.$$  

Since the equality is independent of $m$, it follows that

$$\prod_{|\tau^{(i)}| \neq 1} |2\tau^{(i)} - 2| = \prod_{|\tau^{(i)}| > 1} |2\tau^{(i)} - 1|^{-1}.$$  

Since the right hand side is non-divisible by 2, the product on the left is empty, and we obtain

$$1 = \prod_{|\tau^{(i)}| > 1} |2\tau^{(i)} - 1|^{-1} < 1.$$  

The obtained contradiction proves that $\limsup P(u_n) = \infty$, and we take $p$ any common prime ideal factor of $P(u_n)$ and $(2\tau - 1)^{\tau^n - 1}$. On the other hand,

$$(\alpha, \beta) = T^{n+2}_1 T_0(1, 0) = \left( t, t\frac{t^{n+2} - 1}{t - 1} \right) \in W(p)(t),$$

$$T_0(\alpha, \beta) - (\alpha, \beta) = (\tau^{-n-2}((2\tau - 1)\tau^n - 1), 0).$$  

\[\square\]
Lemma 2. If \( p \nmid \tau \) is a prime ideal of \( F \) satisfying Lemma 1, and the system of linear equations
\[
\frac{1}{2} x_{T_0(\alpha, \beta)} + \frac{1}{2} x_{T_1(\alpha, \beta)} = \lambda x_{(\alpha, \beta)}
\] (7)
holds for all \((\alpha, \beta) \in W_p(t)\) with \( x \neq 0 \) and \( |\lambda| \geq 1 \), then \( \lambda = 1 \).

Proof. Let
\[
|x_{(\alpha_0, \beta_0)}| = \max_{(\alpha, \beta) \in W_p(t)} |x_{(\alpha, \beta)}|.
\]
We infer from (7) that \( |\lambda| = 1 \) and
\[
x_{T_0(\alpha_0, \beta_0)} = x_{T_1(\alpha, \beta)} = \lambda x_{(\alpha_0, \beta_0)},
\]
hence, by induction on the number of steps needed to reach \((\alpha, \beta)\) from \((\alpha_0, \beta_0)\),
\[
x_{T_0(\alpha, \beta)} = x_{T_1(\alpha, \beta)} = \lambda x_{(\alpha, \beta)} \quad \text{and} \quad |x_{(\alpha, \beta)}| = |x_{(\alpha_0, \beta_0)}| > 0,
\]
for all \((\alpha, \beta) \in W_p(t)\), thus, in particular, for \((\alpha, \beta)\) satisfying (5). But (5) implies
\[
x_{T_0(\alpha, \beta)} = x_{(\alpha, \beta)}, \quad \lambda = 1.
\]
□

Lemma 3. Let \( e \) be the order of \( t = \tau^{-1} \mod \tau \) of \( F \) in the multiplicative group \( F^*_q \). Then
\[
e \geq \frac{\log q - [F : Q] \log 2}{\log M(\tau)}. \quad (8)
\]
Proof. It follows from \( t^e \equiv 1 \pmod{p} \) that \( \tau^e \equiv 1 \pmod{p} \) and
\[
quotient{N_{F/Q}(\tau^e - 1)}.
\]
However,
\[
|N_{F/Q}(\tau^e - 1)| \leq \sum_{S \subseteq \{1, 2, \ldots, [F:Q]\}} \prod_{i \in S} |\tau_i^e| \leq q^{[F:Q]} M(\tau)^e,
\]
thus (8) follows. □

Lemma 4. Let \( p \nmid \tau \) be a prime ideal of \( F \) satisfying Lemma 1, and
\[
d_{m,p}(t) = \left\lceil \frac{\left| \{0 \leq n < m : B_n(t) \equiv 0 \pmod{p}\} \right|}{m} \right\rceil.
\]
Then the limit \( \lim_{n \to \infty} d_{2^n, p}(t) \) exists and satisfies the inequality
\[
\lim_{n \to \infty} d_{2^n, p}(t) \leq \frac{\log M(\tau)}{\log q - [F : Q] \log 2}. \quad (9)
\]
Proof. The proof follows that of [1, Theorem 2.5], only instead of [3, Example 8.3.2] we use [3, Theorem 7.10.33] and Lemma 2, together with [3, Exercise 4.4.20 and Formula 8.3.13], and instead of the inequality \( \frac{2}{p} \leq \frac{2}{\log p} \), we use Lemma 3.

Proof of Theorem 1. Let

\[
d_m(t) = \frac{\left| \{0 \leq k < m : B_k(t) = 0 \} \right|}{m}.
\]

Clearly, for every \( p \nmid \tau \),

\[
d_m(t) \leq d_{m,p}(t),
\]

and by (9) and Lemma 1,

\[
\lim_{n \to \infty} d_{2^n}(t) = 0.
\]  \hfill (10)

To show (1), we choose \( n \) by the inequalities

\[
2^{n-1} \leq m < 2^n.
\]  \hfill (11)

Thus

\[
d_m(t) \leq 2d_{2^n}(t),
\]

and (1) follows from (10).

Proof of Corollary 2. For every \( u \in \mathbb{F}_p^* \), all elements \( T_0^j(0,u) \) for \( 0 \leq j < p \) are distinct. Since \( T_0^0(1,0) = (1,0) \), following the proof of Lemma 4, we infer that \( \lim_{n \to \infty} d_{2^n,p}(1) \) exists and satisfies the inequality

\[
\lim_{n \to \infty} d_{2^n,p}(1) \leq 1/p.
\]  \hfill (12)

We choose \( n \) by the inequality (11), and Corollary 2 follows from (12).

Definition 1. For \( n < 0 \), \( B_n(x) = -B_{-n}(x) \).

Definition 2. For \( n \in \mathbb{Z} \),

\[
f_n(x) = \frac{B_{2n}(x)}{x + 1}.
\]

Definition 3. \( e_n = f_n(-1) \).

Lemma 5. For \( n \in \mathbb{Z} \),

\[
B_n(-1) = 3 \left\{ \frac{n}{3} + \frac{1}{2} \right\} - \frac{3}{2}.
\]
Proof. For $n \geq 0$, the formula is known and due to Ulas [8, Theorem 5.1].

For $n < 0$, we have by Definition 1

$$B_n(-1) = -B_{-n}(-1) = 3 \left\{ -\frac{n}{3} + \frac{1}{2} \right\} + \frac{3}{2} = 3 \left\{ \frac{n}{3} + \frac{1}{2} \right\} - \frac{3}{2}.$$ 

\[\square\]

Lemma 6. For $\alpha \in \mathbb{N}, a, b \in \mathbb{Z}, |b| \leq 2^k$, we have

$$B_{2^\alpha a + b}(x) = B_{2^\alpha - |b|}(x)B_{a}(x) + B_{|b|}(x)B_{a + \text{sgn } b}(x). \quad (13)$$

Proof. For $a, b \in \mathbb{N}$, (13) follows from [5, Lemma 1]. For $a \in \mathbb{N} \setminus \{0\}, b \leq 0$, we have

$$2^\alpha a + b = 2^\alpha (a - 1) + 2^\alpha - |b|,$$

and (13) follows again from [5, Lemma 1] with $a' = a - 1, b' = 2^\alpha - |b|$. For $a = 0, b < 0$, we have by Definition 1

$$B_{2^\alpha a + b}(x) = -B_{-b}(x) = B_{|b|}(x)B_{-1}(x).$$

For $a < 0$, we have by the already proved cases

$$B_{2^\alpha a + b}(x) = -B_{-2^\alpha a - b}(x) = -B_{2^\alpha - |b|}(x)B_{-a}(x) - B_{|b|}(x)B_{a - \text{sgn } b}(x)$$

$$= B_{2^\alpha - |b|}(x)B_{a}(x) + B_{|b|}(x)B_{a + \text{sgn } b}(x). \quad \square$$

Lemma 7. For $k \in \mathbb{N}, a, b \in \mathbb{Z}, 3|b| < 4^k$, we have

$$e_{4^k a + b} = e_a + e_b. \quad (14)$$

Proof. By Definition 2 and Lemmas 5 and 6, we have

$$f_{4^k a + b}(x) = \frac{B_{2^k 3a + b}(x)}{x + 1} = B_{4^k - 3|b|}(x)f_a(x) + f_{|b|}(x)B_{3a + \text{sgn } b}(x),$$

$$f_{4^k a + b}(-1) = f_a(-1) + f_{|b|}(-1) \text{ sgn } b = f_a(-1) + f_b(-1), \quad \square$$

hence, by Definition 3 we obtain (14).

Lemma 8. If $k \in \mathbb{N} \setminus \{0\}$, 

$$n = \sum_{i=1}^{k} c_i 4^{k-i} > 0, \quad \bigcup_{i=1}^{k} \{c_i\} \subset \{-1, 1\}, \quad (15)$$

then

$$e_n = |\{i : c_i = 1\}| - |\{i : c_i = -1\}|. \quad (16)$$
Proof. We proceed by induction on \( n \). For \( n = 1 \), (16) holds. Assume that \( n > 1 \) is given by (15), and that (16) holds for all integers in question less than \( n \). Then, applying Lemma 7 with \( k = 1 \), \( a = \sum_{i=1}^{k-1} c_i 4^{k-i-1} < n \), \( b = c_k \), we obtain

\[
e_n = e_a + e_b = e_a + c_k,
\]

and (16) follows from the inductive assumption. \( \square \)

Lemma 9. If \( k \in \mathbb{N} \setminus \{0\} \),

\[
n = \sum_{i=1}^{k} c_i 4^{k-i}, \quad \bigcup_{i=1}^{k} \{c_i\} \subset \{1, 2\},
\]

then

\[
e_n = \left|\{i : c_i = 1\}\right| - \left|\{i : c_i = 2\}\right|.
\]

Proof. We proceed by induction on \( n \). For \( n = 1 \), (18) holds. Assume that \( n > 1 \) is given by (17), and that (18) holds for all integers in question less than \( n \). Then, if \( c_k = 1 \), applying Lemma 7 with \( k = 1 \), \( a = \sum_{i=1}^{k-1} c_i 4^{k-i-1} < n \), \( b = 1 \), we have

\[
e_n = e_a + e_b = e_a + 1,
\]

and (18) follows from the inductive assumption. If \( c_k = 2 \), we have for \( a = \sum_{i=1}^{k-1} c_i 4^{k-i-1} \), by Definition 2

\[
f_n(x) = f_{4a+2}(x) = \frac{B_{12a+6}(x)}{x+1} = \frac{x B_{6a+3}(x)}{x+1} = x f_{2a+1}(x),
\]

hence, by Definition 3

\[
e_n = -e_{n/2}.
\]

If for a strictly increasing sequence of integers \( 0 \leq l_1 < l_2 < \cdots < l_{2h} = k \), \( c_i = 1 \) \((0 < i \leq l_1)\), \( c_i = 2 \) \((l_1 < i \leq l_2)\), \ldots , \( c_i = 1 \) \((l_{2h-2} < i \leq l_{2h-1})\), \( c_i = 2 \) \((l_{2h-1} < i \leq l_{2h})\), we have

\[
n/2 = \sum_{i=1}^{k} d_i 4^{k-i},
\]

where \( d_1 = 1 \), \( d_i = -1 \) \((1 < i \leq l_1 + 1)\), \ldots , \( d_i = -1 \) \((l_{2h-2} < i \leq l_{2h-1} + 1)\), \( d_i = 1 \) \((l_{2h-1} + 1 < i \leq l_{2h})\), (18) follows from (19) and the inductive assumption. \( \square \)
Lemma 10. If \( k \in \mathbb{N} \setminus \{0\} \),

\[
    n = \sum_{i=1}^{k} c_i 4^{k-i}, \quad \bigcup_{i=1}^{k} \{c_i\} \subset \{-1, 0, 1, 2\},
\]

then

\[
    e_n = |\{i : c_i = 1\}| - |\{i : c_i = -1\}| - |\{i : c_i = 2\}| + 3|\{i : \exists j \geq 0 \ c_i = -1, c_{i+1} = \ldots = c_{i+j} = 1, c_{i+j+1} = 2\}|.
\]

Proof. We proceed by induction on \( n \). For \( n = 0 \), (21) holds. Assume that \( n > 0 \) is given by (20), and that (21) holds for all non-negative integers less than \( n \).

If \( \{0, -1\} \cap \bigcup_{i=1}^{k} \{c_i\} = \emptyset \), (21) holds by Lemma 9. If \( \{0, -1\} \cap \bigcup_{i=1}^{k} \{c_i\} \neq \emptyset \), let \( j \) be the greatest index such that \( c_j \in \{0, -1\} \). If \( c_j = 0 \), we take \( a = \sum_{i=1}^{j-1} c_i 4^{j-i-1}, \quad b = \sum_{i=j+1}^{k} c_i 4^{k-i} \) in Lemma 7. Since \( 3|b| \leq 3 \sum_{i=j+1}^{k} 2 \cdot 4^{k-i} = 2(4^{k-j} - 1) < 4^{k-j+1} \), we obtain

\[
    e_n = e_a + e_b,
\]

and (21) follows from the inductive assumption. If \( c_j = -1 \), we take \( a = \sum_{i=1}^{j-1} c_i 4^{j-i-1}, \quad b = \sum_{i=j+1}^{k} c_i 4^{k-i} \) in Lemma 7. Since \( 3|b| \leq 3 \sum_{i=j+1}^{k} 4^{k-i} = 4^{k-j+1} - 1 < 4^{k-j+1} \), we obtain

\[
    e_n = e_a + e_b = e_a - e|b|.
\]

If \( j = k \), then \( e|b| = 1 \), and (21) follows from the inductive assumption.

If \( j < k \), and for an increasing sequence of integers \( j = l_0 < l_1 < l_2 < \cdots < l_{2h-1} \leq l_{2h} = k \) \((h > 0)\), \( c_i = 2 \) \((j < i \leq l_1)\), \( c_i = 1 \) \((l_1 < i \leq l_2)\), \ldots, \( c_i = 2 \) \((l_{2h-2} < i \leq l_{2h-1})\), \( c_i = 1 \) \((l_{2h-1} < i \leq k)\), then, for \( h = 1, l_1 = j \), it holds that \( |b| = 4^{k-j} - \sum_{i=j+1}^{k} 4^{k-i} \)

otherwise

\[
    |b| = \sum_{i=j+1}^{k} d_i 4^{k-i},
\]

where, for \( h = 1, l_1 > j \), it holds that \( d_i = 1 \) \((j < i < l_1)\), \( d_i = 2, d_i = -1 \) \((l_1 < i \leq k)\), otherwise, \( d_i = 1 \) \((j < i \leq l_1)\), \( d_i = 2 \) \((l_1 < i \leq l_2)\), \ldots, \( d_i = 1 \) \((l_{2h-2} < i < l_{2h-1})\), \( d_{l_{2h-1}} = 2, d_i = -1 \) \((l_{2h-1} < i \leq k)\). Hence, by the inductive assumption, if \( h = 1, l_1 = j \), then \( e|b| = -k + j + 1 \), otherwise

\[
    e|b| = 2 \sum_{\mu=1}^{h} l_{2\mu-1} - l_{2\mu} + k - j - 2,
\]

and (21) follows from (22) and the inductive assumption. \( \square \)
Proof of Theorem 2. If \((x + 1)^2 \mid B_n(x)\), then by Definition 2 and 3
\[ e_n = 0. \] (23)
Consider \(n\) satisfying the inequality
\[-\frac{4^k - 1}{3} \leq n \leq 2 \cdot \frac{4^k - 1}{3}. \] (24)
then every expansion \(n = \sum_{i=1}^{k} c_i 4^{k-i}, c_i \in \{-1, 0, 1, 2\}\) is equally probable.
By the Bernoulli law of large numbers, for every \(\varepsilon \in (0, 1/6)\) and sufficiently large \(k\), the number of \(n\)'s in the interval (24) such that
\[ \left| \{i : c_i = 1\} - \frac{k}{4} \right| > \varepsilon k \]
or
\[ \left| \{i : c_i = -1\} - \frac{k}{4} \right| > \varepsilon k \]
is less than \(\varepsilon 4^k\). If, on the other hand,
\[ \left| \{i : c_i = 1\} - \frac{k}{4} \right| \leq \varepsilon k \quad \text{and} \quad \left| \{i : c_i = -1\} - \frac{k}{4} \right| \leq \varepsilon k \]
and \(e_n = 0\), then, by Lemma 10,
\[ l = \left| \{i \geq 0 : c_i = -1, c_{i+1} = \cdots = c_{i+j} = 1, c_{i+j+1} = 2\} \right| \in (k/12 - \varepsilon k, k/12 + \varepsilon k), \]
and the number of \(n\)'s in the interval (24) satisfying (23) does not exceed
\[ \sum_{k/12 - \varepsilon k < l < k/12 + \varepsilon k} 4^{k-2l} \left( \frac{\lfloor k/2 \rfloor}{l} \right) < (2\varepsilon k + 1)k \cdot 4^{k-k/6 + 2\varepsilon k} \cdot \left( \frac{\lfloor k/2 \rfloor}{\lfloor k/12 + \varepsilon k \rfloor} \right) = L. \]
Since \(L/4^k\) tends to 0, when \(\varepsilon\) is small enough and \(k\) tends to infinity, the theorem follows. \(\square\)

Proof of Theorem 3. \(B_k(t) = 0\) implies \(\Phi_k(x) \mid B_k(x)\). Thus \(\Phi_k(2) \mid B_k(2) = k\) and (3) follows. Moreover, if \(e = 2^a\), then \(0 = B_k(t) \equiv B_k(1) \pmod{1-t}\), and since \(N_{F/Q}(1-t) = 2\), \(B_k(t) \equiv 0 \pmod{2}\), thus by Lemma 5, \(k \equiv 0 \pmod{3}\). Since for \(a > 1\), \((\Phi_{2^a}(2), 2) = 1\), (4) follows. If \(e = 2 \cdot 3^a\) and \(B_k(t) = 0\), then \(k \equiv 0 \pmod{3}\), thus \(x+1 \mid B_k(x)\), and since for \(a > 0\), \((x+1, \Phi_k(x)) = 1\), we obtain \((x+1)\Phi_k(x) \mid B_k(x)\), thus \(3\Phi_k(2) \mid k\) and (4) follows. \(\square\)

Proof of Theorem 4. Since \(B_n(0) = 0\) for \(n\) even, \(B_n(1) = 1\) for \(n\) odd, if \(B_n\) is reciprocal, it is monic, but by [6, Corollary 2] for almost all \(n\), in the sense of density, \(B_n\) is not monic. \(\square\)
References


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