Publ. Math. Debrecen
92/1-2 (2018), 17–57
DOI: 10.5486/PMD.2018.7698

Isometries of vector-valued function spaces preserving the kernel of a linear operator

By KAZUHIRO KAWAMURA (Tsukuba)

Abstract. We prove Banach–Stone type theorems for isometries of continuously differentiable (continuously twice differentiable resp.) functions on the unit interval equipped with norms related to the first derivative (a linear differential operator of second-order resp.). They are consequences of a general theorem on isometries of function spaces, on which a linear operator is defined, over compact Hausdorff spaces.

1. Introduction

The Banach–Stone theorem and its variants on isometries of function spaces is a subject of extensive study. They state that every surjective linear isometry between function spaces is an operator of special type, often referred to as a (generalized) weighted composition operator, whenever the function spaces satisfy certain conditions. Various extensions of the classical theorem due to Banach and Stone, which deals with isometries of continuous function spaces over compact Hausdorff spaces with the supremum norm, have been obtained by many authors. Among these extensions, the following are of importance to the present paper. They deal with isometries between function spaces of:

(i) vector-valued continuous functions [1], [2], [3], [10], [16], [19], [20], [21], [30], [34],

Mathematics Subject Classification: 46E15, 55R25.
Key words and phrases: isometry, weighted composition operator, Banach–Stone theorem, continuously differentiable functions, linear operator, differential operator.
The author is supported by JSPS KAKENHI Grant Number 26400080.

(ii) scalar/vector-valued continuously differentiable functions on $[0,1]$ or subsets of $\mathbb{R}$ [5], [9], [10], [11], [17], [18], [31], [35], [36], and

(iii) scalar/vector-valued Lipschitz functions on compact metric spaces [4], [6], [7], [12], [28].

Also monographs [14] and [15] are valuable sources of information. A real-linear isometry of scalar-valued continuous function spaces which fail to be a weighted composition operator was explicitly constructed and studied in [29] and [33]. Miura and the author [27] studied such isometries and the study continued in [22] for vector-valued continuous function spaces, and in [23] for spaces of continuous sections of vector bundles. A subsequent paper [26] by Koshimizu, Miura and the author studied isometries with respect to various norms on the continuously differentiable function space on the unit interval and gave a single framework to characterize these isometries.

The present paper is a continuation of these papers. Throughout the paper, $\mathbb{F}$ denotes a scalar field $\mathbb{R}$ or $\mathbb{C}$. For a compact Hausdorff space X and a Banach space E over $\mathbb{F}$, $C(X,E)$ denotes the space of all E-valued continuous functions on X with the supremum norm $\|\cdot\|_\infty$. We consider the following set up:

> X and Y are compact Hausdorff spaces and E and F are *finite dimensional Hilbert spaces* over $\mathbb{F}$. A linear operator $D : A \to C(Y,F)$ is defined on a subspace A of $C(X,E)$. For p with $1 \le p \le \infty$, $\|\cdot\|_{D,p}$ denotes the norm on A defined by, for $f \in A$,
>
> $$\|f\|_{D,p} = \begin{cases} (\|f\|_\infty^p + \|Df\|_\infty^p)^{1/p}, & \text{if } p < \infty, \\ \max(\|f\|_\infty, \|Df\|_\infty), & \text{if } p = \infty. \end{cases}$$

Let $\mathcal{U}(E)$ be the group of all linear isometries on the Hilbert space E. We show that, under certain conditions on A and D that guarantee the "abundance of functions in A," every surjective linear isometry $T : A \to A$ such that $T(\operatorname{Ker} D) = T^{-1}(\operatorname{Ker} D) = \operatorname{Ker} D$ is always a generalized weighted composition operator in the following sense:

> there exist homeomorphisms $\varphi : X \to X$ and $\psi : Y \to Y$, and maps $U : X \to \mathcal{U}(E); x \mapsto U_x$ and $V : Y \to \mathcal{U}(F); y \mapsto V_y$ such that for each $f \in A$ and for each $(x,y) \in X \times Y$ we have the equalities
>
> $$Tf(x) = U_x(f(\varphi(x))), \quad D(Tf)(y) = V_y(Df(\psi(y))).$$

The Main Theorem, stated in Section 2 after some preliminaries, is somewhat technical, yet some of its consequences are worth mentioning here.

The first consequence generalizes a result of [26] on the norms of $C^1([0,1],\mathbb{C})$, the space of all complex-valued continuously differentiable function space on $[0,1]$. It supplies a “continuous interpolation” between the norm $\|f\|_\Sigma = \|f\|_\infty + \|f'\|_\infty$ and the norm $\|f\|_M = \max(\|f\|_\infty, \|f'\|_\infty)$ on $C^1([0,1],\mathbb{C})$, where f' denotes the derivative of f. The result for $p = 2$ gives a partial answer to a question posed by Rao and Roy in [36]. A part of the result on the norm $\|\cdot\|_M$ follows from a result of Jarosz [17].

Theorem 1.1. *Let $p \in [1,\infty]$, and let $\|\cdot\|_p$ be the norm on $C^1([0,1])$ defined by*

$$\|f\|_p = (\|f\|_\infty^p + \|f'\|_\infty^p)^{1/p}, \quad f \in C^1([0,1]). \tag{1.1}$$

Let $T : C^1([0,1],\mathbb{C}) \to C^1([0,1],\mathbb{C})$ be a surjective real-linear $\|\cdot\|_p$-isometry and assume, if $p > 1$, that T satisfies

$$f \equiv \text{const } \textit{if and only if } Tf \equiv \text{const}. \tag{1.2}$$

Then there exist a constant c with $|c| = 1$ and $\epsilon, \gamma \in \{\pm 1\}$ such that

$$Tf(t) = c[f(\varphi_\gamma(t))]^\epsilon, \quad f \in C^1([0,1]),$$

where $\varphi_1(t) = t$ and $\varphi_{-1} = 1 - t$. Also $[f(x)]^1 = f(x)$ and $[f(x)]^{-1} = \overline{f(x)}$(= the complex conjugate of $f(x)$). If, moreover, T is complex-linear, then $\epsilon = 1$.

Remark. When $p = 1$, condition (1.2) is a consequence of other assumptions.

Another application of the Main Theorem yields a novel Banach–Stone type theorem. Let $L : C^n([0,1],\mathbb{F}) \to C([0,1],\mathbb{F})$ be a linear differential operator of order at most n. We consider the solution space $\mathcal{S}_L$ of the homogeneous differential equation

$$Lf = 0, \quad f \in C^n([0,1],\mathbb{F}),$$

and define the following norm for each $p \in [1,\infty]$:

$$\|f\|_{L,p} = (\|f\|_\infty^p + \|Lf\|_\infty^p)^{1/p}, \quad f \in C^n([0,1],\mathbb{F}). \tag{1.3}$$

Then we may ask whether every linear surjective $\|\cdot\|_{L,p}$-isometry which makes the space $\mathcal{S}_L$ invariant must be of a special form. In this direction, we prove the following.

Theorem 1.2. *Let $\alpha \in (0,\pi/2)$, and let $\|\cdot\|_{<\alpha>}$ be the norm on $C^2([0,1],\mathbb{C})$ defined by*

$$\|f\|_{<\alpha>} = \max(\|f\|_\infty, \|f'' + \alpha^2 f\|_\infty). \tag{1.4}$$

Let $\mathcal{S}_\alpha = \{a\exp(i\alpha t) + b\exp(-i\alpha t)|a, b \in \mathbb{C}\}$. Then every complex-linear surjective $\|\cdot\|_{<\alpha>}$-isometry $T : C^2([0,1],\mathbb{C}) \to C^2([0,1],\mathbb{C})$ satisfying $T(\mathcal{S}_\alpha) = T^{-1}(\mathcal{S}_\alpha) = \mathcal{S}_\alpha$ is of the form

$$Tf(t) = \kappa f(\varphi_\gamma(t)), \quad f \in C^2([0,1],\mathbb{C}),$$

where $\kappa \in \mathbb{C}$ is a constant with $|\kappa| = 1$, $\gamma = \pm 1$, and φ_γ is the function defined in Theorem 1.1.

The formalism of the Main Theorem looks more complicated than necessary for the proofs of these two results. If we are content with proving Theorems 1.1 and 1.2, then simpler arguments seem to be available. While the scheme presented by the Main Theorem provides a convenient framework to obtain a Banach–Stone type theorem in other contexts as well. Recently the author applied the scheme in the vector bundle setup to prove Banach–Stone type theorems for isometries of C^1 function spaces over compact Riemannian manifolds ([24],[25]).

The whole proof is based on the extreme point method in the setting of vector-valued function spaces (cf. [13]).

2. Preliminaries and statement of the Main Theorem

Throughout the paper, $\mathbb{T}$ denotes the unit circle on the complex plane: $\mathbb{T} = \{z \in \mathbb{C}||z| = 1\}$. As in Section 1, for a Banach space E over a scalar $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $C(X, E)$ denotes the space of all continuous E-valued functions defined on a compact Hausdorff space X with the supremum norm $\|\cdot\|_\infty$. Also $B(E)$ and $S(E)$ denote the unit ball and the unit sphere of E respectively. For two normed linear spaces L_1, L_2 over $\mathbb{F}$, let $\mathcal{L}_\mathbb{F}(L_1, L_2)$ denote the space of the $\mathbb{F}$-linear maps $L_1 \to L_2$ with the operator norm. The dual space $\mathcal{L}_\mathbb{F}(L,\mathbb{F})$ is denoted by L^*. An *extreme point* of a convex set C of a linear space L is a point $\xi \in C$ with the property that the equality $\xi = \frac{\eta+\zeta}{2}$ with $\eta, \zeta \in C$ implies $\eta = \zeta = \xi$. The set of all extreme points of C is denoted by $\mathrm{ext}(C)$. By abuse of notation, the extreme points of the unit ball $B(E)$ of a Banach space E are denoted by $\mathrm{ext}(E)$ for simplicity. Under this notation, a Banach space E is strictly convex if and only if $\mathrm{ext}(E) = S(E)$ (see [32]).

Let A be a (not necessarily closed) linear subspace of $C(X, E)$, and let $D : A \to C(Y, F)$ be a (not necessarily bounded) linear operator on A to the space $C(Y, F)$, for a compact Hausdorff space Y and a Banach space F. For each p with $1 \leq p \leq \infty$, we define a norm on A as follows:

$$\begin{aligned}\|f\|_{D,p} &= (\|f\|_\infty^p + \|Df\|_\infty^p)^{1/p} \\ &= \sup_{(x,y)\in X\times Y} (\|f(x)\|^p + \|Df(y)\|^p)^{1/p}, \quad f \in A. \end{aligned} \tag{2.1}$$

As in Section 1, we follow the standard convention: $\|f\|_{D,\infty}=\max(\|f\|_\infty,\|Df\|_\infty)$. We seek a set of conditions on A and D which implies that every surjective linear $\|\cdot\|_{D,p}$-isometry is a generalized weighted composition operator:

$$Tf(x) = V_x(f(\varphi(x))), \quad f \in A,\ x \in X, \tag{2.2}$$

where $\varphi : X \to X$ is a homeomorphism and $V_x : E \to E$ is a linear isometry for $x \in X$. We consider the following conditions on A and D.

(C1) The set $M_A = \{f \times Df | f \in A\}$ is closed in $C(X \times Y, E \times F)$.

(C2-1) For each $x \in X$ and for each $u \in E$, we have the following:

(C2-1a) there exists an $f \in \operatorname{Ker} D$ such that $f(x) = u$ and $\|f\|_\infty = \|u\|$.

(C2-1b) for each finite subset J of $X \setminus \{x\}$, there exists a $g \in A$ such that $g(x) = u$ and $g|J \equiv 0$.

(C2-2) For each $y \in Y$ and for each $v \in F$, we have the following:

(C2-2a) for each $\epsilon > 0$ there exists an $f \in A$ such that $Df(y) = v$, $\|Df\|_\infty = \|v\|$ and $\|f\|_\infty < \epsilon$.

(C2-2b) for each finite subset J of $Y \setminus \{y\}$, there exists a $g \in A$ such that $Dg(y) = v$ and $Dg|J \equiv 0$.

(C2-3) For each $u \in E$, for each $x \in X$, for each finite set $J \subset X \setminus \{x\}$ and for each finite set $K \subset Y$, there exists an $f \in A$ such that $f(x) = u$, $f|J \equiv 0$, and $Df|K \equiv 0$.

The next conditions are related to the identification of the extreme points of the dual spaces. For a point $(x, y) \in X \times Y$, "(Peak)" refers to the following condition on (x, y):

(Peak) for each $(u, v) \in E \times F$, there exists an $f \in A$ such that $(f(x), Df(y)) = (u, v)$ and

$$\|f(x')\| < \|u\|, \quad \|Df(y')\| < \|v\|,$$

for each $(x', y') \in (X \setminus \{x\}) \times (Y \setminus \{y\})$.

Also, for a point $x \in X$ and a point $y \in Y$, the conditions "(P_X)" and "(P_Y)" refer to:

(P_X) for each $u \in E$, there exists an $f \in A$ such that $f(x) = u$, $\|f(x')\| < \|u\|$ for each $x' \in X \setminus \{x\}$, and further $\|Df\|_\infty < \|u\|$, and

(P_Y) for each $v \in F$, there exists a $g \in A$ such that $Dg(y) = v$, $\|Dg(y')\| < \|v\|$ for each $y' \in Y \setminus \{y\}$, and further $\|g\|_\infty < \|v\|$.

Then we consider:

(C3-1) The set $P = \{(x, y) \in X \times Y | (x, y)$ satisfies (Peak)$\}$ is dense in $X \times Y$.

(C3-2) The sets $P_X = \{x \in X | x$ satisfies(P_X)$\}$ and $P_Y = \{y \in Y | y$ satisfies (P_Y)$\}$ are dense in X and in Y respectively.

The next conditions describe the connection of $\operatorname{Ker} D$ with the constant functions. For a vector $u \in E$ and a vector $v \in F$, c_u and c_v denote the constant functions on X and Y which take value u and v respectively. The subspace of constant functions on X is denoted by Const_X.

(C4-1) The subspace A contains the subspace of constant functions Const_X and $D|\mathrm{Const}_X$ is a bounded operator, i.e. there exists a constant $K > 0$ such that $\|Dc_u\|_\infty \le K\|u\|$ for each $u \in E$.

(C4-2) There exists a constant $L > 0$ with the following property: for each $v \in F$, there exists an $h_v \in A$ such that $Dh_v = c_v$ and $\|h_v\|_\infty \le L\|v\|$.

As was pointed out by a referee, when A is a closed subspace of $C(X, E)$, condition (C1) is equivalent to the boundedness of the operator D (by the Closed Graph Theorem).

We are now ready to state the main theorem. As in Section 1, $\mathcal{U}(E)$ denotes the group of all linear isometries on E.

Main Theorem. *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let E and F be finite dimensional Hilbert spaces. For compact Hausdorff spaces X and Y, let A be an $\mathbb{F}$-subspace of $C(X, E)$, and let $D : A \to C(Y, F)$ be a linear operator satisfying conditions* (C1), (C2-1), (C2-2), (C4-1) *and* (C4-2). *For $p \in [1, \infty]$, let $\| \cdot \|_{D,p}$ be the norm defined by* (2.1) *by p and D, and let $T : A \to A$ be a surjective $\mathbb{F}$-linear $\| \cdot \|_{D,p}$-isometry.*

(1) *Assume $p \in (1, \infty)$, and assume that A and D also satisfy condition* (C3-1), *and further $T : A \to A$ satisfies*

$$T(\operatorname{Ker} D) = T^{-1}(\operatorname{Ker} D) = \operatorname{Ker} D. \tag{2.3}$$

Then T is a generalized weighted composition operator in the sense of (2.2)*: there exist homeomorphisms $\varphi : X \to X$ and $\psi : Y \to Y$, and maps $U : X \to \mathcal{U}(E); x \mapsto U_x$ and $V : Y \to \mathcal{U}(F); y \mapsto V_y$ such that*

$$Tf(x) = U_x(f(\varphi(x))), \quad D(Tf)(y) = V_y(Df(\psi(y))),$$

for each $f \in A$ and for each $(x, y) \in X \times Y$. Moreover, the maps U and V are continuous with respect to the strong operator topology on $\mathcal{U}(E)$ and $\mathcal{U}(F)$ respectively.

(2) *Assume that $p = 1$, $\mathbb{F} = \mathbb{R}$, $\dim E = \dim F$, and X, Y are connected. Also assume that A and D further satisfy conditions* (C2-3) *and* (C3-1). *Then the same conclusion as that of* (1) *holds for every $\mathbb{R}$-linear surjective $\|\cdot\|_{D,1}$-isometry $T : A \to A$. Moreover, equality* (2.3) *always holds for such T.*

(3) *Assume that $p = \infty$. If A and D further satisfy the conditions* (C3-2), *then the same conclusion as that of (1) holds for each linear surjective $\|\cdot\|_{D,\infty}$-isometry $T : A \to A$ satisfying equality* (2.3).

Remark. Conditions (C4-1) and (C4-2) guarantee the continuity of U and V. We made use of this observation in [24].

In Section 3, we examine the extreme point of the dual spaces and apply the result in Section 4 to prove the Main Theorem. Theorems 1.1 and 1.2 are proved in Section 5. In order to clarify the whole structure of the proof, some proofs are postponed to Section 6.

3. Extreme points of dual spaces

This preliminary section studies extreme points of dual spaces of products of Banach spaces. Let E and F be strictly convex Banach spaces, and let $p \in [1, \infty]$. We define a norm $\|\cdot\|_p$ on $E \times F$ by

$$\|(u, v)\|_p = (\|u\|^p + \|v\|^p)^{1/p}, \quad (u, v) \in E \times F, \tag{3.1}$$

where $\|u\|$ and $\|v\|$ denote the norms of u and v in E and F respectively. The product space $E \times F$ equipped with the above norm is denoted by $E \times_p F$. The next two propositions play basic roles in studying extreme points of the dual spaces. For completeness, we give a proof of Proposition 3.1 in Section 6.

Proposition 3.1. *Let $1 \le p \le \infty$.*

(1) *The space $(E \times_p F)^*$ is isometric to $E^* \times_q F^*$, where $\frac{1}{p} + \frac{1}{q} = 1$ with the convention $\frac{1}{\infty} + \frac{1}{1} = 1$.*

(2) *If E^* and F^* are strictly convex, then*

$$\operatorname{ext}(E \times_p F)^* \cong \begin{cases} S(E^* \times_q F^*) & \text{if } 1 < p < \infty, \\ S(E^*) \times S(F^*) & \text{if } p = 1, \\ S(E^*) \times \{0\} \cup \{0\} \times S(F^*) & \text{if } p = \infty. \end{cases}$$

Observe that $(E \times F)^* \cong E^* \times F^*$ as linear spaces, and also recall $S(E^* \times_q F^*) = \{(\xi, \eta) \in E^* \times F^* | \|\xi\|^q + \|\eta\|^q = 1\}$.

For a subspace A of $C(X)$ and a point $x \in X$, let $\delta_x^A : A \to \mathbb{F}$ denote the point-evaluation functional at x defined by

$$\delta_x^A(f) = f(x), \quad f \in A.$$

When no confusion occurs, δ_x^A is simply denoted by δ_x.

Proposition 3.2 ([14, Theorem 2.3.5, Corollary 2.3.6]). *Let X be a compact Hausdorff space and E be a Banach space.*

(1) $\mathrm{ext}(C(X,E)^*) = \{\xi \circ \delta_x^{C(X,E)} | \xi \in \mathrm{ext}(E^*)\}$.

(2) *For each subspace A of $C(X,E)$, we have the inclusion*

$$\mathrm{ext}(A^*) \subset \{\xi \circ \delta_x^A | \xi \in \mathrm{ext}(E^*)\}.$$

For a compact Hausdorff space X and for a Banach space E, let A be a linear subspace of $C(X,E)$. For a linear operator $D : A \to C(Y,F)$, where Y is a compact Hausdorff space and F is a Banach space, and for $p \in [1,\infty]$, consider the norm (2.1) on A. Define a subspace M_A of $C(X \times Y, E \times_p F)$ by

$$M_A = \{f \times Df : X \times Y \to E \times_p F | f \in A\},$$

and define $\Omega : (A, \|\cdot\|_{D,p}) \to (M_A, \|\cdot\|_\infty)$ by

$$\Omega(f) = f \times Df, \quad f \in A.$$

It is easy to see that Ω is a linear isometry, and hence the dual A^* is linearly isometric to M_A^*. When A and D satisfy conditions mentioned in Section 2, the set $\mathrm{ext}(M_A^*)$ is identified as in the next theorem. For $(x,y) \in X \times Y$, let $\delta_{(x,y)}^{E\times F} : M_A \to E \times F$ be defined by

$$\delta_{(x,y)}^{M_A}(f \times Df) = (f(x), Df(y)), \quad f \in A.$$

Most likely, the next theorem still holds without the finite-dimensionality assumption of E and F, while the assumption simplifies the proof of Lemma 3.6 and the result is sufficient for our purpose. In what follows, $\overline{\mathrm{ext} M_A^*}$ denotes the closure of $\mathrm{ext} M_A^*$ with respect to the weak*-topology. The idea of considering the closure of the extreme point-set was first presented in [18].

Theorem 3.3. *Let E and F be finite dimensional Hilbert spaces, and let A be a subspace of $C(X, E)$ with a linear operator $D : A \to C(Y, F)$. For $p \in [1, \infty]$, we have the following.*

(1) *If $1 < p < \infty$ and A satisfies conditions* (C1) *and* (C3-1)*, then we have the following equality:*

$$\overline{\mathrm{ext} M_A^*} = \{(\xi, \eta) \circ \delta_{(x,y)}^{M_A} | (\xi, \eta) \in S(E^* \times_q F^*), (x, y) \in X \times Y\}. \tag{3.2}$$

(2) *If $p = 1$ and A satisfies conditions* (C1) *and* (C3-1)*, then we have the following equality:*

$$\overline{\mathrm{ext} M_A^*} = \{(\xi, \eta) \circ \delta_{(x,y)}^{M_A} | (\xi, \eta) \in S(E^*) \times S(F^*), (x, y) \in X \times Y\}. \tag{3.3}$$

(3) *If $p = \infty$ and A satisfies conditions* (C1) *and* (C3-2)*, then we have the following equality:*

$$\begin{aligned} \overline{\mathrm{ext} M_A^*} = \{&(\xi, 0) \circ \delta_{(x,y)}^{M_A} | \xi \in S(E^*), (x, y) \in X \times Y\} \\ &\cup \{(0, \eta) \circ \delta_{(x,y)}^{M_A} | \eta \in S(F^*), (x, y) \in X \times Y\}. \end{aligned} \tag{3.4}$$

For the proof of the above theorem, we reduce the detection of extreme points above to that of extreme points of the dual of the *scalar-valued* function space and appeal to the next theorem. This reduction follows the idea of Cambern–Botelho and Jamison ([5] and [9]). The following theorem was proved by de Leeuw for real-valued function spaces. For completeness, we supply a proof for complex-valued function spaces in Section 6.

Theorem 3.4 (cf. [12, p. 61])**.** *Let A be a closed linear subspace of the space $C(Z)$ of the complex-valued continuous functions over a compact Hausdorff space Z, and let z be a point of Z. If there exists an $f \in A$ such that*

$$\|f\|_\infty = f(z) = 1, \quad |f(x)| \leq 1 \text{ for each } x \neq z,$$

and further, for a point $x \in Z$, the equality

$$|f(x)| = 1$$

holds if and only if there exists a $c \in \mathbb{T}$ such that

$$g(x) = cg(z) \text{ for each } g \in A.$$

Then $\delta_z^A \in A^*$, defined by $\delta_z^A(f) = f(z)$ $(f \in A)$, is an extreme point of the unit ball of A^*.

In order to apply the above theorem, we recall the isometry $(E \times_p F)^* \cong E^* \times_q F^*$ with $\frac{1}{p} + \frac{1}{q} = 1$: every element of $S((E \times_p F)^*)$ is identified with a pair $(\xi, \eta) \in E^* \times F^*$ such that $\|\xi\|^q + \|\eta\|^q = 1$ (Proposition 3.1). Define a map $\Lambda : (A, \|\cdot\|_{D,p}) \to C(S((E \times_p F)^*) \times (X \times Y), \mathbb{F})$ by

$$\Lambda(f)((\xi, \eta), (x, y)) = \xi(f(x)) + \eta(Df(y)),$$

$$(\xi, \eta) \in E^* \times F^*, \quad \|\xi\|^q + \|\eta\|^q = 1, \quad (x, y) \in X \times Y, \quad f \in A.$$

Notational Convention. By the end of this section, a point $((\xi, \eta), (x, y)) \in S((E \times_p F)^*) \times (X \times Y)$ is written as (ξ_x, η_y) in order to keep in mind that "ξ and η evaluate points of the form $f(x)$ and $Df(y)$ respectively".

The proofs of the next two lemmas are postponed to Section 6. The finite-dimensionality assumption of the spaces E and F are used in the proof of Lemma 3.6.

Lemma 3.5. *For each $f \in A$, we have $\|\Lambda(f)\|_\infty = \|f\|_{D,p}$.*

Lemma 3.6. *Let $N_A = \Lambda(A)$. Then N_A is a closed subspace of*

$$C(S((E \times_p F)^*) \times (X \times Y), \mathbb{F}).$$

Proof of Theorem 3.3. (1) First, we assume that $1 < p < \infty$ and A satisfies condition (C-1) and (C3-1). By the previous two lemmas, $\Lambda : M_A \to N_A$ is an isometric isomorphism onto the closed subspace N_A of $C(S((E \times_p F)^*) \times (X \times Y), \mathbb{F})$.

For a point $(\xi_x, \eta_y) \in (E \times_p F)^* \times (X \times Y) \cong (E^* \times_q F^*) \times (X \times Y)$, let $\Delta_{(\xi_x, \eta_y)} : N_A \to \mathbb{F}$ be the evaluation functional at (ξ_x, η_y) defined by

$$\Delta_{(\xi_x, \eta_y)}(\Lambda(f)) = \Lambda(f)(\xi_x, \eta_y) = \xi(f(x)) + \eta(Df(y)).$$

Noticing the equality

$$\Delta_{(\xi_x, \eta_y)}(\Lambda(f)) = \xi(f(x)) + \eta(Df(y)) = (\xi, \eta) \circ \delta^{M_A}_{(x,y)}(f), \tag{3.5}$$

we see that the adjoint Λ^* satisfies $\Lambda^*(\Delta_{(\xi,\eta)}) = (\xi, \eta) \circ \delta^{M_A}_{(x,y)}$. Hence by Lemma 3.5, $(\xi, \eta) \circ \delta^{M_A}_{(x,y)} \in \mathrm{ext}(M_A^*)$ if and only if $\Delta_{(\xi_x, \eta_y)} \in \mathrm{ext}(N_A^*)$. We show that $\Delta_{(\xi_{x_0}, \eta_{y_0})}$ is an extreme point of $B(N_A^*)$ whenever the point (x_0, y_0) satisfies the condition (Peak). In view of (3.5) and Theorem 3.4, it suffices to prove the following.

($\sharp$1) Let $(x_0, y_0) \in X \times Y$ be a pair of points satisfying the condition (Peak). For each $(\xi^0, \eta^0) \in S((E \times_p F)^*)$, there exists an $f \in A$ such that

(1) $\Delta_{(\xi^0_{x_0},\eta^0_{y_0})}(\Lambda(f)) = \Lambda(f)(\xi^0_{x_0}, \eta^0_{y_0}) = 1$,

(2) for each $(\xi_x, \eta_y) \in S((E \times_p F)^*) \times (X \times Y)$, we have

$$|\Lambda(f)(\xi_x, \eta_y)| \leq 1,$$

(3) $|\Lambda(f)(\xi_x, \eta_y)| = 1$ if and only if there exists $c \in \mathbb{T}$ such that

$$\Lambda(g)(\xi_x, \eta_y) = c\Lambda(g)(\xi^0_{x_0}, \eta^0_{y_0})$$

for each $g \in A$.

Proof of ($\sharp$1). Take $(u_0, v_0) \in E \times F$ so that $\xi^0 = \langle \cdot, u_0 \rangle, \eta^0 = \langle \cdot, v_0 \rangle$. We have $\|u_0\|^q + \|v_0\|^q = 1$. By the condition (Peak), we find an $f \in A$ such that

$$\begin{aligned} f(x_0) = \|u_0\|^{q-2}u_0, Df(y_0) = \|v_0\|^{q-2}v_0, \\ \|f(x)\| < \|u_0\|^{q-1} \text{ for each } x \neq x_0 \\ \|Df(y)\| < \|v_0\|^{q-1} \text{ for each } y \neq y_0. \end{aligned} \tag{3.6}$$

Let $(\xi, \eta) \in S((E \times_p F)^*) \times (X \times Y)$ with $(\xi_x, \eta_y) \neq (\xi^0_{x_0}, \eta^0_{y_0})$. We divide our consideration into two cases.

Case 1. $(x, y) \neq (x_0, y_0)$.
Applying condition (3.6) and the Hölder inequality, we proceed as follows:

$$\begin{aligned} |\Lambda(f)(\xi_x, \eta_y)| &= |\xi(f(x)) + \eta(Df(y))| \leq \|\xi\| \cdot \|f(x)\| + \|\eta\| \cdot \|Df(y)\| \\ &\leq (\|\xi\|^q + \|\eta\|^q)^{1/q} \cdot (\|f(x)\|^p + \|Df(y)\|^p)^{1/p} \\ &< (\|\xi\|^q + \|\eta\|^q)^{1/q} \cdot (\|u_0\|^q + \|v_0\|^q)^{1/p} = 1. \end{aligned}$$

Case 2. $(x, y) = (x_0, y_0)$.
Let $\xi = \langle \cdot, u \rangle, \eta = \langle \cdot, v \rangle$, and notice $\|u\|^q + \|v\|^q = 1$. Then we have

$$\begin{aligned} |\Lambda(f)(\xi_{x_0}, \eta_{y_0})| &= |\langle f(x_0), u \rangle + \langle Df(y_0), v \rangle| \\ &\leq \|u_0^{q-1}\|\|u\| + \|v_0\|^{q-1}\|\|v\| && (3.7) \\ &\leq (\|u_0\|^{(q-1)p} + \|v_0\|^{(q-1)p})^{1/p}(\|u\|^q + \|v\|^q)^{1/q} && (3.8) \\ &= 1, \end{aligned}$$

where the equality $|\Lambda(f)(\xi_{x_0}, \eta_{y_0})| = 1$ holds if and only if the equalities in both (3.7) and (3.8) hold, which is equivalent to the following. There exist a and b such that

$$u = au_0, \quad v = bv_0, \tag{3.9}$$

and also

$$\|u_0\|^{(q-1)p}\|v\|^q = \|v_0\|^{(q-1)p}\|u\|^q \ (\Leftrightarrow \|u_0\|\|v\| = \|v_0\|\|u\|).$$

The latter condition implies $|a| = |b| = 1$, which further implies

$$\begin{aligned}1 &= \Big|\|u_0\|^{q-2}\langle u_0, au_0\rangle + \|v_0\|^{q-2}\langle v_0, bv_0\rangle\Big| \\ &= \Big|\|u_0\|^q\bar{a} + \|v_0\|^q\bar{b}\Big| = \Big|\|u_0\|^q + \|v_0\|^q(\bar{b}/\bar{a})\Big|,\end{aligned}$$

which yields $a = b := \bar{c}$. Then for each $g \in A$, we have

$$\begin{aligned}\Delta_{(\xi_{x_0},\eta_{y_0})}(\Lambda(g)) &= \langle g(x_0), u\rangle + \langle Dg(y_0), v\rangle \\ &= c(\langle g(x_0), u_0\rangle + \langle Dg(y_0), v_0\rangle) = c \cdot \Delta_{(\xi^0_{x_0},\eta^0_{y_0})}(\Lambda(g)).\end{aligned}$$

This proves ($\sharp$1). □

The above ($\sharp$1), Proposition 3.2 (2) and Proposition 3.1 (2) imply the following inclusions:

$$\begin{aligned}&\{(\xi,\eta)\circ\delta^{E\times F}_{(x,y)} | (\xi,\eta) \in S((E\times_p F)^*),\ (x,y) \text{ satisfies (Peak)}\} \\ &\qquad \subset \mathrm{ext}(M_A^*) \subset \{(\xi,\eta)\circ\delta^{E\times F}_{(x,y)} | (\xi,\eta) \in S((E\times_p F)^*),\ (x,y) \in X\times Y\}.\end{aligned}$$

By condition (C3-1), the first set is dense in the last one with respect to the weak*-topology, and hence we have the desired equality.

(2) For $p = 1$, we prove the following:

($\sharp$2) Let $(x_0, y_0) \in X \times Y$ be a pair of points such that (x_0, y_0) satisfies the condition (Peak). Then $(\xi^0, \eta^0) \circ \delta^{E\times F}_{(x_0,y_0)}$ is an extreme point of M_A^*, for each $(\xi^0, \eta^0) \in S(E^*) \times S(F^*)$.

Having proved the above, we obtain, again by Proposition 3.2 (2) and Proposition 3.1 (2), the following inclusions:

$$\begin{aligned}&\{(\xi,\eta)\circ\delta^{E\times F}_{(x,y)} | (\xi,\eta) \in S(E^*)\times S(F^*),\ (x,y) \text{ satisfies (Peak)}\} \\ &\qquad \subset \mathrm{ext}(M_A^*) \subset \{(\xi,\eta)\circ\delta^{E\times F}_{(x,y)} | (\xi,\eta) \in S(E^*)\times S(F^*),\ (x,y) \in X\times Y\}.\end{aligned}$$

By (C3-1), we obtain the desired equality.

Proof of ($\sharp$2). The proof is similar to that of ($\sharp$1). Let $\xi^0 = \langle \cdot, u_0 \rangle, \eta^0 = \langle \cdot, v_0 \rangle$. We have $\|u_0\| = \|v_0\| = 1$. By the condition (Peak), we find an $f \in A$ such that

$$\begin{aligned} f(x_0) = u_0/2, \quad Df(y_0) = v_0/2, \\ \|f(x)\| < \|u_0\|/2 (= 1/2) \text{ for each } x \neq x_0, \\ \|Df(y)\| < \|v_0\|/2 (= 1/2) \text{ for each } y \neq y_0. \end{aligned} \tag{3.10}$$

Let $(\xi_x, \eta_y) \in (S(E^*) \times S(F^*)) \times (X \times Y)$ with $(\xi_x, \eta_y) \neq (\xi^0_{x_0}, \eta^0_{y_0})$.

Case 1. $(x, y) \neq (x_0, y_0)$.
We have

$$\begin{aligned} |\Lambda(f)(\xi_x, \eta_y)| &= |\xi(f(x)) + \eta(Df(y))| \leq \|\xi\|\|f(x)\| + \|\eta\|\|Df(y)\| \\ &= \|f(x)\| + \|Df(y)\| < \|f(x_0)\| + \|Df(y_0)\| = 1. \end{aligned}$$

Case 2. $(x, y) = (x_0, y_0)$.
Let $\xi = \langle \cdot, u \rangle, \eta = \langle \cdot, v \rangle$ with $\|u\| = \|v\| = 1$. Then we have

$$\begin{aligned} |\Lambda(f)(\xi_{x_0}, \eta_{y_0})| &= |\langle f(x_0), u \rangle + \langle Df(y_0), v \rangle| \\ &\leq \|f(x_0)\|\|u\| + \|f(y_0)\|\|v\| = \frac{1}{2}\|u\| + \frac{1}{2}\|v\| = 1. \end{aligned}$$

When the equality $|\Lambda(f)(\xi_{x_0}, \eta_{y_0})| = 1$ holds, we have

$$u = au_0, \quad v = bv_0,$$

and hence $|a| = |b| = 1$. Also the equality

$$1 = |\langle f(x_0), u \rangle + \langle Df(y_0), v \rangle| = |\bar{a} \cdot \|u_0\|^2/2 + \bar{b} \cdot \|v_0\|^2/2| = |\bar{a} + \bar{b}|/2$$

implies $a = b := \bar{c}$. Then as in ($\sharp$1), we see for each $g \in A$, we have

$$\Delta_{(\xi_{x_0}, \eta_{y_0})}(\Lambda(g)) = c\Delta_{(\xi^0_{x_0}, \eta^0_{y_0})}(\Lambda(g)).$$

This proves ($\sharp$2). $\square$

(3) For $p = \infty$, we prove:

($\sharp$3) Let $(x_0, y_0) \in X \times Y$ be a pair of points such that x_0 and y_0 satisfy the conditions (P_X) and (P_Y) respectively. Then $(\xi^0, 0) \circ \delta^{E \times F}_{(x_0, y_0)}$ and $(0, \eta^0) \circ \delta^{E \times F}_{(x_0, y_0)}$ are extreme points of M^*_A, for each $(\xi^0, \eta^0) \in S(E^*) \times S(F^*)$.

Having this, we see, once again from Proposition 3.1 (2) and Proposition 3.2 (2), the inclusions

$$\begin{aligned}&\{(\xi,0)\circ\delta^{M_A}_{(x,y)}|\xi\in S(E^*), x \text{ satisfies } (P_X)\}\\ &\quad\cup\{(0,\eta)\circ\delta^{M_A}_{(x,y)}|\eta\in S(F^*), y \text{ satisfies } (P_Y)\}\subset \mathrm{ext}(M_A^*)\subset\\ &\{(\xi,0)\circ\delta^{M_A}_{(x,y)}|\xi\in S(E^*),(x,y)\in X\times Y\}\\ &\quad\cup\{(0,\eta)\circ\delta^{M_A}_{(x,y)}|\eta\in S(F^*),(x,y)\in X\times Y\}.\end{aligned}$$

By condition (C3-2), the first set is dense in the last one with respect to the weak*-topology, and we have the desired equality.

Proof of ($\sharp$3). We show that $(\xi^0,0)\circ\delta^{E\times F}_{(x_0,y_0)}$ is an extreme point for each $\xi^0\in S(E^*)$. Let $\xi^0=\langle\cdot,u_0\rangle$ with $\|u_0\|=1$. Apply the condition (P_X) to find $f\in A$ such that

$$f(x_0)=u_0, \|f(x)\|<1 \text{ for each } x\neq x_0, \text{ and } \|Df\|_\infty<1. \tag{3.11}$$

Let $(\xi_x,\eta_y)\in(S(E^*\times_1 F^*))\times(X\times Y)=S((E\times_\infty F)^*)\times(X\times Y)$ with $(\xi_x,\eta_y)\neq(\xi^0_{x_0},\eta^0_{y_0})$. Notice that $\|\xi\|+\|\eta\|=1$.

Case 1. $x\neq x_0$.
We have

$$\begin{aligned}|\Lambda(f)(\xi_x,\eta_y)|&=|\xi(f(x))+\eta(Df(y))|\\ &\leq\|\xi\|\|f(x)\|+\|\eta\|\|Df(y)\|<\|\xi\|+\|\eta\|=1.\end{aligned}$$

Case 2. $x=x_0,\eta\neq 0$.
We see

$$\begin{aligned}|\Lambda(f)(\xi_{x_0},\eta_y)|&=|\xi(f(x_0))+\eta(Df(y_0))|\\ &\leq\|\xi\|\|f(x_0)\|+\|\eta\|\|Df(y_0)\|\leq\|\xi\|\|f(x_0)\|+\|\eta\|\|Df\|_\infty\\ &<\|\xi\|\|u_0\|+\|\eta\|=\|\xi\|+\|\eta\|=1.\end{aligned}$$

Case 3. $x=x_0$, $\eta=0$ (and hence $\|\xi\|=1$).
In this case we have

$$|\Lambda(f)(\xi_{x_0},\eta_y)|=|\xi(f(x_0))+\eta(Df(y))|\leq\|\xi\|\|u_0\|=1.$$

Let $\xi=\langle\cdot,u\rangle$ with $\|u\|=1$. If the equality holds in the above, then we have $u=au_0$, and hence $|a|=1$. For each $g\in A$, we have

$$\Delta_{(\xi_{x_0},\eta_y)}(\Lambda(g))=\xi(g(x_0))=a\xi_0(g(x_0)).$$

This proves ($\sharp$3). □

These prove Theorem 3.3 in all cases. □

4. Proof of the Main Theorem

This section is devoted to the proof of the Main Theorem. Sections 4.1 and 4.2 prove preliminary results, depending on the value p: $p \in (1, \infty]$ and $p = 1$. Our proof for the case $p = 1$ is a straightforward modification of [5, Lemma 2.2], in which we assume that the scalar $\mathbb{F}$ is equal to $\mathbb{R}$ and $\dim E = \dim F < \infty$.

Throughout this section, A is an $\mathbb{F}$-subspace of $C(X, E)$, $D : A \to C(Y, F)$ is a linear operator satisfying conditions (C1), (C2-1) and (C2-2), and also $T : A \to A$ is a surjective linear $\|\cdot\|_{D,p}$-isometry. The proof of the Main Theorem is outlined as follows.

Step 1. In Sections 4.1 and 4.2, we prove that

(1.1) if $T^{-1}(\mathrm{Ker}(D)) = T(\mathrm{Ker}D) = \mathrm{Ker}D$, and A and D satisfy (C3-1) for $1 < p < \infty$ and (C3-2) for $p = \infty$, or

(1.2) if A and D satisfy (C3-1) and (C2-3) and if X, Y are connected, for $p = 1$,

then there exist continuous maps $\varphi : X \to X$, $\psi : Y \to Y$, $\alpha : X \times S(E^*) \to S(E^*)$ and $\lambda : Y \times S(F^*) \to S(F^*)$ such that

$$\xi(Tf(x)) = \alpha(x, \xi)(f(\varphi(x))), \quad \eta(D(Tf)(y)) = \lambda(y, \eta)Df(\psi(y)),$$

for each $((x, \xi), (y, \eta)) \in (X \times S(E^*)) \times (Y \times S(F^*))$ and $f \in A$.

Step 2. In Section 4.3, we show, under the additional hypotheses (C4-1) and (C4-2), that T and $D \circ T$ are weighted composition operators of the desired form.

Recall that $M_A = \{f \times Df | f \in A\}$, and let $\tilde{T} : M_A \to M_A$ be the linear operator defined by

$$\tilde{T}(f \times Df) = (Tf) \times D(Tf) \in C(X \times Y, E \times_p F), \quad f \in A.$$

Then $\tilde{T}$ is a surjective linear $\|\cdot\|_\infty$-isometry. Its adjoint operator $\tilde{T}^* : M_A^* \to M_A^*$ is a surjective linear isometry, which preserves the extreme points of M_A^*. Also it is a homeomorphism with respect to the weak*-topology on M_A^*, and hence, restricts to a homeomorphism $\tilde{T}^*|\overline{\mathrm{ext}M_A^*} : \overline{\mathrm{ext}M_A^*} \to \overline{\mathrm{ext}M_A^*}$. The following lemma will be used repeatedly in the sequel.

Lemma 4.1.

(1) *Assume* $\xi_1 \circ \delta_{x_1}^E = \xi_2 \circ \delta_{x_2}^E$ *with* $\xi_i \in E^*, i = 1, 2$. *If* $\xi_1 \neq 0 \neq \xi_2$, *then we have* $(x_1, \xi_1) = (x_2, \xi_2)$.

(2) *Assume* $\eta_1 \circ \delta_{y_1}^F = \eta_2 \circ \delta_{y_2}^F$ *with* $\eta_i \in F^*, i = 1, 2$. *If* $\eta_1 \neq 0 \neq \eta_2$, *then we have* $(y_1, \eta_1) = (y_2, \eta_2)$.

PROOF. (1) For each $f \in A$, we have

$$\xi_1(f(x_1)) = \xi_2(f(x_2)). \tag{4.1}$$

If $x_1 \neq x_2$, then apply condition (C2-1b) to find $f \in A$ such that

$$\xi_1(f(x_1)) \neq 0, \quad f(x_2) = 0.$$

This contradicts (4.1), and hence $x_1 = x_2 := x$. For each $u \in E$, take an $f \in A$ with $f(x) = u$ by condition (C2-1) (either of (C2-1a) or (C2-1b) will do). Then we have by (4.1) that $\xi_1(u) = \xi_2(u)$.

(2) The proof is the same as that of (1), with the only exception that this time we apply condition (C2-2). $\square$

4.1. The case $p > 1$. Throughout this subsection, we assume that the isometry $T : A \to A$ satisfies equality (2.3): $T^{-1}(\operatorname{Ker} D) = T(\operatorname{Ker} D) = \operatorname{Ker} D$. The goal of this section is to prove the following proposition.

Proposition 4.2. *Let $p \in (1, \infty]$, and let $T : A \to A$ be a linear isometry satisfying equality* (2.3)*. Assume that A and D satisfy* (C1), (C2-1), (C2-2), *and further satisfy* (C3-1) *if $p < \infty$, and* (C3-2) *if $p = \infty$. Then there exist continuous maps $\varphi : X \to X$, $\psi : Y \to Y$, $\alpha : X \times S(E^*) \to S(E^*)$ and $\lambda : Y \times S(F^*) \to S(F^*)$ such that, for each $((x, \xi), (y, \eta)) \in (X \times S(E^*)) \times (Y \times S(F^*))$ and for each $f \in A$, we have the following equalities:*

$$\xi(Tf(x)) = \alpha(x, \xi)(f(\varphi(x))) \tag{4.2}$$

$$\eta(D(Tf)(y)) = \lambda(y, \eta)Df(\psi(y)). \tag{4.3}$$

PROOF. The proof is divided into two cases: $p < \infty$ and $p = \infty$.

(1) $p \in (1, \infty)$. First let

$$\mathcal{S}(M_A^*) = \left\{ \zeta \circ \delta_{(x,y)}^{E \times F} | \zeta \in S((E \times_p F)^*), (x, y) \in X \times Y \right\}.$$

By Theorem 3.3 we have $\mathcal{S}(M_A^*) = \overline{\operatorname{ext} M_A^*}$, and

$$\tilde{T}^* | \mathcal{S}(M_A^*) : \mathcal{S}(M_A^*) \to \mathcal{S}(M_A^*) \tag{4.4}$$

is a homeomorphism. For each $(x, \xi) \in X \times S(E^*), (y, \eta) \in Y \times S(F^*)$, notice $\xi \circ \delta_x^E = (\xi, 0) \circ \delta_{(x,y)}^{E \times F}$ and $\eta \circ \delta_y^E = (0, \eta) \circ \delta_{(x,y)}^{E \times F}$. Then (4.4) yields the following two equalities:

$$\tilde{T}^*(\xi \circ \delta_x^E) = \alpha(x, \xi) \circ \delta_{\varphi_\alpha(x,\xi)}^E + \beta(x, \xi) \circ \delta_{\varphi_\beta(x,\xi)}^F \tag{4.5}$$

$$\tilde{T}^*(\eta \circ \delta_y^F) = \kappa(y, \eta) \circ \delta_{\psi_\kappa(y,\eta)}^E + \lambda(y, \eta) \circ \delta_{\psi_\lambda(y,\eta)}^F, \tag{4.6}$$

for some $(\alpha(x,\xi), \beta(x,\xi)), (\kappa(y,\eta), \lambda(y,\eta)) \in S((E\times F)^*)$ and $(\varphi_\alpha(x,\xi), \varphi_\beta(x,\xi))$, $(\psi_\kappa(y,\eta), \psi_\lambda(y,\eta)) \in X \times Y$.

We examine the above equalities in a series of lemmas. Notice that it is not clear at this moment that they are single-valued maps. After the next lemma it turns out that they are.

Lemma 4.3.

(1) $\beta(x,\xi) = 0$, *for each* $(x,\xi) \in X \times S(E^*)$.

(2) $\kappa(y,\eta) = 0$, *for each* $(y,\eta) \in Y \times S(F^*)$.

Proof. Suppose that $\beta(x,\xi) \neq 0$ in (4.5), and observe that

$$\|\alpha(x,\xi)\| = (1 - \|\beta(x,\xi)\|^p)^{1/p} < 1. \tag{4.7}$$

Take an arbitrary $u \in S(E)$, and apply (C2-1a) to find a $g \in \operatorname{Ker} D$ such that $g(x) = u$ and $\|g\|_\infty = 1$, and let $f = T^{-1}g$. Observe $f \in \operatorname{Ker} D$ by (2.3), and

$$\begin{aligned} \|f\|_\infty &= (\|f\|_\infty^p + \|Df\|_\infty^p)^{1/p} = \|f\|_{D,p} \\ &= \|g\|_{D,p} = (\|g\|_\infty^p + \|Dg\|_\infty^p)^{1/p} = \|g\|_\infty = 1. \end{aligned}$$

Then we have

$$\begin{aligned} |\xi(u)| &= |\xi(g(x))| = |\xi(Tf(x))| = |\tilde{T}^*(\xi \circ \delta_x)(f)| \\ &= |\alpha(x,\xi) f(\varphi_\alpha(x,\xi)) + \beta(x,\xi) Df(\varphi_\beta(x,\xi))| \\ &= |\alpha(x,\xi) f(\varphi_\alpha(x,\xi))| \leq \|\alpha(x,\xi)\| \|f\|_\infty \leq \|\alpha(x,\xi)\|. \end{aligned}$$

Hence we have $1 = \|\xi\| \leq \|\alpha(x,\xi)\|$, but this contradicts (4.7) and proves (1).

(2) By (4.6), we have for each $g \in A$,

$$\eta(D(Tg)(y)) = \kappa(y,\eta)(g(\psi_\kappa(y,\eta))) + \lambda(y,\eta)(Dg(\psi_\lambda(y,\eta))). \tag{4.8}$$

For each $u \in E$, take $g \in \operatorname{Ker} D$ such that $g(\psi_\kappa(y,\eta)) = u$ by (C2-1a). We obtain by (2.3) that $D(Tg) = 0$. Hence $0 = \kappa(y,\eta)(u)$ by (4.8), and therefore $\kappa(y,\eta) \equiv 0$. This completes the proof of Lemma 4.3. □

The above lemma reduces (4.5) and (4.6) to

$$\tilde{T}^*(\xi \circ \delta_x^E) = \alpha(x,\xi) \circ \delta^E_{\varphi_\alpha(x,\xi)}, \tag{4.9}$$

$$\tilde{T}^*(\eta \circ \delta_y^F) = \lambda(y,\eta) \circ \delta^F_{\psi_\lambda(y,\eta)}. \tag{4.10}$$

By Lemma 4.1, we see that $\alpha(x,\xi)$, $\varphi_\alpha(x,\xi)$, $\lambda(\eta,y)$, $\psi_\lambda(y,\eta)$ are all single-valued functions. Next, we apply the finite-dimensionality assumption of E and F to show

Lemma 4.4.

(1) *In (4.9), $\varphi_\alpha(x,\xi)$ does not depend on ξ.*

(2) *In (4.10), $\psi_\lambda(y,\eta)$ does not depend on η.*

PROOF. (1) Fix $x \in X$, and we prove that the map $\phi : S(E^*) \to X$ defined by

$$\phi(\xi) = \varphi_\alpha(x,\xi)$$

is a constant map. The proof below is adopted from the proof of BOTELHO and JAMISON [5, Lemma 2.1]. Let $d = \dim E$, and let $(\epsilon_i)_{i=1}^d$ be an orthonormal basis of E^*. Suppose that there exists a point $z \in \phi(S(E^*)) \setminus \{\phi(\epsilon_i) | i = 1, \dots d\}$, and let $z = \varphi_\alpha(x,\xi)$, $z_i = \varphi_\alpha(x,\epsilon_i), i = 1,\dots,d$. Using (C2-1b), we choose an $f \in A$ so that

$$\alpha(x,\xi)(f(z)) \neq 0, \quad f(z_i) = 0,\ i = 1,\dots d. \tag{4.11}$$

We then have from (4.9) that

$$\begin{aligned} \epsilon_i(Tf(x)) &= \tilde{T}^*(\epsilon_i \circ \delta_x^E) = \alpha(x,\epsilon_i) \circ \delta_{z_i}(f) \\ &= \alpha(x,\epsilon_i)(f(z_i)) = 0, \end{aligned}$$

for each $i = 1,\dots,d$, which implies $Tf(x) = 0$. While we see from (4.9) that $0 = \xi(Tf(x)) = \alpha(x,\xi)(f(z)) \neq 0$, a contradiction. This proves that $\phi(S(E^*)) = \{\phi(\epsilon_i) | i = 1,\dots d\}$. Since this holds for every orthonormal basis $\{\epsilon_i | i = 1,\dots d\}$, we see that ϕ is a constant map. This proves (1).

(2) is proved in exactly the same way as (1). We apply (C2-2b) this time. □

By these two lemmas, we may write

$$\tilde{T}^*(\xi \circ \delta_x^E) = \alpha(x,\xi) \circ \delta_{\varphi(x)}^E, \tag{4.12}$$

$$\tilde{T}^*(\eta \circ \delta_y^F) = \lambda(y,\eta) \circ \delta_{\psi(y)}^F, \tag{4.13}$$

for $(x,\xi) \in X \times S(E^*)$ and $(y,\eta) \in Y \times S(F^*)$. Then the continuity of the maps $\alpha : X \times S(E^*) \to S(E^*)$, $\varphi : X \to X$, $\lambda : Y \times S(F^*) \to S(F^*)$, and $\psi : Y \to Y$ follows from the continuity of $\tilde{T}^*$. The above (4.12) and (4.13) are equivalent to (4.2) and (4.3) respectively.

This proves Proposition 4.2 for the case $1 < p < \infty$.

(2) Next, we assume $p = \infty$. In view of Theorem 3.3 (3), we define

$$\mathcal{S}(M_A^*) = \left\{\xi \circ \delta_x^E | \xi \in S(E^*), x \in X\right\} \cup \left\{\eta \circ \delta_y^F | \eta \in S(F^*), y \in Y\right\}.$$

Again, $\tilde{T}^*$ preserves $\mathcal{S}(M_A^*)$. Hence for each $x \in X$ and for each $\xi \in S(E_x^*)$, we have either

$$\tilde{T}^*(\xi \circ \delta_x^E) = \alpha \circ \delta_z^E, \tag{4.14}$$

for some $\alpha \in S(E^*)$, $z \in X$, or

$$\tilde{T}^*(\xi \circ \delta_x^E) = \lambda \circ \delta_w^F, \tag{4.15}$$

for some $\lambda \in S(F^*), w \in Y$.

We show below that (4.15) does not occur. For this purpose, fix $x \in X$ and define

$$S_1 = \left\{\xi \in S(E^*) | S^*(\xi \circ \delta_x^E) = \alpha \circ \delta_z^E \text{ for some } \alpha, z\right\},$$
$$S_2 = \left\{\xi \in S(E^*) | S^*(\xi \circ \delta_x^E) = \lambda \circ \delta_w^F \text{ for some } \lambda, w\right\}.$$

Lemma 4.5. *S_1 and S_2 are closed subsets of $S(E^*)$.*

Proof. Suppose that (ξ_ν) is a convergent net in S_1 with $\xi = \lim_\nu \xi_\nu$, and set $\tilde{T}^*(\xi_\nu \circ \delta_x) = \alpha_\nu \circ \delta_{z_\nu}^E$. By the compactness of X and $S(E^*)$, we may assume that (α_ν) and z_ν are convergent, and let $\lim_\nu \alpha_\nu = \alpha, \lim z_\nu = z$. By the continuity of $\tilde{T}^*$, we have $\tilde{T}^*(\xi \circ \delta_x^E) = \alpha \circ \delta_z^E$, and hence $\xi \in S_1$. Hence S_1 is closed. The same proof works to prove that S_2 is closed. □

Now, we have $S_1 = \emptyset$ or $S_2 = \emptyset$. This follows from the connectedness of $S(E^*)$ and the above lemma if $\dim_{\mathbb{R}} E > 1$. If $\dim_{\mathbb{R}} E = 1$, this is a direct consequence of the equality $S(E^*) = \{\pm 1\}$ and the real-linearity of $\tilde{T}^*$.

If $S_1 = \emptyset$, then we have, for each $\xi \in S(E^*)$,

$$\tilde{T}^*(\xi \circ \delta_x^E) = \lambda \circ \delta_w^F,$$

for some $w \in Y$ and $\lambda \in S(F^*)$. For each $g \in \operatorname{Ker} D$, we see from (2.3) that $f = T^{-1}g \in \operatorname{Ker} D$. Then for each $\xi \in S(E^*)$, we have

$$\xi(g(x)) = \xi(Tf(x)) = \lambda(Df(w)) = 0,$$

which implies $g(x) = 0$. This violates condition (C2-1a). Thus we have proved $S_2 = \emptyset$.

Therefore, we always have (4.14) for each $x \in X$ and $\xi \in S(E^*)$. Lemma 4.1 implies that $z \in X$ and $\alpha \in S(E^*)$ are uniquely determined by x and ξ, and we may write $\alpha = \alpha(x, \xi)$ and $z = \varphi(x)$ by single-valued maps $\alpha : X \times S(E^*) \to S(E^*)$ and $\varphi : X \to X$.

Similarly, we can prove, for each $y \in Y$ and for each $\eta \in S(F^*)$, there exist $w \in Y, \lambda \in S(F^*)$ such that

$$\tilde{T}^*(\eta \circ \delta^F_y) = \lambda \circ \delta^F_w,$$

and $w \in Y$ and λ are uniquely determined by y, η. Thus we obtain single-valued maps $\lambda : Y \times S(F^*) \to S(F^*)$ and $\psi : Y \to Y$ such that

$$\tilde{T}^*(\eta \circ \delta^F_y) = \lambda(y, \eta) \circ \delta^F_{\psi(y)}.$$

Then the continuity of α, φ, λ and ψ is a direct consequence of that of $\tilde{T}^*$.

This completes the proof of Proposition 4.2. □

4.2. The case $p = 1$. In this subsection, we assume that $p = 1, \mathbb{F} = \mathbb{R}$, X and Y are connected, and follow the proof of [5, Lemma 2.2] to prove the next proposition which corresponds to Proposition 4.2 in the present context. Here we do *not* assume that the $\|\cdot\|_{D,1}$-isometry $T : A \to A$ satisfies (2.3).

Proposition 4.6. *Let $p = 1$ and $\mathbb{F} = \mathbb{R}$, and also let $\dim E = \dim F := d < \infty$. Assume that X, Y are connected, A and D satisfy* (C1), (C2-1), (C2-2) *and further satisfy* (C2-3) *and* (C3-1). *Then we have the following:*

(1) *There exist continuous maps $\varphi : X \to X$, $\psi : Y \to Y$, $\alpha : X \times S(E^*) \to S(E^*)$ and $\lambda : Y \times S(F^*) \to S(F^*)$ such that for each $((x, \xi), (y, \eta)) \in (X \times S(E^*)) \times (Y \times S(F^*))$ and for each $f \in A$, we have*

$$\xi(Tf(x)) = \alpha(x, \xi)(f(\varphi(x))) \tag{4.16}$$

$$\eta(D(Tf)(y)) = \lambda(y, \eta)(Df(\psi(y))). \tag{4.17}$$

(2) *The isometry T satisfies equality (2.3).*

Proof. The proposition is proved in a series of lemmas. In view of Theorem 3.3 (2), we define

$$\mathcal{S}(\mathcal{M}^*_{\mathcal{A}}) = S(E^*) \times S(F^*).$$

The set is again preserved by $\tilde{T}^*$, and we may write for each $(x, y) \in X \times Y$ and for each $(\xi, \eta) \in S(E^*) \times S(F^*)$

$$\tilde{T}^*\left((\xi, \eta) \circ \delta^{E\times F}_{(x,y)}\right) = (\alpha(x, y, \xi, \eta), \lambda(x, y, \xi, \eta)) \circ \delta_{(\varphi(x,y,\xi,\eta),\psi(x,y,\xi,\eta))}. \tag{4.18}$$

Lemma 4.7. *The above $\alpha(x, y, \xi, \eta), \lambda(x, y, \xi, \eta)$, $\varphi(x, y, \xi, \eta)$ and $\psi(x, y, \xi, \eta)$ are all uniquely determined by x, y, ξ and η, and they all define continuous maps.*

PROOF. Equation (4.18) is rephrased as

$$\begin{aligned}\xi(Tf(x)) &+ \eta(D(Tf)(y)) \\ &= \alpha(x,y,\xi,\eta)f(\varphi(x,y,\xi,\eta)) + \lambda(x,y,\xi,\eta)Df(\psi(x,y,\xi,\eta)),\end{aligned} \quad (4.19)$$

for each $f \in A$. In order to prove that α, φ, λ and ψ are uniquely determined, it suffices to prove that the equality

$$\alpha_1(f(x_1)) + \lambda_1(Df(y_1)) = \alpha_2(f(x_2)) + \lambda_2(Df(y_2)), \quad f \in A, \quad (4.20)$$

implies $(\alpha_1, x_1, \lambda_1, y_1) = (\alpha_2, x_2, \lambda_2, y_2)$. Suppose that $x_1 \neq x_2$. Applying (C2-3), we find an $f \in A$ such that

$$\alpha_1(f(x_1)) \neq 0, \quad f(x_2) = 0, \quad Df(y_1) = Df(y_2) = 0,$$

which contradicts (4.20). Thus $x_1 = x_2 := x$. Next, for each $u \in E$, apply (C2-1a) to find a $g \in \operatorname{Ker} D$ such that $g(x) = u$. Equality (4.20) then yields $\alpha_1(u) = \alpha_2(u)$. Hence $\alpha_1 = \alpha_2$. Then equality (4.20) reduces to

$$\lambda_1(Df(y_1)) = \lambda_2(Df(y_2)),$$

for each $f \in A$. As in Lemma 4.1 (2), we apply (C2-2b) and (C2-2a) to conclude $(y_1, \eta_1) = (y_2, \eta_2)$.

The continuity of these four maps $\alpha(x,y,\xi,\eta)$, $\lambda(x,y,\xi,\eta))$, $\varphi(x,y,\xi,\eta)$ and $\psi(x,y,\xi,\eta)$ is a direct consequence of that of $\tilde{T}^*$ and the above. This completes the proof. $\square$

Lemma 4.8. *The points $\varphi(x,y,\xi,\eta)$ and $\psi(x,y,\xi,\eta)$ do not depend on ξ, η.*

PROOF. As in Lemma 4.4, we adopt the proof of [5, Lemma 2.1]. Fix a point $(x,y) \in X \times Y$, and define $\bar{\varphi}_{x,y} : S(E^*) \times S(F^*) \to X$ by

$$\bar{\varphi}_{x,y}(\xi,\eta) = \varphi(x,y,\xi,\eta), \quad (\xi,\eta) \in S(E^*) \times S(F^*).$$

We prove in the sequel that $\bar{\varphi}_{x,y}$ is a constant map. Let $\{\epsilon_1, \ldots \epsilon_d\}$ and $\{\omega_1, \ldots \omega_d\}$ be orthonormal bases of E^* and F^*. Suppose that there exists a point $z \in \bar{\varphi}_{x,y}(S(E^*) \times S(F^*)) \setminus \{\bar{\varphi}_{x,y}(\epsilon_i, \omega_j), \bar{\varphi}_{x,y}(\epsilon_i, -\omega_j) | i,j = 1, \ldots, d\}$, and let $z = \bar{\varphi}_{x,y}(\xi,\eta)$ for some $(\xi,\eta) \in S(E^*) \times S(F^*)$. Also let

$$z_{ij} = \bar{\varphi}_{x,y}(\epsilon_i, \omega_j), \ w_{ij} = \bar{\varphi}_{x,y}(\epsilon_i, -\omega_j).$$

We apply (C2-3) to find an $f \in A$ such that

$$\alpha(x, y, \xi, \eta)(f(z)) \neq 0, \tag{4.21}$$

$$Df(\psi(x, y, \xi, \eta)) = 0, \tag{4.22}$$

$$f(z_{ij}) = f(w_{ij}) = 0, \tag{4.23}$$

$$Df(\psi(x, y, \epsilon_i, \omega_j)) = Df(\psi(x, y, \epsilon_i, -\omega_j)) = 0, \tag{4.24}$$

for each $i, j = 1, \ldots, d$. By (4.19), (4.23) and (4.24), we have

$$\epsilon_i(Tf(x)) + \omega_j(D(Tf)(y)) = \epsilon_i(Tf(x)) - \omega_j(D(Tf)(y)) = 0,$$

and hence $\epsilon_i(Tf(x)) = \omega_j(D(Tf)(y)) = 0$ for each $i, j = 1, \ldots d$. Thus we have $Tf(x) = D(Tf)(y) = 0$. Using (4.19) and (4.22) again, we have the equality $\alpha(x, y, \xi, \eta)(f(z)) = 0$, which contradicts (4.21).

Thus we have $\bar{\varphi}_{x,y}(S(E^*) \times S(F^*)) = \{\bar{\varphi}_{x,y}(\epsilon_i, \omega_j), \bar{\varphi}_{x,y}(\epsilon_i, -\omega_j) | i, j = 1, \ldots, d\}$. Since (ϵ_i) and (ω_j) are arbitrary orthonormal bases, we see that $\bar{\varphi}_{x,y}$ is a constant map, as desired. This proves the lemma. □

The next lemma depends on the assumption that $\dim E = \dim F = d < \infty$ and the connectedness of X and Y.

Lemma 4.9. *In* (4.19), $\alpha(x, y, \xi, \eta)$ *does not depend on* η *and* $\lambda(x, y, \xi, \eta)$ *does not depend on* ξ.

Sketch of Proof. The proof is a straightforward modification of those of [5, Lemmas 2.2 and 3.2] and is sketched below. First, we show that, for each $(x, y) \in X \times Y$ and for each $\xi \in S(E^*)$, the map

$$\eta \mapsto \alpha(x, y, \xi, \eta), \quad S(F^*) \to S(E^*)$$

is a constant map.

For this, fix a point $(x, y) \in X \times Y$ and $\xi \in S(E^*)$, let $L_\xi^{x,y}(\eta) = \alpha(x, y, \xi, \eta) + \alpha(x, y, \xi, -\eta) \in E^*$, and let $L_\xi^{x,y}(\eta) = \langle \cdot, \ell_\xi^{x,y}(\eta) \rangle$ for a vector $\ell_\xi^{x,y}(\eta) \in E$. For each $f \in \mathrm{Ker}(D)$, we have from (4.19)

$$2\xi(Tf(x)) = L_\xi^{x,y}(\eta)(f(\varphi(x, y))).$$

By (C2-1a), each $u \in E$ admits a function $f_u \in \mathrm{Ker}(D)$ such that $f_u(\varphi(x, y)) = u$. We see from this that $L_\xi^{x,y}(\eta)(u) = 2\xi(Tf_u(x))$, and hence $L_\xi^{x,y}(\eta)$ does not depend on η. Hence as in [5, p. 43], we have

$$\ell_\xi^{x,y}(\eta) + \ell_\xi^{x,y}(-\eta) \equiv c(x, y, \xi), \tag{4.25}$$

a constant vector with respect to η. As in [5, Lemma 2.2], we can conclude from (4.25) that either of the following holds:

(i) for each $\eta \in S(F^*)$ we have $\ell_\xi^{x,y}(\eta) = \ell_\xi^{x,y}(-\eta)$;

(ii) for each $\eta \in S(F^*)$ we have $\ell_\xi^{x,y}(\eta) = -\ell_\xi^{x,y}(-\eta)$.

We define a map $\theta : X \times Y \times S(F^*) \to C(S(F^*), S(E^*))$ by

$$\theta(x, y, \xi) : \eta \mapsto \ell_\xi^{x,y}(\eta), \quad \eta \in S(F^*).$$

The above shows that either

(iii) $\theta(x, y, \xi)$ is a constant map, or

(iv) $\theta(x, y, \xi)$ is an antipodal map in the sense that $\theta(x, y, \xi)(-\eta) = -\theta(x, y, \xi)(\eta)$ for each $\eta \in S(F^*)$.

Observe that $S(E^*)$ and $S(F^*)$ are both homeomorphic to the $(d-1)$-dimensional sphere: The map $\theta(x, y, \xi)$ of the type (iv) above is not homotopic to a constant map; this follows directly for $d = 1$, and follows from the proof of Borsuk–Ulam theorem for $d \geq 2$ ([8, Chapter IV, Theorem 20.1]). Hence such a map belongs to a different component than that of the type-(iii)-constant-maps in the space $C(S(F^*), S(E^*))$. By the connectedness of $X \times Y \times S(F^*)$ if $d \geq 2$, and by a direct argument for $d = 1$, we see that either of the following holds:

(a) $\theta(x, y, \xi) \equiv$ constant for each $(x, y, \xi) \in X \times Y \times S(E^*)$, or

(b) $\theta(x, y, \xi)$ is an antipodal map for each $(x, y, \xi) \in S(E^*)$.

If (b) holds, then we have $\alpha(x, y, \xi, -\eta) = -\alpha(x, y, \xi, \eta)$ for each $(x, y, \xi, \eta) \in X \times Y \times S(E^*) \times S(F^*)$. Using this to the equality (4.19), we see, for each $f \in \mathrm{Ker}(D)$

$$\begin{aligned}\xi(Tf(x)) + \eta(D(Tf)(y)) &= \alpha(x, y, \xi, \eta) f(\varphi(x, y)),\\ \xi(Tf(x)) - \eta(D(Tf)(y)) &= -\alpha(x, y, \xi, \eta) f(\varphi(x, y)).\end{aligned}$$

These two imply $\xi(Tf(x)) = 0$ for each $x \in X$ and $\xi \in S(E^*)$. Thus $Tf \equiv 0$, and hence $f \equiv 0$ for each $f \in \mathrm{Ker}\, D$, a contradiction. Therefore (a) holds, which implies (i) above, and thus $L_\xi^{x,y}$ is a constant map. This proves that the map $\alpha(x, y, \xi, \eta)$ does not depend on $\eta \in S(F^*)$, We may now write in (4.19) as

$$\alpha(x, y, \xi) := \alpha(x, y, \xi, \eta). \tag{4.26}$$

In order to prove that $\lambda(x, y, \xi, \eta)$ does not depend on ξ, we exchange the roles of E^* and F^*, ξ and η respectively, and proceed in exactly the same way as above. In the final step, we derive a contradiction by assuming that

$$\lambda(x, y, -\xi, \eta) = -\lambda(x, y, \xi, \eta),$$

for each $(x, y, \xi, \eta) \in X \times Y \times S(E^*) \times S(F^*)$. Indeed, under the notation (4.26) we have from (4.19) the following:

$$\begin{aligned} \xi(Tf(x)) + \eta(D(Tf)(y)) &= \alpha(x, y, \xi)f(\varphi(x, y)) + \lambda(x, y, \xi, \eta)Df(\psi(x, y)), \\ -\xi(Tf(x)) + \eta(D(Tf)(y)) &= \alpha(x, y, -\xi)f(\varphi(x, y)) - \lambda(x, y, \xi, \eta)Df(\psi(x, y)), \end{aligned}$$

from which we have

$$2\eta(D(Tf)(y)) = (\alpha(x, y, \xi) + \alpha(x, y, -\xi))\,(f(\varphi(x, y))).$$

The right hand side of the above does not depend on η, and thus $\eta(D(Tf)(y)) = 0$ for each $\eta \in S(F^*)$. This shows $D(Tf) \equiv 0$ for each $f \in A$, and hence $D = 0$, a contradiction. As in the proof for $\alpha(x, y, \xi, \eta)$, this yields the desired conclusion on $\lambda(x, y, \xi, \eta)$. These prove Lemma 4.9. □

The above lemma enables us to rewrite (4.19) as

$$\xi(Tf(x)) + \eta(D(Tf)(y)) = \alpha(x, y, \xi)f(\varphi(x, y)) + \lambda(x, y, \eta)Df(\psi(x, y)), \quad (4.27)$$

for each $f \in A$, where $\lambda(x, y, \eta) := \lambda(x, y, \xi, \eta)$ for an arbitrary $\xi \in S(E^*)$.

Lemma 4.10.

(1) $T(\mathrm{Ker}(D)) \subset \mathrm{Ker}(D)$.

(2) *For each* $(x, y, \xi, \eta) \in X \times Y \times S(E^*) \times S(F^*)$, *we have*

$$\alpha(x, y, -\xi) = -\alpha(x, y, \xi), \quad \lambda(x, y, -\eta) = -\lambda(x, y, \eta).$$

(3) *For each* $((x, \xi), (y, \eta)) \in (X \times S(E^*)) \times (Y \times S(F^*))$, *we have*

$$\xi(Tf(x)) = \alpha(x, y, \xi)(f(\varphi(x, y))), \quad (4.28)$$
$$\eta(D(Tf)(y)) = \lambda(x, y, \eta)(Df(\psi(x, y))), \quad (4.29)$$

for each $f \in A$.

Proof. (1): By (4.27), we have for each $f \in \mathrm{Ker}(D)$,

$$2\eta(D(Tf)(y)) = (\alpha(x, y, \xi) + \alpha(x, y, -\xi))\,f(\varphi(x, y)).$$

The right hand side of the above does not depend on η, from which we conclude $\eta(D(Tf)(y)) = 0$ for each $\eta \in S(F^*)$. Thus we have $D(Tf) \equiv 0$.

(2) and (3): By (1), we have, for each $f \in \mathrm{Ker}(D)$, the equality

$$\xi(Tf(x)) = \alpha(x, y, \xi)(f(\varphi(x, y))), \quad (x, y) \in X \times Y,\ \xi \in S(E^*).$$

Fix a point $(x, y) \in X \times Y$. By condition (C2-1a), each $u \in E$ admits an $f_u \in \mathrm{Ker}(D)$ such that $u = f_u(\varphi(x, y))$. It follows from the above that

$$-\alpha(x, y, \xi)(u) = -\xi(Tf_u(x)) = \alpha(x, y, -\xi)(u),$$

and thus $\alpha(x, y, -\xi) = -\alpha(x, y, \xi)$. Applying this to (4.27), we have

$$-\xi(Tf(x)) + \eta(D(Tf)(y)) = -\alpha(x, y, \xi)f(\varphi(x, y)) + \lambda(x, y, \eta)Df(\psi(x, y)),$$

for each $f \in A$. Combining this with (4.27), we have

$$\xi(Tf(x)) = \alpha(x, y, \xi)f(\varphi(x, y)),$$

and then

$$\eta(D(Tf)(y)) = \lambda(x, y, \eta)Df(\psi(x, y)),$$

for each $(x, y, \xi, \eta) \in X \times Y \times S(E^*) \times S(F^*)$. The equality $\lambda(x, y, -\eta) = -\lambda(x, y, \eta)$ follows directly from the above. $\square$

In order to complete the proof of (1) of Proposition 4.6, we show that $\varphi(x, y)$ and $\psi(x, y)$ do not depend on y and x respectively.

Lemma 4.11. *The map $\varphi(x, y)$ does not depend on y, and the map $\psi(x, y)$ does not depend on x. Hence they define maps $\varphi : X \to X$ and $\psi : Y \to Y$ by $\varphi(x) := \varphi(x, y)$ and $\psi(y) := \psi(x, y)$.*

Proof. We show that $\varphi(x, y_1) = \varphi(x, y_2)$ for each $y_1, y_2 \in Y$. If it is not the case, apply (C2-1b) to find $f \in A$ such that $\alpha(x, y, \xi)(f(\varphi(x, y_1)) \neq 0$ and $f(\varphi(x, y_2)) = 0$. Then we have

$$0 \neq \alpha(x, y_1, \xi)f(\varphi(x, y_1) = \xi(Tf(x)) = \alpha(x, y_2, \xi)f(\varphi(x, y_2)) = 0,$$

a contradiction which implies the desired conclusion. With the help of (C2-2b), the second assertion can be proved in the same way. $\square$

The equations

$$\xi(Tf(x)) = \alpha(x, y, \xi)f(\varphi(x)) \quad \text{and} \quad \eta(D(Tf)(y)) = \lambda(x, y, \eta)Df(\psi(y))$$

easily imply that $\alpha(x, y, \xi)$ and $\lambda(x, y, \eta)$ do not depend on y and x respectively. Now Lemmas 4.7–4.11 prove (1) of Proposition 4.6. Applying the same argument to T^{-1}, we see that $T^{-1}(\mathrm{Ker}(D)) \subset \mathrm{Ker}(D)$, which implies equality (2.3). This completes the proof of Proposition 4.6. $\square$

4.3. Proof of the Main Theorem. Let $T : A \to A$ be a surjective linear $\|\cdot\|_{D,p}$-isometry, where A and D satisfy conditions (C1), (C2-1), (C2-2), (C4-1) and (C4-2). Assume further that

(i) $p \in (1, \infty)$, A and D satisfy condition (C3-1), T satisfies (2.3), or

(ii) $p = \infty$, A and D satisfy condition (C3-2), or

(iii) $p = 1$, $\mathbb{F} = \mathbb{R}$, $\dim E = \dim F$, X and Y are connected, and further A and D satisfy conditions (C2-3) and (C3-2).

In all cases, we have obtained, in Section 4.1 and 4.2, continuous maps $\varphi : X \to X$, $\psi : Y \to Y$, $\alpha : X \times S(E^*) \to S(E^*)$ and $\lambda : Y \times S(F^*) \to S(F^*)$ such that

$$\xi(Tf(x)) = \alpha(x, \xi)(f(\varphi(x)))), \tag{4.30}$$

$$\eta(D(Tf)(y)) = \lambda(y, \eta)(Df(\psi(y))), \tag{4.31}$$

for each $f \in A$. Equality (4.30) implies

$$f(\varphi(x)) = 0 \Rightarrow Tf(x) = 0, \tag{4.32}$$

for each $x \in X$. In fact $f(\varphi(x)) = 0$ implies $\xi(Tf(x)) = \alpha(\xi)(f(\varphi(x))) = 0$, for each $\xi \in S(E^*)$. Hence $Tf(x) = 0$, as desired.

For each $u \in E$, choose an $f \in A$ such that $f(\varphi(x)) = u$. Let

$$U_x(u) := Tf(x). \tag{4.33}$$

The above (4.32) guarantees that $U_x : E \to E$ is a well-defined linear map, for which we have

$$U_x(f(\varphi(x))) = Tf(x), \tag{4.34}$$

for each $f \in A$. Next we prove

$$\|U_x(u)\| \leq \|u\|, \tag{4.35}$$

for each $x \in X$ and for each $u \in E$. For the proof, we apply (C2-1a) to choose an $f \in \operatorname{Ker} D$ such that $f(\varphi(x)) = u$ and $\|f\|_\infty = \|u\|$. Notice that $D(Tf) = 0$. Then we have the desired inequality as follows:

$$\|U_x(u)\| = \|Tf(x)\| \leq \|Tf\|_{D,p} = \|f\|_{D,p} = (\|f\|_\infty^p + \|Df\|_\infty^p)^{1/p} = \|f\|_\infty = \|u\|.$$

This proves (4.35). The continuity of the map $U : X \to \mathcal{U}(E); x \mapsto U_x$ is proved in the next lemma, where $\mathcal{U}(E)$ is the group of all linear isometries on E with the strong operator topology.

Lemma 4.12. *The map $U : X \to \mathcal{U}(E)$ defined by $U(x) = U_x,\ x \in X$, is continuous.*

Proof. As before, let $c_u : X \to E$ be the constant function on X taking the value $u \in E$. Fix a vector $u \in E$, and let (x_μ) be a net in X such that $\lim_\mu x_\mu = x_0$. We show

$$\lim_\mu \|U_{x_\mu}(u) - U_{x_0}(u)\| = 0. \tag{4.36}$$

Let $f_0 \in A$ such that $f_0(\varphi(x_0)) = u$. For an arbitrary μ, let f_μ be the function defined by

$$f_\mu = f_0 + c_{u - f_0(\varphi(x_\mu))}.$$

By (C4-1), we have $c_{u-f_0(\varphi(x_\mu))} \in A$, and hence we see $f_\mu \in A$. It follows directly from the definition that $f_\mu(\varphi(x_\mu)) = u$. By the definition (4.33), we have

$$U_0(u) = Tf_0(x_0), \quad U_{x_\mu}(u) = Tf_\mu(x_\mu).$$

We estimate $\|f_\mu - f_0\|_{D,p}$ as follows:

$$\begin{aligned} \|f_\mu - f_0\|_\infty &= \|c_{u-f_0(\varphi(x_\mu))}\|_\infty = \|u - f_0(\varphi(x_\mu))\|, \text{ and} \\ \|Df_\mu - Df_0\|_\infty &= \|Dc_{u-f_0(\varphi(x_\mu))}\|_\infty \le K\|u - f_0(\varphi(x_\mu))\| \quad \text{(by (C4-1))}. \end{aligned}$$

Hence

$$\|f_\mu - f_0\|_{D,p} \le (1 + K^p)^{1/p}\|u - f_0(\varphi(x_\mu))\|. \tag{4.37}$$

Thus we have the following inequality:

$$\begin{aligned} \|Tf_\mu(x_\mu) - Tf_0(x_\mu)\| &\le \|Tf_\mu - Tf_0\|_\infty \le \|Tf_\mu - Tf_0\|_{D,p} \\ &= \|f_\mu - f_0\|_{D,p} \le (1 + K^p)^{1/p}\|u - f_0(\varphi(x_\mu))\|. \end{aligned}$$

Then we have

$$\begin{aligned} \|Tf_\mu(x_\mu) - Tf_0(x_0)\| &\le \|Tf_\mu(x_\mu) - Tf_0(x_\mu)\| + \|Tf_0(x_\mu) - Tf_0(x_0)\| \\ &\le (1 + K^p)^{1/p}\|u - f_0(\varphi(x_\mu))\| + \|Tf_0(x_\mu) - Tf_0(x_0)\|, \end{aligned}$$

from which we conclude $\lim_\mu \|Tf_\mu(x_\mu) - Tf_0(x_0)\| = 0$. By (4.33), this is equivalent to (4.36). This proves lemma 4.13. □

Similarly, we use (4.31) to find, for each $y \in Y$, a linear operator $V_y : F \to F$ such that

$$V_y(Df(\psi(y))) = D(Tf)(y),$$

for each $y \in Y$. We show that

$$\|V_y(v)\| \le \|v\| \tag{4.38}$$

for each $y \in Y$ and for each $v \in F$.

For this, we take an arbitrarily $\epsilon > 0$, and choose $g \in A$ such that $Dg(y) = v, \|Dg\|_\infty = \|v\|$, and $\|g\|_\infty < \epsilon$ by means of (C2-2a). This time we proceed as follows:

$$\begin{aligned}\|V_y(v)\| &= \|D(Tg)(y)\| \le \|Tg\|_p = \|g\|_p \\ &= (\|g\|_\infty^p + \|Dg\|_\infty^p)^{1/p} < (\epsilon^p + \|Dg\|_\infty^p)^{1/p} = (\epsilon^p + \|v\|^p)^{1/p}.\end{aligned}$$

Since ϵ is an arbitrary positive number, we obtain the desired inequality.

As for the continuity of $y \mapsto V_y$, we have

Lemma 4.13. *The map $V : Y \to \mathcal{U}(F)$; $y \mapsto V_y$ is continuous with respect to the strong operator topology on $\mathcal{U}(F)$.*

Proof. The proof is identical to that of the previous lemma. For a net (y_ν) in Y with $\lim_\nu y_\nu = y_0$ and $v \in F$, we take a $g_0 \in A$ such that $Dg_0(\psi(y_0)) = v$ by applying (C2-2a). We apply (C4-2) to find an $h_\nu \in A$ such that

$$Dh_\nu = c_{v - Dg_0(\psi(y_\nu))}, \quad \|h_\nu\|_\infty \le L\|v - Dg_0(\psi(y_0))\|.$$

Let $g_\nu = g_0 + h_\nu$, then we can show $\lim_\nu \|D(Tg_\nu)(y_\nu) - D(Tg_0)(y_0)\| = 0$. □

Repeating the same argument to the inverse operator T^{-1}, we obtain continuous maps $\hat{\varphi} : X \to X$, $\hat{\psi} : Y \to Y$, $\hat{U} : E \to E$ and $\hat{V} : F \to F$ such that

$$(T^{-1}h)(x) = \hat{U}_x(h(\hat{\varphi}(x))), \quad D(T^{-1}h)(y) = \hat{V}_y(Dh(\hat{\psi}(y))),$$

for each $h \in A$ and for each $(x, y) \in X \times Y$. Also we have

$$\|\hat{U}_x(u)\| \le \|u\|, \tag{4.39}$$

$$\|\hat{V}_y(v)\| \le \|v\|, \tag{4.40}$$

for each $(u, v) \in E \times F$. We can straightforwardly verify that

$$\hat{\varphi} = \varphi^{-1}, \quad \hat{\psi} = \psi^{-1},$$

and

$$\hat{U}_{\varphi(x)} = (U_x)^{-1}, \quad \hat{V}_{\psi(y)} = (V_y)^{-1},$$

for each $x \in X$ and for each $y \in Y$. Then inequalities (4.35), (4.38), (4.39) and (4.40) imply U_x and V_y are isometries.

This completes the proof of the Main Theorem. $\square$

5. Applications of the Main Theorem

5.1. Proof of Theorem 1.1. Let $p \in [1, \infty]$, and let $\| \cdot \|_p$ be the norm on $C^1([0,1])$ defined by (1.1). Let $T : C^1([0,1]) \to C^1([0,1])$ be a surjective real-linear $\| \cdot \|_p$-isometry. Assume, if $p > 1$, that T satisfies condition (1.2).

First, we identify $\mathbb{C}$ with the 2-dimensional real Euclidean space $\mathbb{R}^2$ (as real vector spaces) and $C([0,1], \mathbb{C})$ with $C([0,1], \mathbb{R}^2)$. Let $X = Y = [0,1]$, $E = F = \mathbb{R}^2$, $A = C^1([0,1], \mathbb{R}^2) \subset C([0,1], \mathbb{R}^2)$, and let $Df = f' : C^1([0,1], \mathbb{R}^2) \to C([0,1], \mathbb{R}^2)$, the derivative of f. First, we verify conditions (C1)–(C4): The verification of conditions (C1), (C2-1), (C2-2), (C2-3), (C4-1) and (C4-2) are straightforward. For (C3-1) observe that

> for each pair of distinct points (s,t) of $[0,1]$ and for each $u, v \in \mathbb{R}^2$, there exists $f \in C^1([0,1], \mathbb{R}^2)$ such that $(f(s), f'(t)) = (u, v)$ and
>
> $$\|f(x)\| < \|u\|, \quad \|f'(y)\| < \|v\|,$$
>
> for each $x \neq s$ and for each $y \neq t$.

This shows that each point $(s,t) \in [0,1] \times [0,1]$ with $s \neq t$ satisfies (Peak). By the same reasoning, we see that each point of $[0,1]$ satisfies both (P_X) and (P_Y). Thus conditions (C3-1) and (C3-2) are satisfied as well.

Under these notations, the above $T : C^1([0,1], \mathbb{R}^2) \to C^1([0,1], \mathbb{R}^2)$ is a real-linear $\| \cdot \|_{D,p}$-isometry that preserves, if $p > 1$, the subspace of constant functions $\mathrm{Const}_{[0,1]} = \mathrm{Ker}\, D$. By the Main Theorem, we obtain homeomorphisms $\varphi, \psi : [0,1] \to [0,1]$ and continuous maps $U : [0,1] \to \mathcal{U}(\mathbb{R}^2)$, $t \mapsto U_t$, and $V[0,1] \to \mathcal{U}(\mathbb{R}^2)$, $t \mapsto V_t$, such that

$$Tf(t) = U_t(f(\varphi(t))), \tag{5.1}$$

$$(Tf)'(t) = V_t(f'(\psi(t))), \tag{5.2}$$

for each $f \in C^1([0,1], \mathbb{R}^2)$ and for each $t \in [0,1]$. From the remark to the Main Theorem, T preserves the constant functions for the case $p = 1$ as well. It follows easily that U_t does not depend on t. Thus (5.1) is written as

$$Tf(t) = U_*(f(\varphi(t))), \quad t \in [0,1],$$

for some orthogonal transformation U_*.

Case 1. $\det U_* = 1$.
Then U_* is expressed as a 2×2 matrix of the form

$$U_* = R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

for some θ. Transforming this back to the original function space $C^1([0,1],\mathbb{C})$, we see that T must be of the form

$$Tf(t) = \exp(i\theta) f(\varphi(t)), \quad f \in C^1([0,1],\mathbb{C}).$$

Case 2. $\det U_* = -1$.
In this case, U_* is the multiplication $U_* = R(\theta)J$, where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the complex-valued function space $C^1([0,1],\mathbb{C})$, J represents the complex conjugation, and hence we have $Tf(t) = \exp(i\theta)\overline{f(\varphi(t))}$, $f \in C^1([0,1],\mathbb{C})$.

Summing up these two cases, the isometry T of (5.1) is written as

$$Tf(t) = c[f(\varphi(t))]^{\epsilon}, \tag{5.3}$$

with $|c| = 1$.

In order to identify the map φ, we take the derivative of (5.3) and compare it with (5.2). By the same reasoning as above, (5.2) is rephrased as

$$(Tf)'(t) = \tilde{c}(t)[f'(\psi(t))]^{\tilde{\epsilon}},$$

for some continuous function $\tilde{c} : [0,1] \to \mathbb{T}$ and $\tilde{\epsilon} \in \{\pm 1\}$. Notice that unlike (5.3), it is not clear at this stage that $\tilde{c}(t)$ is a constant function. We have

$$(Tf)'(t) = c[f'(\varphi(t))]^{\epsilon}\varphi'(t) = \tilde{c}[f'(\psi(t))]^{\tilde{\epsilon}}, \tag{5.4}$$

which implies $c\varphi'(t) = \tilde{c}(t)$, $\epsilon = \tilde{\epsilon}$. In particular, we have $|\varphi'(t)| = 1$, and since $\varphi(t)$ is a real-valued function, we have $\varphi'(t) = \pm 1$. By the connectedness of $[0,1]$, we conclude that $\varphi' \equiv 1$ or $\varphi' \equiv -1$.

This completes the proof of Theorem 1.1. □

Observe that the equality

$$\lim_{p\to\infty} \|f\|_p = \max(\|f\|_\infty, \|f'\|_\infty)$$

holds for each $f \in C^1([0,1],\mathbb{C})$. Hence the collection $(\|\cdot\|_p)_{p\in[1,\infty]}$ is a collection of norms on $C^1([0,1],\mathbb{C})$ which is continuous in the sense of [26, Section 3]. Thus the collection supplies a continuous (in the above sense) interpolation between the norm $\|f\|_\Sigma = \|f\|_\infty + \|f'\|_\infty$ and $\|f\|_M = \max(\|f\|_\infty, \|f'\|_\infty)$. This is a complementary result to the main theorem of [26] and was announced in that paper.

5.2. Proof of Theorem 1.2. Let $\alpha \in (0, \pi/2)$, and let $\|\cdot\|_{<\alpha>}$ be the norm defined in (1.4) on $C^2([0,1])$. Let $L : C^2([0,1],\mathbb{C}) \to C([0,1],\mathbb{C})$ be the operator defined by $Lf = f'' + \alpha^2 f$. We study a real-linear $\|\cdot\|_{<\alpha>}$-isometry $T : C^2([0,1]) \to C^2([0,1])$ which makes the solution space $\mathcal{S}_L = \{f \in C^2([0,1]) | Lf = 0\}$ invariant.

We start with an elementary lemma whose proof is omitted. The choice of α in $(0, \pi/2)$ guarantees the first statement.

Lemma 5.1.

(1) *For each* $a \in [0,1]$, *the real-valued function* $f(t) = \cos(\alpha(t-a))$ *on* $[0,1]$ *achieves its maximum only at* a.

(2) *For each* $\omega \neq \alpha$, *the function* $f(t) = \frac{A}{\alpha^2 - \omega^2} \cos \omega(t-a)$ *satisfies* $Lf = A \cos \omega(t-a)$.

(3) *For each* $g \in C([0,1],\mathbb{C})$, *every solution* $f \in C^1([0,1],\mathbb{C})$ *of the equation* $Lf = g$ *is written as*

$$f(t) = (A + I_g(t)) \exp(i\alpha t) + (B - J_g(t)) \exp(-i\alpha t), \tag{5.5}$$

where A *and* B *are constants given by*

$$A = \frac{1}{2}\left(f(0) + \frac{1}{i\alpha} f'(0)\right), \quad B = \frac{1}{2}\left(f(0) - \frac{1}{i\alpha} f'(0)\right),$$

and $I_g(t)$ *and* $J_g(t)$ *are the functions defined by*

$$I_g(t) = \int_0^t \frac{\exp(-i\alpha s) g(s)}{2i\alpha} ds, \quad J_g(t) = \int_0^t \frac{\exp(i\alpha s) g(s)}{2i\alpha} ds.$$

In order to apply the Main Theorem, let $X = Y = [0,1]$, $E = F = \mathbb{C}$, $A = C^2([0,1],\mathbb{C}) \subset C([0,1],\mathbb{C})$ and $D = L$. Then we have $\|\cdot\|_{<\alpha>} = \|\cdot\|_{D,\infty}$.

Lemma 5.2. *The above* A *and* D *satisfy conditions* (C1), (C2-1), (C2-2), (C3-2), (C4-1) *and* (C4-2).

Proof. In order to verify (C1), let (f_n) be a sequence of C^2-functions in $C^2([0,1],\mathbb{C})$ such that

$$\lim_n \|f_n - f\|_\infty = 0, \quad \lim_n \|Lf_n - g\|_\infty = 0, \tag{5.6}$$

for some $f, g \in C([0,1],\mathbb{C})$, and let $g_n = Lf_n$. We see from Lemma 5.1 (3), f_n is written by g_n as in (5.5):

$$f_n(t) = (A_n + I_{g_n}(t)) \exp(i\alpha t) + (B_n - J_{g_n}(t)) \exp(-i\alpha t).$$

Then we have

$$A_n \exp(i\alpha t) + B_n \exp(-i\alpha t) = f_n(t) - I_{g_n}(t)\exp(i\alpha t) + J_{g_n}(t)\exp(-i\alpha t).$$

It follows from (5.6) that $A := \lim_n A_n$ and $B := \lim_n B_n$ both exist, and the function $f(t)$ is written as in (5.5). Then we see that f is actually a C^2-function and satisfies $Lf = g$. This verifies (C1).

For (C2-1a), take an arbitrary $u \in \mathbb{C}$ and $x \in [0,1]$, and let $f(t) = u\cos\alpha(t-x)$, $t \in [0,1]$. Then f satisfies the required condition by Lemma 5.1 (1). For (C2-2a), take an arbitrary $\epsilon > 0$ and $v \in \mathbb{C}$, and choose $\omega > 0$ such that $|\frac{v}{\alpha^2-\omega^2}| < \epsilon$. For each $x \in [0,1]$, the function $f(t) = \frac{v}{\alpha^2-\omega^2}\cos\omega(t-x)$ satisfies $Lf(x) = v$, $\|Lf\|_\infty = |v|$ and $\|f\|_\infty < \epsilon$ by Lemma 5.1 (2). This verifies condition (C2-2a). The verification of (C2-1b) and (C2-2b) is straightforward.

For (C3-2), we observe that each point x of $[0,1]$ satisfies the conditions (P_X) and (P_Y): for each $u, v \in \mathbb{C}$ and for each $\epsilon > 0$, choose ω so that $|\frac{v}{\alpha^2-\omega^2}| < \epsilon$. Then we see

$$f(t) = u\cos\alpha(t-x), \quad g(t) = \frac{v}{\alpha^2-\omega^2}\cos\omega(t-x), \quad t \in [0,1]$$

are the functions that witness the conditions (P_X) and (P_Y), respectively, for x (see Lemma 5.1 (1) and (2)).

For (C4-1) and (C4-2), observe $L(c_u) = \alpha^2 c_u$. □

Applying the Main Theorem (3), we have a homeomorphism $\varphi : [0,1] \to [0,1]$ and a continuous function $\kappa : [0,1] \to \mathbb{T}$ such that

$$Tf(t) = \kappa(t) f(\varphi(t)), \quad f \in C^2([0,1],\mathbb{C}).$$

In order to identify $\varphi(t)$ and $\kappa(t)$, we use the assumption that $T(\mathcal{S}_\alpha) = \mathcal{S}_\alpha$. From this, we find constants $a, b, c, d \in \mathbb{C}$ satisfying the following equalities:

$$\kappa(t)\exp(i\alpha\varphi(t)) = a\exp(i\alpha t) + b\exp(-i\alpha t), \tag{5.7}$$

$$\kappa(t)\exp(-i\alpha\varphi(t)) = c\exp(i\alpha t) + d\exp(-i\alpha t). \tag{5.8}$$

By (5.7) and (5.8), we see that $|a\exp(i\alpha t) + b\exp(-i\alpha t)| = |b + a\exp(2i\alpha t)| = 1$ for each $t \in [0,1]$, and similarly, see that $|d + c\exp(2i\alpha t)| = 1$ for each $t \in [0,1]$. We then easily obtain

$$ab = cd = 0 \quad \text{and} \quad |a| + |b| = |c| + |d| = 1.$$

There are three cases to consider.

Case 1. $(b, c) = (0, 0)$.
Equalities (5.7) and (5.8) are reduced to

$$\kappa(t)\exp(i\alpha\varphi(t)) = a\exp(i\alpha t), \quad \kappa(t)\exp(-i\alpha\varphi(t)) = d\exp(-i\alpha t),$$

from which we have $\kappa(t)^2 = ad$. Hence $\kappa(t)$ is a constant function. Let $\kappa(t) \equiv \kappa$. Then by (5.7), we see

$$\exp(i\alpha\varphi(t)) = a\kappa^{-1}\exp(i\alpha t).$$

Since φ is a homeomorphism on $[0, 1]$, we have $\varphi(0) \in \{0, 1\}$. If $\varphi(0) = 0$, then by the above, $a\kappa^{-1} = 1$ and $\exp(i\alpha\varphi(t)) = \exp(i\alpha t)$. This implies $\varphi(t) - t \in \frac{2\pi}{\alpha}\mathbb{Z}$ for each $t \in [0, 1]$. It follows from the connectedness of $[0, 1]$ that the function $t \mapsto (\varphi(t) - t)$ is a constant function, and hence is equal to zero. Thus we conclude $\varphi(t) = t$. On the other hand, if $\varphi(0) = 1$, then $\exp(i\alpha) = a\kappa^{-1}$ and we obtain $\exp(i\alpha\varphi(t)) = \exp(i\alpha(t + 1))$. However, this implies $\varphi(t) = t + 1$, which is impossible.

Case 2. $(b, d) = (0, 0)$ or $(a, c) = (0, 0)$.
If $(b, d) = (0, 0)$, then equalities (5.7) and (5.8) are reduced to

$$\kappa(t)\exp(i\alpha\varphi(t)) = a\exp(i\alpha t), \quad \kappa(t)\exp(-i\alpha\varphi(t)) = c\exp(i\alpha t),$$

which implies $\exp(2i\alpha\varphi(t)) = ac^{-1}$, and hence φ is a constant, a contradiction. The same argument applies to the case $(a, c) = (0, 0)$ to derive a contradiction.

Case 3. $(a, d) = (0, 0)$.
We have

$$\kappa(t)\exp(i\alpha\varphi(t)) = b\exp(-i\alpha t), \quad \kappa(t)\exp(-i\alpha\varphi(t)) = c\exp(i\alpha t).$$

As in Case 1, we conclude $\kappa(t)$ is a constant function $\equiv \kappa$ and $\exp(i\alpha\varphi(t)) = b\kappa^{-1}\exp(-i\alpha t)$. If $\varphi(0) = 0$, we conclude $b\kappa^{-1} = 1$ and $\exp(i\alpha\varphi(t)) = \exp(-i\alpha t)$. This yields an equality $\varphi(t) = -t$, a contradiction. If $\varphi(0) = 1$, then we have $b\kappa^{-1} = \exp(i\alpha)$ and $\exp(i\alpha\varphi(t)) = \exp(i\alpha(1 - t))$. We conclude in the same way as in Case 1 that $\varphi(t) = 1 - t$.

This completes the proof of Theorem 1.2. $\square$

6. Proofs of Proposition 3.1, Theorem 3.4, Lemma 3.5 and Lemma 3.6

This section supplies proofs of Proposition 3.1, Theorem 3.4, Lemmas 3.5 and Lemma 3.6.

6.1. Proof of Propostion 3.1.

Proof of Propostion 3.1. We prove

(1) the Banach spaces $(E \times_p F)^*$ and $E^* \times_q F^*$ are isometric with $\frac{1}{p} + \frac{1}{q} = 1$,

(2) the isometric isomorphisms

$$\mathrm{ext}(E \times_p F)^* \cong \begin{cases} S(E^* \times_q F^*) & \text{if } 1 < p < \infty, \\ S(E^*) \times S(F^*) & \text{if } p = 1, \\ S(E^*) \times \{0\} \cup \{0\} \times S(F^*) & \text{if } p = \infty, \end{cases}$$

hold when E^* and F^* are strictly convex.

(1) The proof is an adoption of the standard duality $\ell_p^* \cong \ell_q$. Let $\omega : E^* \times_q F^* \to (E \times_p F)^*$ be the linear map defined by

$$\omega(\alpha, \beta)(u, v) = \alpha(u) + \beta(v), \quad (\alpha, \beta) \in E^* \times F^*, (u, v) \in E \times F.$$

By the Hölder inequality, we see

$$|\omega(\alpha, \beta)(u, v)| \leq \|\alpha\|\|u\| + \|\beta\|\|v\| \leq (\|\alpha\|^q + \|\beta\|^q)^{1/q}(\|u\|^p + \|v\|^p)^{1/p}.$$

Hence we have

$$\|\omega(\alpha, \beta)\| \leq (\|\alpha\|^q + \|\beta\|^q)^{1/q} = \|(\alpha, \beta)\|_q.$$

In order to show the equality in the above, first we assume $\alpha \neq 0 \neq \beta$. For each $\epsilon > 0$, take a vector $u_0 \in E$ so that $\|u_0\| = 1, |\alpha(u_0)| \geq \|\alpha\| - (\epsilon/\|\alpha\|^{q-1})$, and let $u = \|\alpha\|^{q-1}u_0$. Then

$$\|u\| = \|\alpha\|^{q-1}, \quad |\alpha(u)| > \|\alpha\|^q - \epsilon.$$

Similarly, take $v \in F$ so that

$$\|v\| = \|\beta\|^{q-1}, \quad |\beta(v)| > \|\beta\|^q - \epsilon.$$

By multiplying appropriate scalars with modulus one, we may assume that $\alpha(u)$ and $\beta(v)$ are positive. Then

$$\begin{aligned} \|\alpha\|^q + \|\beta\|^q - 2\epsilon \leq |\alpha(u)| + |\beta(v)| &= \alpha(u) + \beta(v) = \omega(\alpha, \beta)(u, v) \\ &\leq \|\omega(\alpha, \beta)\|\|(u, v)\|_p = \|\omega(\alpha, \beta)\|(\|\alpha\|^{(q-1)p} + \|\beta\|^{(q-1)p})^{1/p} \\ &= \|\omega(\alpha, \beta)\|(\|\alpha\|^q + \|\beta\|^q)^{1/p}. \end{aligned}$$

Since ε is an arbitrary positive number, we see from the above that $\|\omega(\alpha,\beta)\| \geq (\|\alpha\|^q + \|\beta\|^q)^{1-(1/p)} = (\|\alpha\|^q + \|\beta\|^q)^{1/q}$. When $\alpha = 0$ or $\beta = 0$, this inequality directly follows from the definition. This proves that ω is an isometry and proves (1).

(2) (2.1) First, we assume $1 < p < \infty$ and show that $(E \times_p F)^*$ is strictly convex, from which the conclusion follows directly.

In view of (1), it suffices to prove the following.

If L and M are strictly convex normed linear space, then $L \times_q M$ is strictly convex for each $q \in (1,\infty)$.

Take $(x_1,y_1),(x_2,y_2) \in S(L \times_q M)$, and note that $(\|x_1\|,\|y_1\|),(\|x_2\|,\|y_2\|) \in S(\ell_q^2)$.

Case 1. $(\|x_1\|,\|y_1\|) = (\|x_2\|,\|y_2\|)$.
By the strict convexity of L and M, we have

$$\|\frac{x_1+x_2}{2}\| < \frac{\|x_1\|+\|x_2\|}{2} = \|x_1\| \quad \text{and} \quad \|\frac{y_1+y_2}{2}\| < \frac{\|y_1\|+\|y_2\|}{2} = \|y_1\|.$$

Then we have

$$\|\frac{x_1+x_2}{2}\|^q + \|\frac{y_1+y_2}{2}\|^q < \|x_1\|^q + \|y_1\|^q = 1,$$

which proves the desired inequality.

Case 2. $(\|x_1\|,\|y_1\|) \neq (\|x_2\|,\|y_2\|)$.

By the strict convexity of ℓ_q^2, we see

$$\left(\frac{\|x_1\|+\|x_2\|}{2}\right)^q + \left(\frac{\|y_1\|+\|y_2\|}{2}\right)^q < 1.$$

Then we have

$$\|\frac{x_1+x_2}{2}\|^q + \|\frac{y_1+y_2}{2}\|^q \leq \left(\frac{\|x_1\|+\|x_2\|}{2}\right)^q + \left(\frac{\|y_1\|+\|y_2\|}{2}\right)^q < 1,$$

which is to be shown.

(2.2) Next, we assume that $p = 1$, and observe by (1) that $(E \times_1 F)^* \cong E^* \times_\infty F^*$. In particular, we have the inclusion $S(E^*) \times S(F^*) \subset S((E \times_1 F)^*)$. If $(\xi,\eta) \in S((E \times_1 F)^*)$ with $\eta \notin S(F^*) = \mathrm{ext}(F^*)$ (the last equality follows from the strict convexity of F^*), then $\eta = \frac{1}{2}(\beta_1+\beta_2)$ for some $\beta_1,\beta_2 \in S(F^*)$

with $\beta_1 \neq \eta \neq \beta_2$. Then the equality $(\xi, \eta) = \frac{1}{2}((\xi, \beta_1) + (\xi, \beta_2))$ with $(\xi, \beta_i) \in S((E \times_1 F)^*)$, $i = 1, 2$, shows that (ξ, η) is not an extreme point. Similarly, (ξ', η') is not an extreme point if $\xi' \notin S(E^*)$. This proves the inclusion

$$\mathrm{ext}(E \times_1 F)^* \subset S(E^*) \times S(F^*).$$

The reverse inclusion follows from the strict convexity of E^* and F^*. This proves the desired equality.

(2.3) Finally, we assume $p = \infty$. Then $(E \times_\infty F)^* \cong E^* \times_1 F^*$. If $(\xi, \eta) \in S(E^* \times_1 F^*)$ with $\xi \neq 0 \neq \eta$, then define

$$\alpha = ((1 - \|\eta\|)\xi, (1 + \|\xi\|)\eta) \neq (\xi, \eta) \quad \text{and} \quad \beta = ((1 + \|\eta\|)\xi, (1 - \|\xi\|)\eta) \neq (\xi, \eta).$$

Then $\alpha, \beta \in S(E^* \times_1 F^*)$ and $(\xi, \eta) = \frac{1}{2}(\alpha + \beta)$, and hence (ξ, η) is not an extreme point. Thus we see $\mathrm{ext}(E^* \times_1 F^*) \subset S(E^*) \times \{0\} \cup \{0\} \times S(F^*)$.

On the other hand, if $(\xi, 0) = \frac{1}{2}((\alpha_1, \alpha_2) + (\beta_1, \beta_2))$ with $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in S(E^* \times_1 F^*)$, then by the strict convexity of E^*, we have $\xi = \alpha_1 = \beta_1$, and thus $\|\alpha_1\| = \|\beta_1\| = 1$. Also, we have $\alpha_2 + \beta_2 = 0$. These two imply $\alpha_2 = \beta_2 = 0$, and hence $(\alpha_1, \alpha_2) = (\beta_1, \beta_2) = (\xi, 0)$. Thus $(\xi, 0)$ is an extreme point. In this way, we obtain the inclusion $S(E^*) \times \{0\} \cup \{0\} \times S(F^*) \subset \mathrm{ext}(E^* \times_1 F^*)$. This proves the desired conclusion. $\square$

6.2. Proof of Theorem 3.4. We prove de Leeuw's Lemma for complex-scalar function spaces.

Proof of Theorem 3.4. Let $\delta_z = \delta_z^A$ for simplicity. We prove that δ_z is an extreme point of A^*. Assume that $\delta_z = \frac{1}{2}(\psi_1 + \psi_2)$ for $\psi_1, \psi_2 \in S(A^*)$. We take norm-one extensions of ψ_1, ψ_2 respectively to functionals on $C(Z)$ by the Hahn–Banach theorem, and apply the Riesz representation theorem to take complex-valued Borel regular measures μ_1, μ_2 on Z with total variations $|\mu_1| = |\mu_2| = 1$ so that

$$\psi_i(g) = \int_X g d\mu_i, \quad g \in A,$$

for $i = 1, 2$. For $\mu = \mu_1, \mu_2$, we notice

$$\left| \int_X f d\mu \right| \leq \int_X |f| d|\mu| \leq 1.$$

Since

$$1 = f(z) = \frac{1}{2}(\psi_1(f) + \psi_2(f)) = \frac{1}{2}\left(\int_X f d\mu_1 + \int_X f d\mu_2 \right),$$

we have

$$\int_X f d\mu_i = 1, \quad i = 1, 2. \tag{6.1}$$

Let $\mu = \mu_1, \mu_2$ for simplicity, and let

$$Z_1 = \{x \in Z | |f(x)| = 1\}, \quad Z_0 = \{x \in Z | |f(x)| < 1\}.$$

We show

$$|\mu|(Z_0) = 0. \tag{6.2}$$

To show (6.2), we first observe

$$1 = \left| \int_Z f d\mu \right| \leq \int_Z |f| d|\mu| \leq 1,$$

and hence $\int_X |f| d|\mu| = 1$. Let $g = 1 - |f|$. It is a non-negative function such that

$$\int_Z g d|\mu| = 1 - \int_Z |f| d|\mu| = 0,$$

due to the assumption $|\mu|(Z) = 1$. Then

$$0 = \int_Z g d|\mu| = \int_{Z_1} g d|\mu| + \int_{Z_0} g d|\mu| = \int_{Z_0} g d|\mu|,$$

which implies the desired equality (6.2).

Combining (6.1) and (6.2), we obtain

$$\int_{Z_1} f d\mu = 1. \tag{6.3}$$

For each $x \in Z_1$, we have $c \in \mathbb{T}$ such that $g(x) = cg(z)$ for each $g \in A$. In particular, $f(x) = c$, and hence the above implies $g(x) = f(x)g(z)$. This proves the equality

$$g(x) = f(x)g(z), \quad x \in Z_1.$$

Therefore, for $i = 1, 2$, we have

$$\begin{aligned} \psi_i(g) &= \int_Z g d\mu_i = \int_{Z_1} g d\mu_i \quad \text{(by (6.2))} \\ &= \int_{Z_1} g(z) f(x) d\mu_i(x) = g(z) \int_{Z_1} f d\mu_i = g(z) \quad \text{(by (6.3))}. \end{aligned}$$

Therefore, $\psi_1 = \psi_2 = \delta_z$, which is to be proved. $\square$

6.3. Proofs of Lemma 3.5 and Lemma 3.6.

Proof of Lemma 3.5. We prove that the map

$$\Lambda : (A, \|\cdot\|_{D,p}) \to C(S((E \times_p F)^*) \times (X \times Y), \mathbb{F})$$

defined by

$$\Lambda(f)((\xi, \eta), (x, y)) = \xi(f(x)) + \eta(Df(y)),$$
$$(\xi, \eta) \in E^* \times F^*, \quad \|\xi\|^q + \|\eta\|^q = 1, \quad (x, y) \in X \times Y, \quad f \in A,$$

satisfies the equality $\|\Lambda(f)\|_\infty = \|f\|_{D,p}$ for each $f \in A$. For $(\xi_x, \eta_y) \in S((E \times_p F)^*) \times (X \times Y)$ with $\|\xi\|^q + \|\eta\|^q = 1$, we have, by Hölder's inequality,

$$\begin{aligned}|\Lambda(f)(\xi_x, \eta_y)| &\le \|\xi\|\|f(x)\| + \|\eta\|\|Df(y)\| \\ &\le (\|\xi\|^q + \|\eta\|^q)^{1/q}(\|f(x)\|^p + \|Df(y)\|^p)^{1/p} \le \|f\|_{D,p}.\end{aligned}$$

Hence $\|\Lambda(f)\|_\infty \le \|f\|_p$. In order to prove the reverse inequality, take a point $(x_0, y_0) \in X \times Y$ so that $\|f\|_{D,p} = (\|f(x_0)\|^p + \|Df(y_0)\|^p)^{1/p}$, and let $(u_0, v_0) = (f(x_0), Df(y_0))$. Choose $(\xi, \eta) \in E^* \times F^*$ such that

$$\begin{aligned}\xi(u_0) &= \|u_0\|^p \cdot \|f\|_{D,p}^{1-p}, &\qquad \|\xi\| &= \|u_0\|^{p-1} \cdot \|f\|_{D,p}^{1-p}, \\ \eta(v_0) &= \|v_0\|^p \cdot \|f\|_{D,p}^{1-p}, &\qquad \|\eta\| &= \|v_0\|^{p-1} \cdot \|f\|_{D,p}^{1-p}.\end{aligned}$$

Then we have $\|\xi\|^q + \|\eta\|^q = 1$, and

$$|\Lambda(f)(\xi_{x_0}, \eta_{y_0})| = (\|u_0\|^p + \|v_0\|^p)\|f\|_{D,p}^{1-p} = \|f\|_{D,p},$$

which proves the desired equality. □

Proof of Lemma 3.6. We show that the subspace $N_A = \Lambda(A)$ is closed in $C(S((E \times_p F)^*) \times (X \times Y), \mathbb{F})$. Let $\dim E = d$, fix an orthonormal basis $(e_i)_{1 \le i \le d}$ of E, and let $(\epsilon_i)_{1 \le i \le d}$ be its dual basis of E^*. Take a net $(f_\alpha) \subset A$, and assume that $\lim_\alpha \Lambda(f_\alpha) = h \in C(S((E \times_p F)^*) \times (X \times Y), \mathbb{F})$. Then for each $i = 1, \ldots, d$ and for each $(x, y) \in X \times Y$, we have $\lim_\alpha \epsilon_i(f_\alpha(x)) = \lim_\alpha \Lambda(f_\alpha)(\epsilon_{i,x}, 0_y) = h(\epsilon_{i,x}, 0_y)$. Since $f_\alpha(x) = \sum_{i=1}^d \epsilon_i(f_\alpha(x))e_i$, we see $\lim_\alpha f_\alpha(x)$ exists and is equal to $\sum_{i=1}^d h(\epsilon_{i,x}, 0_y)e_i$. In particular, the vector $\sum_{i=1}^d h(\epsilon_{i,x}, 0_y)e_i$ does not depend on the choice of the orthonormal basis and a point $y \in Y$. Fixing a point $y \in Y$, define $f(x) = \sum_{i=1}^d h(\epsilon_{i,x}, 0_y)e_i$ for $x \in X$. Then we see

$$\lim_\alpha \|f_\alpha - f\|_\infty = 0. \tag{6.4}$$

Indeed, we have for each $x \in X$,

$$\begin{aligned}\|f_\alpha(x)-f(x)\| &\leq \sum_{i=1}^{d} \|(\epsilon_i(f_\alpha(x))-h(\epsilon_{i,x},0_y))e_i\| \leq \sum_{i=1}^{d} \|\Lambda(f_\alpha)(\epsilon_{i,x},0_y)-h(\epsilon_{i,x},0_y)\| \\ &\leq \sum_{i=1}^{d} \|\Lambda(f_\alpha)-h\|_\infty = d\|\Lambda(f_\alpha)-h\|_\infty,\end{aligned}$$

and the last term tends to 0. This proves (6.4), and in particular, f is continuous.

By the same reason, we obtain a continuous map $g : Y \to F$ such that $\lim_\alpha Df_\alpha = g$. By condition (C1), we have $f \in A$ and $g = Df$. For each $(\xi_x, \eta_y) \in S((E \times_p F)^*) \times (X \times Y)$, we have

$$\begin{aligned}\Lambda(f)(\xi_x, \eta_y) &= \xi(f(x)) + \eta(Df(y)) \\ &= \lim_\alpha(\xi(f_\alpha(x)) + \eta(Df_\alpha(y))) = \lim_\alpha \Lambda(f_\alpha)(\xi_x, \eta_y) = h(\xi_x, \eta_y).\end{aligned}$$

Hence we obtain $\Lambda(f) = h$ as desired. $\square$

Acknowledgements. The author expresses his deep and sincere appreciation to the referees of the paper, whose comments considerably improved the paper.

References

[1] H. Al-Halees and R. J. Fleming, Extreme point methods and Banach–Stone theorems, *J. Aust. Math. Soc.* **75** (2003), 125–143.

[2] J. Araujo and J. J. Font, Linear isometries between subspaces of continuous functions, *Trans. Amer. Math. Soc.* **349** (1997), 413–428.

[3] E. Behrends, M-Structure and the Banach–Stone Theorem, Lecture Notes in Mathematics, Vol. **736**, *Springer*, *Berlin*, 1979.

[4] F. Botelho, R. J. Fleming and J. Jamison, Extreme points and isometries on vector-valued Lipschitz spaces, *J. Math. Anal. Appl.* **381** (2011), 821–832.

[5] F. Botelho and J. Jamison, Surjective isometries of differentiable vector-valued functions, *Studia Math.* **192** (2009), 39–50.

[6] F. Botelho and J. Jamison, Surjective isometries on spaces of vector valued continuous and Lipschitz functions, *Positivity* **17** (2013), 395–405.

[7] F. Botelho, J. Jamison and B. Zheng, Isometries of spaces of vector valued Lipschitz fuctions, *Positivity* **17** (2013), 47–65.

[8] G. E. Bredon, Topology and Geometry, Graduate Texts in Mathematics, Vol.**139**, *Springer-Verlag*, *New York*, 1993.

[9] M. Cambern, Isometries of certain Banach algebras, *Studia Math.* **25** (1964–1965), 217–225.

[10] M. Cambern, A Holsztyński theorem for spaces of continuous vector-valued functions, *Studia Math.* **63** (1978), 213–217.

[11] M. Cambern and V. D. Pathak, Isometries of spaces of differentiable functions, *Math. Japon.* **26** (1981), 253–260.

[12] K. de Leeuw, Banach spaces of Lipschitz functions, *Studia Math.* **21** (1961–1962), 55–66.

[13] N. Dunford and J. T. Schwarz, Linear Operators. I. General Theory, *Interscience*, *New York – London*, 1958.

[14] R. J. Fleming and J. E. Jamison, Isometries on Banach Spaces: Function Spaces, *Chapman & Hall/CRC*, *Boca Raton, FL*, 2003.

[15] R. Fleming and J. Jamison, Isometries on Banach Spaces. Vol. 2. Vector-Valued Function Spaces, *Chapman & Hall/CRC*, *Boca Raton, FL*, 2008.

[16] J. J. Font, Linear isometries between certain subspaces of continuous vector-valued functions, *Illinois J. Math.* **42** (1998), 389–397.

[17] K. Jarosz, Isometries in semisimple commutative Banach algebras, *Proc. Amer. Math. Soc.* **94** (1985), 65–71.

[18] K. Jarosz and V. D. Pathak, Isometries between function spaces, *Trans. Amer. Math. Soc.* **305** (1988), 193–206.

[19] J.-S. Jeang and N.-C. Wong, Into isometries of $C_0(X, E)$s, *J. Math. Anal. Appl.* **207** (1997), 286–290.

[20] J. S. Jeang and N. C. Wong, On the Banach–Stone problem, *Studia Math.* **155** (2003), 95–105.

[21] M. Jerison, The space of bounded maps into a Banach space, *Ann. of Math. (2)* **52** (1950), 309–327.

[22] K. Kawamura, Linear surjective isometries between vector-valued function spaces, *J. Aust. Math. Soc.* **100** (2016), 349–373.

[23] K. Kawamura, Banach–Stone theorems for spaces of vector bundle continuous sections, *Topology Appl.* **227** (2017), 118–134.

[24] K. Kawamura, Isometries of function spaces over Riemannian manifolds, (preprint).

[25] K. Kawamura, Banach–Stone type theorems for C^1 function spaces over Riemannian manifolds, (preprint).

[26] K. Kawamura, H. Koshimizu and T. Miura, Norms on $C^1([0, 1])$ and their isometries, (preprint).

[27] K. Kawamura and T. Miura, Real-linear surjective isometries between function spaces, *Topology Appl.* **226** (2017), 66–85.

[28] H. Koshimizu, Finite codimensional linear isometries on spaces of differentiable and Lipschitz functions, *Cent. Eur. J. Math.* **9** (2011), 139–146.

[29] H. Koshimizu, T. Miura, H. Takagi and S.-E. Takahasi, Real-linear isometries between subspaces of continuous functions, *J. Math. Anal. Appl.* **413** (2014), 229–241.

[30] K-S. Lau, A representation theorem for isometries of $C(X, E)$, *Pacific J. Math.* **60** (1975), 229–233.

[31] L. Li and R. Wang, Surjective isometries on the vector-valued differentiable function spaces, *J. Math. Anal. Appl.* **427** (2015), 547–556.

[32] R. Megginson, An Introduction to Banach Space Theory, Graduate Texts in Mathematics, Vol. **183**, *Springer-Verlag*, *New York*, 1998.

[33] T. Miura, Surjective isometries between function spaces, *Contemp. Math.* **645** (2015), 231–239.

[34] W. Novinger, Linear isometries of subspaces of spaces of continuous functions, *Studia Math.* **53** (1975), 273–276.

[35] V. D. Pathak, Isometries of $C^{(n)}[0,1]$, *Pacific J. Math.* **94** (1981), 211–222.

[36] N. V. Rao and A. K. Roy, Linear isometries of some function spaces, *Pacific J. Math.* **38** (1971), 177–192.

KAZUHIRO KAWAMURA
INSTITUTE OF MATHEMATICS
UNIVERSITY OF TSUKUBA
TSUKUBA, IBARAKI, 305-8571
JAPAN

E-mail: kawamura@math.tsukuba.ac.jp

(Received May 6, 2016; revised February 10, 2017)