On the simultaneous equations
\[ \sigma(2^a) = p^{f_1}q^{g_1}, \sigma(3^b) = p^{f_2}q^{g_2}, \sigma(5^c) = p^{f_3}q^{g_3} \]

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Abstract. Let \( \sigma(N) \) denote the sum of divisors of \( N \). We shall solve the simultaneous equations
\[ \sigma(2^a) = p^{f_1}q^{g_1}, \sigma(3^b) = p^{f_2}q^{g_2}, \sigma(5^c) = p^{f_3}q^{g_3} \]
with \( p,q \) distinct primes.

1. Introduction

As usual, let \( \sigma(N) \) denote the sum of divisors of \( N \), and \( \omega(N) \) the number of distinct prime factors of \( N \). In [18], the author has shown that there are only finitely many odd superperfect numbers (i.e., the number satisfying \( \sigma(\sigma(N)) = 2N \)) with bounded numbers of distinct prime factors, by proving that the simultaneous equations \( \sigma(p_1^{e_1} \cdots p_k^{e_k}) = q_1^{f_1} \cdots q_k^{f_k} \) for \( k+1 \) prime powers
\[ p_i^{e_i}(i=1,2,\ldots,k+1) \]
cannot have solutions with \( p_1,\ldots,p_{k+1} \) all small. In this paper, we use the method developed in [18] to solve the simultaneous equations \( \sigma(2^a) = \sigma(5^c) \) with \( p,q \) distinct primes.

Wakulicz [16] has shown that all solutions of the purely exponential diophantine equation \( 2^n - 5^m = 3 \) are \( (n,m) = (2,0),(3,1) \) and \( (7,3) \), from which Makowski and Schinzel [9] derived that the equation \( \sigma(2^n) = \sigma(5^c) \) has only the solution \( (a,c) = (4,2) \). We note that it is easy to show that \( \sigma(2^n) = \sigma(3^b) \) has no nontrivial solution, and \( \sigma(3^b) = \sigma(5^c) \) also has no nontrivial solution. Bugeaud and Mignotte [3] have shown that neither of \( \sigma(2^n), \sigma(3^b), \sigma(5^c) \) can be a perfect power except \( \sigma(3) = 2^2 \) and \( \sigma(3^4) = 11^2 \). Moreover, they have shown

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that the only perfect powers \( \frac{x^n - 1}{x - 1} \), with \( x = z^t \), \( z \leq 10 \), are \( (3^5 - 1)/2 = 11^2 \) and \( (7^4 - 1)/6 = 20^2 \).

Now we shall state our result.

**Theorem 1.1.** The simultaneous equations

\[
\sigma(2^a) = p^f_1 q^{g_1}, \quad \sigma(3^b) = p^f_2 q^{g_2}, \quad \sigma(5^c) = p^f_3 q^{g_3},
\]

with \( a,b,c > 0 \), \( f_1, f_2, f_3, g_1, g_2, g_3 \geq 0 \) and \( p, q \) distinct primes, have only the following solutions:

(i) \( (a,b,c) = (1,1,1) \),
(ii) \( (a,b,c) = (4,1,2) \),
(iii) \( (a,b,c) = (4,4,2) \) and
(iv) \( (a,c) = (4,2) \) and \( \sigma(3^b) \) is prime.

In other words, if \( \omega(\sigma(2^a3^b5^c)) \leq 2 \), then \( (a,b,c) \) must satisfy one of the above conditions.

Our result is related to the Nagell–Ljunggren equation

\[
\frac{x^m - 1}{x - 1} = y^n, \quad x \geq 2, \quad y \geq 2, \quad m \geq 3, \quad n \geq 2,
\]

which has been conjectured to have only three solutions \( (x,y,m,n) = (3,11,5,2), (7,20,4,2) \) and \( (18,7,3,3) \). Some of recent remarkable results concerning to the Nagell–Ljunggren equation are [2], [3], [4], [11] and [12]. Our result leads us to conjecture that the diophantine equation

\[
\frac{x^\ell - 1}{x - 1} = y^m z^n
\]

has only finitely many solutions in integers \( x \geq 2, \quad z \geq y \geq 2 \) and \( \ell, m, n \geq n_0 \) for some constant \( n_0 \). The abc-conjecture, which MOCHIZUKI [13] claims to prove, would allow us to take \( n_0 = 3 \). More exactly, assuming the abc-conjecture, we could prove that any integer solution of (3) with \( \ell \geq 3, m \geq 1, n \geq 2, \ 1 \leq y < z \), and \( x^\ell \) sufficiently large must satisfy \( (\ell, m, n) = (4,1,2), (3,1,3) \) or \( (\ell, n) = (3,2) \).

Indeed, applying the abc-conjecture to the equation \( 1 + (x - 1)y^m z^n = x^\ell \), we see that for any given \( \epsilon > 0 \), the inequality

\[
x^2 y z > x(x - 1)y z \geq \prod_{p|\ell(x - 1)yz^m z^n} p > x^{\ell(1-\epsilon/2)}
\]
would hold for a sufficiently large $x^\ell$. Hence, the inequality
\[
\frac{2}{\ell} + \frac{\ell - 1}{\ell} \times \frac{2}{n + \min\{n, m\}} > 1 - \epsilon
\] (5)
would also hold for a sufficiently large $x^\ell$. In particular, taking $\epsilon = 1/15$, we see
that the left of (5) must be greater than $14/15$ for a sufficiently large $x^\ell$. Recalling
that $n \geq 2$ and $m \geq 1$, we obtain that $\ell \leq 4$. We must have $n + \min\{n, m\} = 3$
for $\ell = 4$, and $n + \min\{n, m\} \leq 4$ for $\ell = 3$. So, either of $(\ell, m, n) = (4, 1, 2)$,
$(3, 1, 3)$ or $(\ell, n) = (3, 2)$ should hold for a sufficiently large $x^\ell$.

2. Preliminary lemmas

In this section, we introduce some preliminary lemmas. One is Matveev’s
lower bound for linear forms of logarithms [10].

Lemma 2.1. Let $a_1, a_2, \ldots, a_n$ be positive integers such that $\log a_1, \ldots,$
log $a_n$ are not all zero and $A_j \geq \max\{0.16, \log a_j\}$ for each $j = 1, 2, \ldots, n$. Moreover, we put
\[
B = \max\{1, b_1|A_1/A_n, |b_2|A_2/A_n, \ldots, |b_n|\},
\]
\[
\Omega = A_1A_2 \ldots A_n, \quad C_0 = 1 + \log 3 - \log 2,
\]
\[C_1(n) = \frac{8}{(n - 1)!}e^{\pi n + 2(n + 2)(4(n + 1))} \times (4.4n + 5.5 \log n + 7), \quad (6)\]
and
\[
\Lambda = b_1 \log a_1 + \ldots + b_n \log a_n. \quad (7)
\]
Then we have
\[
\log |\Lambda| > -C_1(n)(C_0 + \log B) \max \left\{1, \frac{n}{6} \right\} \Omega. \quad (8)
\]

The next lemmas deal with some arithmetical properties of values of cyclotomic polynomials. Lemma 2.2 is a basic and well-known result of this area. Lemma 2.2 has been proved by Zsigmondy [19], and rediscovered by many authors such as Dickson [6] and Kanold [7]. We need only the special case $b = 1$, for which this lemma had already been proved by Bang [1]. See also Theorem 6.4A.1 in [14].

Lemma 2.2. If $a > b \geq 1$ are coprime integers, then $a^n - b^n$ has a prime
factor which does not divide $a^m - b^m$ for any $m < n$, unless $(a, b, n) = (2, 1, 6)$,
$a - b = n = 1$, or $n = 2$, and $a + b$ is a power of 2.
Let $o_p(a)$ denote the residual order of $a \pmod{p}$. Lemma 2.2 immediately gives the following result.

**Lemma 2.3.** If $(a^e - 1)/(a - 1) = p^f q l^2$ for some integers $a, e, f_1, f_2$ and primes $p < q$, then we have $(a, e, p, q, f_1, f_2) = (2, 6, 3, 7, 2, 1)$, $e = r$ or $e = r^2$ for some prime $r$. Moreover, in the case $e = r$, then we have $p = r$, $o_p(a) = r$ or $a_p(a) = o_p(a) = r$. In the case $e = r^2$, we have $(r^{f_1}, q l^2) = ((a^r - 1)/(a - 1), (a^{r^2} - 1)/(a^{r^2} - 1))$, $(r^{f_1}, q l^2) = ((a^r - 1)/(a^r - 1), (a^r - 1)/(a - 1))$ or $(a, e, p, f_1) = (2^m - 1, 4, 2, m + 1)$ for some integer $m$.

**Proof.** If $e$ has at least two distinct prime factors and $(a, e) \neq (2, 6)$, then $e$ must have at least four distinct divisors. By Lemma 2.2, for each divisor $d > 1$ of $e$, $(a^d - 1)/(a - 1)$ has a prime factor which does not divide $(a^m - 1)/(a - 1)$ for any $m < d$, and therefore $(a^e - 1)/(a - 1)$ must have at least three distinct prime factors, contrary to the assumption. Hence, $(a, e) = (2, 6)$ or $e$ must be a prime power. If $e = r^l$ is power of a prime $r$, then, for each $k \leq l$, $(a^k - 1)/(a - 1)$ has a prime factor which divides none of $(a^m - 1)/(a - 1)$ with $m < r^k$. Thus we must have $l \leq 2$.

If $e = r$, then $o_p(a) = 1$ or $r$. If $o_p(a) = 1$, then $a \equiv 1 \pmod{p}$ and $(a^r - 1)/(a - 1) \equiv r \pmod{p}$. Hence, we must have $p = r$. Now we see that $q \equiv p = r$ and $o_q(a) = r$, since $o_q(a) = 1$ should yield $q = r$ as before, which is clearly a contradiction. If $o_p(a) = r$, then $r \equiv 1 \pmod{p}$, and therefore $q > p > r$. Hence, we must have $o_p(a) = o_q(a) = r$.

If $e = r^2$, then $(a^{r^2} - 1)/(a - 1)$ must have a prime factor not dividing $(a^r - 1)/(a - 1)$. Hence, $(a^r - 1)/(a - 1)$ must be a prime power. If a prime $p$ (or $q$) divides both $(a^r - 1)/(a - 1)$ and $(a^{r^2} - 1)/(a^{r^2} - 1)$, then, by Lemma 6.4A.2 in [14], we must have $o_p(a) = 1$ and $p = r$ (or $o_q(a) = 1$ and $q = r$). However, by Lemma 2.2, this occurs only if $r = 2$ and $a + 1 = 2^m$ for some integer $m$. □

The following lemma is proved in [3], as mentioned in the Introduction.

**Lemma 2.4.** Let $a, e, x, f$ be positive integers with $a, x, f > 1$ and $e > 2$. The equation $(a^e - 1)/(a - 1) = x^f$ has no solution but $(a, e, x, f) = (3, 5, 11, 2), (7, 4, 20, 2)$ in integers $2 \leq a \leq 10$, $e > 2$, $x > 1$, $f > 1$.

Using results mentioned in the Introduction, we can immediately solve some special case of our main theorem.

**Lemma 2.5.** Choose $a < b$ from the first three primes $2, 3, 5$. If $(a^{e_1} - 1)/(a - 1) = p^k$ and $(b^{f_1} - 1)/(b - 1) = p^l$ for some integers $e, f, k, l$ and some prime $p$, then $(a^e, b^f) = (2^5, 5^3)$ and $p = 31$, $k = l = 1$. 
On some simultaneous equations

PROOF. In the case \( k = l = 1 \) and \( (a^e - 1)/(a - 1) = (b^f - 1)/(b - 1) \), as observed in the Introduction, we have \( (a^e, b^f) = (2^5, 5^3) \).

Lemma 2.4 yields that the perfect power case must arise from \((3^5 - 1)/2 = 11^2\) or \((3^2 - 1)/2 = 2^2\). In this case, we must have \(2^e - 1 = 2\) or \(11\) or \((5^f - 1)/4 = 2\) or \(11\), which is clearly impossible. \(\square\)

3. Bounding the smallest exponent

For convenience, we put \(a_1 = 2\), \(a_2 = 3\), \(a_3 = 5\) and \(e_1 = a_1 + 1\), \(e_2 = b_1 + 1\), \(e_3 = c + 1\). In this section, we would like to give an absolute and explicit upper bound for the smallest one among \(a_i^{e_i}\)'s, which is the main part of our argument.

Lemma 3.1. For each \(i = 1, 2, 3\), we have

\[ e_i \log a_i < E_i = C_i \log p \log q (\log \log p \log q) + \log(1 - a_i^{e_i}) \]

where \(c_1 = 1.422 \times 10^{10}\), \(c_2 = 1.226 \times 10^{12}\), \(c_4 = 1.957 \times 10^{12}\), \(c_5 = 23.3\), \(c_6 = 27.8\), \(c_7 = 28.1\).

PROOF. Let \(\Lambda_i = f_i \log p + g_i \log q + \log(a_i - 1) - e_i \log a_i = \log(1 - a_i^{-e_i})\) for \(i = 1, 2, 3\). It immediately follows from Matveev’s theorem that

\[ -\log |\Lambda_1| < C(3) \left( C_0 + \log \left( \frac{e_1 \log 2}{\log q} \right) \right) \log 2 \log p \log q, \]

and

\[ -\log |\Lambda_j| < C(4) \left( C_0 + \log \left( \frac{e_j \log a_j}{\log q} \right) \right) \log 2 \log a_j \log p \log q, \]

for \(j = 2, 3\).

Now we shall prove (9) in the case \(i = 1\). We may assume that \(e_1 > 10^{10} \log q / \log 2\). Since \(0 < |\Lambda_1| = -\log(1 - 2^{-e_1}) < 1/(2^{e_1} - 1)\), we have

\[ -\log |\Lambda_1| > \log(2^{e_1} - 1) > (1 - 10^{-10})e_1 \log 2. \]

Combining upper and lower bounds for \(\Lambda_1\), we obtain

\[ \frac{e_1 \log 2}{\log q} < (1 + 10^{-10}) \left( C_0 + \log \left( \frac{e_1 \log 2}{\log q} \right) \right) C(3) \log 2 \log p \]

\[ < 1.244 \times 10^{10} \log p \log \left( \frac{e_1 \log 2}{\log q} \right). \]
Hence, observing that $1.244 \times 10^{10} \log p \geq 1.244 \times 10^{10} \log 2$, we obtain
\[
\frac{e_1 \log 2}{\log q} < 1.143 \times (1.244 \times 10^{10} \log p) \log(1.244 \times 10^{10} \log p)
\]
\[
< 1.422 \times 10^{10} (\log \log p + 23.3),
\]
giving (9) in the case $i = 1$.

Next we shall prove (9) in the case $i = 2$. We may assume that $e_2 > 10^{10} \log q / \log 3$ as in the previous case. From $0 < |\Lambda_2| = -\log(1 - 3^{-e_2}) < 1/(3^{e_2} - 1)$, we see that
\[
-\log |\Lambda_2| > \log(3^{e_2} - 1) > (1 - 10^{-10})e_2 \log 3,
\]
and therefore
\[
\frac{e_2 \log 3}{\log q} < (1 + 10^{-10}) \left( C_0 + \log \left( \frac{e_2 \log 3}{\log q} \right) \right) C(4) \log 2 \log 3 \log p
\]
\[
< 1.089 \times 10^{12} \log p \log \left( \frac{e_2 \log 3}{\log q} \right).
\]
This gives (9) in the case $i = 2$.

Similarly, (9) in the case $i = 3$ follows from
\[
\frac{e_3 \log 5}{\log q} < (1 + 10^{-10}) \left( C_0 + \log \left( \frac{e_3 \log 5}{\log q} \right) \right) C(4) \log 2 \log 5 \log p
\]
\[
< 1.595 \times 10^{12} \log p \log \left( \frac{e_3 \log 5}{\log q} \right).
\]
This completes the proof of the lemma.

Next, we shall show that we cannot have all of $a_i^{e_i}$'s large.

**Lemma 3.2.** Let $x$ be the smallest among $a_i^{e_i}$'s. Let $h_1 = f_2g_3 - f_3g_2$, $h_2 = f_3g_1 - f_1g_3$ and $h_3 = f_1g_2 - f_2g_1$ and $H = \max |h_i|$. Then
\[
\log x \leq \log \left( \frac{7H}{4} \right) + C(3)(C_0 + \log((e_1 + 3)H)) \log 2 \log 3 \log 5.
\]

**Proof.** We begin by observing that
\[
(2^{e_1} - 1)^{h_1} \left( \frac{3^{e_2} - 1}{2} \right)^{h_2} \left( \frac{5^{e_3} - 1}{4} \right)^{h_3} = 1.
\]
Now we put
\[ \Lambda = (e_1 - 2h_2 - 2h_3) \log 2 + e_2 h_2 \log 3 + e_3 h_3 \log 5 \]
\[ = h_1 \log \left( \frac{2e_1}{2e_1 - 1} \right) + h_2 \log \left( \frac{3e_2}{3e_2 - 1} \right) + h_3 \left( \log \frac{5e_3}{5e_3 - 1} \right). \tag{20} \]

Then we have
\[ 0 < |\Lambda| \leq H \left( \frac{1}{2e_1 - 1} + \frac{1}{3e_2 - 1} + \frac{1}{5e_3 - 1} \right) \leq \frac{7H}{4x}. \tag{21} \]

and therefore
\[ \log |\Lambda| \leq -\log x + \log \left( \frac{7H}{4} \right). \tag{22} \]

It follows from the assumption \( e_i > 0 \) that \( \Lambda \neq 0 \). Hence, Matveev's lower bound gives
\[ \log |\Lambda| \geq -C(3)(C_0 + \log((e_1 + 3)H)) \log 2 \log 3 \log 5. \tag{23} \]

Combining (22) and (23), we obtain (18). \( \square \)

The third step is to obtain upper bounds for each \( e_i \).

**Lemma 3.3.** Unless \( x = p = 31 \), we have \( e_1 < 4.44 \times 10^{52}, e_2 < 2.54 \times 10^{54} \) and \( e_3 < 2.55 \times 10^{54}, \) and \( H < 2.89 \times 10^{68} \).

**Proof.** We may assume without the loss of generality that \( p < q \). We begin by considering the case \( q \mid x \). In this case, we have
\[ \log q < \log x < \log \left( \frac{7H}{4} \right) + C(3)(C_0 + \log((e_1 + 3)H)) \log 2 \log 3 \log 5. \tag{24} \]

We note that
\[ H \leq C_2 C_3 \log p \log q \log \log p + C_5 (\log \log p + C_6), \tag{23} \]

since it follows from Lemma 3.1 that
\[ f_i < e_i \log a_i / \log p_i < C_i \log q (\log \log p + C_i + 3) \tag{26} \]

and
\[ g_i < e_i \log a_i / \log q_i < C_i \log p (\log \log p + C_i + 3). \tag{27} \]
Hence, we obtain $\log p < \log q < 4.35 \times 10^{12}$.

Now we consider the case $p < q$ and $q \nmid x$. Put $i$ to be the index such that $x = (a_i^{e_i} - 1)/(a_i - 1)$, $j, k$ be the other two and

$$
\Lambda' = e_j h_j \log a_j + e_k h_k \log a_k - h_j \log(a_j - 1) - h_k \log(a_k - 1) + h_i \log x
$$

$$
= h_j \log \left( \frac{a_j^{e_j}}{a_j^{e_j} - 1} \right) + h_k \log \left( \frac{a_k^{e_k}}{a_k^{e_k} - 1} \right).
$$

(28)

It follows from Lemma 2.5 that if $(a_j^{e_j} - 1)/(a_j - 1) = p^{f_j}$ or $(a_k^{e_k} - 1)/(a_k - 1) = p^{f_k}$, then $a_i^{e_i} = 2^{f_i}$ or $5^{f_i}$ and $x = p = 31$. Hence, we see that both numbers $(a_j^{e_j} - 1)/(a_j - 1), (a_k^{e_k} - 1)/(a_k - 1)$ must be divisible by $q$ unless $x = p = 31$.

Thus we obtain

$$
0 < \Lambda' < H \left( \frac{1}{a_j^{e_j} - 1} + \frac{1}{a_k^{e_k} - 1} \right) \leq \frac{3H}{2q}.
$$

(29)

As in the previous case, Matveev’s theorem now gives

$$
\log |\Lambda'| \geq -C(4) \left( C_0 + \log \left( \frac{E_3 H}{\log x} \right) \right) \log 2 \log 3 \log 5 \log x.
$$

(30)

Combining (29) and (30), we obtain

$$
\log q \leq \log \left( \frac{3H}{2} \right) + C(4) \left( C_0 + \log \left( \frac{E_3 H}{\log x} \right) \right) \log 2 \log 3 \log 5 \log x.
$$

(31)

Since

$$
E_3 = C_3 \log p \log q (\log \log p + C_6) \leq C_3 \log x \log q (\log \log x + C_6)
$$

(32)

and

$$
H < C_2 C_3 (\log q)^2 (\log \log q + C_5) (\log \log q + C_6),
$$

(33)

combining (18) and (31), we obtain $\log q < 3.45 \times 10^{27}$. Moreover, we have

$$
\log p = \log x < \log \left( \frac{7H}{4} \right) + C(3) (C_0 + \log((e_1 + 3)H)) \log 2 \log 3 \log 5
$$

$$
< 7.22 \times 10^{12}.
$$

(34)

So, we conclude that in both cases, we have $\log p < 7.22 \times 10^{12}$ and $\log q < 3.45 \times 10^{27}$. Now Lemma 3.1 immediately gives that $e_1 < 4.44 \times 10^{52}$, $e_2 < 2.54 \times 10^{54}$ and $e_3 < 2.55 \times 10^{54}$. Finally, the upper bound $H < 2.89 \times 10^{68}$ follows from $H < C_2 C_3 (\log p)(\log q)(\log \log p + C_6)(\log \log q + C_5)$. □
Now, using the lattice reduction algorithm, we shall obtain feasible upper bounds.

**Lemma 3.4.** We have \( \log x < 354.8 \). Moreover, if \( p < q \) and \( q \) divides \( x \), then \( \log x < 249.5 \).

**Proof.** We begin by noting that we can assume \( x \neq 31 \) without the loss of generality.

In order to reduce our upper bounds, we use the LLL lattice reduction algorithm introduced in [8]. Let \( M \) be the matrix defined by

\[
\begin{align*}
    m_{12} &= m_{13} = m_{21} = m_{23} = 0, \\
    m_{11} &= m_{22} = \gamma \\
    m_{3i} &= \lfloor C \gamma \log a_i \rfloor 
\end{align*}
\]

for \( i = 1, 2, 3 \), where \( C \) and \( \gamma \) are constants chosen later. Let \( L \) denote the LLL-reduced matrix of \( M \), and \( l(L) \) the shortest length of vectors in the lattice generated by the column vectors of \( L \).

From the previous lemma, we know that \( \Lambda \) has coefficients with absolute values at most \( H \max \{e_1 + 3, e_2, e_3\} < 7.37 \times 10^{122} \). It is implicit in the proof of Lemma 3.7 of de Weger’s book [17] that if \( l(\Gamma) > X \sqrt{16 + 4 \gamma} \) and \( X \geq 7.37 \times 10^{122} \), then \( |\Lambda| > X / (C \gamma) \).

Taking \( C = 10^{370} \), \( \gamma = 2 \), we can confirm that \( l(\Gamma) > X \sqrt{16 + 4 \gamma} \), and therefore we obtain that \( |\Lambda| > 3.685 \times 10^{-248} \). Hence, we have

\[
\log x < \log \left( \frac{7H}{4} \right) - \log |\Lambda| < 727.94. 
\]

We choose the index \( i \) such that \( x = (a_i e_i - 1) / (a_i - 1) \), and let \( j, k \) be the others. From the above estimate for \( x \), we derive that

\[
e_i \leq \left\lfloor \frac{\log 2x}{\log a_i} \right\rfloor \leq 1051. 
\]  

We consider the case \( p < q \) and \( q \) does not divide \( x \). From (31) we obtain \( \log q < 3.337 \times 10^{17} \). Lemma 3.1 gives that

\[
|h_i| < C_2 C_3 \log x \log q (\log \log x + C_6) (\log \log q + C_5) < 1.264 \times 10^{48}, \\
|h_j| = |f_i g_k| < C_3 \log x (\log \log q + C_6) < 8.944 \times 10^{16}, \\
|e_j| < C_3 \log x \log q (\log \log q + C_6) / \log 2 < 4.306 \times 10^{34},
\]

and similar upper bounds hold for \( |h_k| \) and \( |e_k| \), respectively. Hence, \( \Lambda \) has coefficients with absolute values at most \( 3.852 \times 10^{351} \). Using the LLL-reduction again with \( C = 10^{157} \) and \( \gamma = 2 \), we obtain \( |\Lambda| > 1.926 \times 10^{-106} \), and therefore \( \log x < \log (7H/4) - \log |\Lambda| < 354.8 \).
Next, we consider the case $p < q$ and $q$ divides $x$. In this case, we have $\log p < \log q \leq \log x < 727.94$. We choose the index $i$ such that $x = (a_i^e - 1)/(a_i - 1)$ and let $j, k$ be the other two. Lemma 3.1 gives that

\[
|h_i| < C_2 C_3 \log^2 x (\log \log x + C_5) (\log \log x + C_6) < 1.392 \times 10^{33},
\]
\[
|h_j| \leq \max |f_i g_k, f_k g_i| < C_3 \log x (\log \log x + C_6) < 4.533 \times 10^{16},
\]
\[
|e_j| < C_3 \log^2 x (\log \log x + C_6)/\log 2 < 4.761 \times 10^{19},
\]

and similar upper bounds hold for $|h_k|$ and $|e_k|$, respectively. Combining these upper bounds with (36), we see that $\Lambda$ has coefficients with absolute values at most $2.159 \times 10^{36}$. We use the LLL-reduction again with $C = 10^{111}$ and $\gamma = 2$, we obtain $|\Lambda| > 1.079 \times 10^{-75}$, and therefore $\log x < \log(7H/4) - \log |\Lambda| < 249.5$. This proves the lemma. \hfill \square

4. Checking the small ranges

The final step is checking all possibilities of $x$. We note that from The Cunningham Project (see [15] or [5]), we know all prime factors of $x$'s below our upper bounds.

For $x = (a_i^e - 1)/(a_i - 1)$, we should check the residual orders of the other prime $a_i$ modulo $x$. A summary is given in Tables 1–6, where $P_n$ such as P13 in the row $e_1 = 49$ denotes a prime with $n$ digits, and $(n)$ indicates that the residual order is a multiple of $n$. For example, putting $x = 2^{447} - 1 = pq$ with $p < q$, $\alpha_q(3)$ is divisible by 6, since $q - 1$ is divisible by $2^3 \times 3^2$ and $3^{(q-1)/3}, 3^{(q-1)/3} \neq 1 \pmod q$, although $3^{(q-1)/4} \equiv 1 \pmod q$, which yields that $(3^{e_2} - 1)/2 = p^{g_2} q^{g_2}$ with $g_2 > 0$ is impossible.

If $p = x = 2^{e_1} - 1$ is prime, then $e_1 \leq 511$, and therefore $e_1$ must belong to the set

\[
\{2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127\}.
\]

Among them, there exists no $e_1$ such that $\alpha_q(3) = 1$, being a prime or the square of a prime, as we can see from Table 1. Hence, by Lemma 2.3, we must have $(3^{e_2} - 1)/2 = q^{g_2}$. By Lemma 2.5, $(5^{e_3} - 1)/4$ must be divisible by $p = x$. Hence, by Lemma 2.3, $\alpha_q(5) = 1$ or $\alpha_q(5)$ must be a prime or the square of a prime, and therefore, from Table 1, $e_1 = e_3 = 2$ or $e_1 = 5, e_3 = 3$. If $e_1 = e_3 = 2$, then $(5^{e_3} - 1)/4 = 6 = 2 \times 3$, and therefore $(3^{e_2} - 1)/2$ must be a power of 2, yielding that $e_2 = 2$. If $e_1 = 5$ and $e_3 = 3$, then $p = 31$ and $(3^{e_2} - 1)/2 = q^{g_2}$, yielding that $e_2 = 5$ or $(3^{e_2} - 1)/2 = q$. 

If \( x = 2^{e_1} - 1 \) is not a prime power, then \( e_1 \leq 359 \), and therefore \( e_1 \) must belong to the set

\[
\{4, 6, 9, 11, 23, 37, 41, 49, 59, 67, 83, 97, 101, 103, 109, 131, 137, 139, 149, 167, 197, 199, 227, 241, 269, 271, 281, 293, 347\}.
\]

Hence, we can write \( x = 2^{e_1} - 1 = pq \) for distinct primes \( p < q \). By Lemma 2.5, \( (3^{e_2} - 1)/2 = p^{q2} \) and \( (5^{e_3} - 1)/4 = p^{q3} \) cannot simultaneously hold. In other words, at least one of these two integers must be divisible by \( q \). But, for no \( e_1 \) in the above set, \( o_q(5) \) is 1, a prime or prime-square, as can be seen from Table 2. Hence, \( (3^{e_2} - 1)/2 \) must be divisible by \( q \). The only \( e_1 \) for which \( o_q(3) \) is 1, a prime or prime-square is \( e_1 = 4 \). Then we must have \( x = 2^4 - 1 = 3 \times 5 \) and \( (p, q) = (3, 5) \). But this implies that \( e_2 \) is divisible by 4, and \( (3^{e_2} - 1)/2 \) must be divisible by 2. Hence, \( (3^{e_2} - 1)/2 \) cannot be of the form \( p^{f_2} q^{g_2} \). Hence, it cannot occur that \( x = 2^{e_1} - 1 \) is not a prime power.

If \( x = (3^{e_2} - 1)/2 = p^{f_2} \) is prime or prime power, then

\[
e_2 \in \{2, 3, 5, 7, 13, 71, 103\}.
\]

For none of them, \( o_p(2) = 1, 6 \) or a prime power. Hence, as above, \( (5^{e_3} - 1)/4 \) must be divisible by \( p \). Since \( o_p(5) \) must be 1 or a prime power, we must have \( e_2 \in \{2, 3, 5\} \). If \( e_2 = 2 \), then \( p = 2 \) and \( e_3 = 2 \), which yields that \( q = 3 \) and \( e_1 = 2 \). If \( e_2 = 3 \), then \( p = 13 \) and \( e_3 = 4 \), which is impossible since \( (5^{e_3} - 1)/4 = 781 = 11 \times 71 \). This implies that \( 2^{e_1} - 1 = 11^{f_1} 71^{g_1} \), which is impossible since \( 2^{10} - 1 = 3 \times 11 \times 31 \) and \( 2^{35} - 1 = 31 \times 71 \times 127 \times 122921 \).

If \( x = (3^{e_2} - 1)/2 \) is not a prime power, then

\[
e_2 \in \{9, 11, 17, 19, 23, 37, 43, 59, 61, 223\}.
\]

Hence, we can write \( x = (3^{e_1} - 1)/2 = pq \) for distinct primes \( p < q \) with \( p, q \neq 31 \). However, \( o_q(2) \) or \( o_q(5) \) can never be \( 1, 6 \), or a prime power among the above \( e_2 \)'s. Hence, both \( 2^{e_1} - 1 \) and \( (5^{e_3} - 1)/4 \) must be a power of \( p \). By Lemma 2.5, we must have \( p = 31 \), which is impossible as mentioned above.

If \( x = (5^{e_3} - 1)/4 \) is a prime power, then

\[
e_3 \in \{3, 7, 11, 13, 47, 127, 149, 181\}.
\]

Among them, no \( e_3 \) gives a prime power (or one) residual order 3 \( \pmod{x} \), and only \( e_3 = 3 \) makes the residual order 2 \( \pmod{x} \) acceptable in view of Lemma 2.3.
Hence, $p = 31$, $e_3 = 3$, $e_1 = 5$ and $(3^{e_2} - 1)/2 = q^{f_2}$, which implies that $e_2 = 5$ or $(3^{e_2} - 1)/2 = q$.

If $x = (5^{e_3} - 1)/4$ is not a prime power, then

$$e_3 \in \{2, 5, 17, 23, 31, 41, 43, 59, 71\}.$$  

Hence, we can write $x = (5^{e_3} - 1)/4 = pq$ for distinct primes $p < q$. None of such $e_3 > 2$ gives an acceptable residual order $2 \pmod{q}$ or $3 \pmod{q}$ in view of Lemma 2.3. Hence, we see that neither $2^{e_1} - 1$ nor $(3^{e_2} - 1)/2$ can be divisible by $q$, and both must be a power of $p$, contrary to Lemma 2.5. Hence, we must have $e_3 = 2$, $(p, q) = (2, 3)$. This yields that $e_1 = e_2 = 2$.

This completes the proof of Theorem 1.1. $\square$

Table 1. The residual orders of $3, 5$ modulo $p$ for $p = 2^{e_1} - 1$.

<table>
<thead>
<tr>
<th>$e_1$</th>
<th>$o_p(3)$</th>
<th>$o_p(5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>N/A</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>30</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>126</td>
<td>42</td>
</tr>
<tr>
<td>13</td>
<td>910</td>
<td>1365</td>
</tr>
<tr>
<td>17</td>
<td>131070</td>
<td>65535</td>
</tr>
<tr>
<td>19</td>
<td>524286</td>
<td>74898</td>
</tr>
<tr>
<td>31</td>
<td>715827882</td>
<td>195225780</td>
</tr>
<tr>
<td>61</td>
<td>(10)</td>
<td>(15)</td>
</tr>
<tr>
<td>89</td>
<td>(6)</td>
<td>(84)</td>
</tr>
<tr>
<td>107</td>
<td>(6)</td>
<td>(6)</td>
</tr>
<tr>
<td>127</td>
<td>(6)</td>
<td>(6)</td>
</tr>
</tbody>
</table>

Table 2. The residual orders of $3, 5$ modulo $p, q$ for $pq = 2^{e_1} - 1$, $p < q$.

<table>
<thead>
<tr>
<th>$e_1$</th>
<th>$2^{e_1} - 1 = pq$</th>
<th>$o_p(3)$</th>
<th>$o_q(3)$</th>
<th>$o_p(5)$</th>
<th>$o_q(5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>$7 \times 73$</td>
<td>6</td>
<td>12</td>
<td>6</td>
<td>72</td>
</tr>
<tr>
<td>11</td>
<td>$23 \times 89$</td>
<td>11</td>
<td>88</td>
<td>22</td>
<td>44</td>
</tr>
<tr>
<td>23</td>
<td>$47 \times 178481$</td>
<td>23</td>
<td>178480</td>
<td>46</td>
<td>44620</td>
</tr>
<tr>
<td>37</td>
<td>$223 \times 616318177$</td>
<td>222</td>
<td>308159088</td>
<td>222</td>
<td>616318176</td>
</tr>
<tr>
<td>41</td>
<td>$13367 \times 164511353$</td>
<td>6683</td>
<td>164511352</td>
<td>13366</td>
<td>164511352</td>
</tr>
<tr>
<td>49</td>
<td>$127 \times P13$</td>
<td>126</td>
<td>(8)</td>
<td>42</td>
<td>(8)</td>
</tr>
<tr>
<td>59</td>
<td>$179951 \times P13$</td>
<td>89975</td>
<td>(8)</td>
<td>89975</td>
<td>(8)</td>
</tr>
<tr>
<td>67</td>
<td>$193707721 \times P12$</td>
<td>96853860</td>
<td>(6)</td>
<td>8071155</td>
<td>(6)</td>
</tr>
<tr>
<td>83</td>
<td>$167 \times P23$</td>
<td>83</td>
<td>(10)</td>
<td>166</td>
<td>(166)</td>
</tr>
</tbody>
</table>

Continued on next page
Table 2 – Continued from previous page

<table>
<thead>
<tr>
<th>(e_1)</th>
<th>(2^{e_1} - 1 = pq)</th>
<th>(o_p(3))</th>
<th>(o_p(5))</th>
<th>(o_p(5))</th>
</tr>
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<tr>
<td>97</td>
<td>11447 \times P26</td>
<td>5723</td>
<td>(194)</td>
<td>11446</td>
</tr>
<tr>
<td>101</td>
<td>P13 \times P14</td>
<td>(303)</td>
<td>(303)</td>
<td>(303)</td>
</tr>
<tr>
<td>103</td>
<td>2550183799 \times P22</td>
<td>(166)</td>
<td>(296)</td>
<td>(249)</td>
</tr>
<tr>
<td>109</td>
<td>745988807 \times P24</td>
<td>(11663)</td>
<td>(118)</td>
<td>(214)</td>
</tr>
<tr>
<td>131</td>
<td>263 \times P38</td>
<td>131</td>
<td>(74)</td>
<td>262</td>
</tr>
<tr>
<td>137</td>
<td>P20 \times P22</td>
<td>(274)</td>
<td>(66290053)</td>
<td>(1202723)</td>
</tr>
<tr>
<td>139</td>
<td>P13 \times P30</td>
<td>(6)</td>
<td>(6)</td>
<td>(6)</td>
</tr>
<tr>
<td>149</td>
<td>P20 \times P25</td>
<td>(745)</td>
<td>(16)</td>
<td>(745)</td>
</tr>
<tr>
<td>167</td>
<td>2349023 \times P44</td>
<td>(26)</td>
<td>(22)</td>
<td>(26)</td>
</tr>
<tr>
<td>197</td>
<td>7487 \times P56</td>
<td>(3743)</td>
<td>(394)</td>
<td>(38)</td>
</tr>
<tr>
<td>199</td>
<td>P12 \times P49</td>
<td>(14)</td>
<td>(1393)</td>
<td>(8)</td>
</tr>
<tr>
<td>227</td>
<td>P18 \times P52</td>
<td>(8)</td>
<td>(35)</td>
<td>(8)</td>
</tr>
<tr>
<td>241</td>
<td>22000409 \times P66</td>
<td>(8)</td>
<td>(5114261)</td>
<td>(482)</td>
</tr>
<tr>
<td>269</td>
<td>13822297 \times P74</td>
<td>(6)</td>
<td>(6)</td>
<td>(6)</td>
</tr>
<tr>
<td>271</td>
<td>15242475217 \times P72</td>
<td>(8)</td>
<td>(542)</td>
<td>(8)</td>
</tr>
<tr>
<td>281</td>
<td>80929 \times P80</td>
<td>(8)</td>
<td>(278)</td>
<td>(6)</td>
</tr>
<tr>
<td>293</td>
<td>P26 \times P63</td>
<td>(6)</td>
<td>(6)</td>
<td>(8)</td>
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<tr>
<td>347</td>
<td>P23 \times P82</td>
<td>(6)</td>
<td>(6)</td>
<td>(21)</td>
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</tbody>
</table>

Table 3. The residual orders of 2, 5 modulo \(p\) for \(p^{1/2} = (3^{e_2} - 1)/2\).

<table>
<thead>
<tr>
<th>(e_2)</th>
<th>(o_p(2))</th>
<th>(o_p(5))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>N/A</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>7</td>
<td>1092</td>
<td>364</td>
</tr>
<tr>
<td>13</td>
<td>398580</td>
<td>30660</td>
</tr>
<tr>
<td>71</td>
<td>(8)</td>
<td>(8)</td>
</tr>
<tr>
<td>103</td>
<td>(12)</td>
<td>(14)</td>
</tr>
</tbody>
</table>
Table 4. The residual orders of $2, 5$ modulo $p, q$ for $pq = (3^2 - 1)/2$, $p < q$.

<table>
<thead>
<tr>
<th>$e_2$</th>
<th>$(3^2 - 1)/2 = pq$</th>
<th>$o_p(2)$</th>
<th>$o_q(2)$</th>
<th>$o_p(5)$</th>
<th>$o_q(5)$</th>
</tr>
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<tbody>
<tr>
<td>9</td>
<td>$13 \times 757$</td>
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<td>756</td>
<td>4</td>
<td>756</td>
</tr>
<tr>
<td>11</td>
<td>$23 \times 3851$</td>
<td>11</td>
<td>3850</td>
<td>22</td>
<td>1925</td>
</tr>
<tr>
<td>17</td>
<td>$1871 \times 3451$</td>
<td>935</td>
<td>595</td>
<td>935</td>
<td>3451</td>
</tr>
<tr>
<td>19</td>
<td>$1597 \times 363889$</td>
<td>532</td>
<td>181944</td>
<td>532</td>
<td>22743</td>
</tr>
<tr>
<td>23</td>
<td>$47 \times 1001523179$</td>
<td>23</td>
<td>(46)</td>
<td>46</td>
<td>(1073)</td>
</tr>
<tr>
<td>37</td>
<td>$1300927 \times P12$</td>
<td>(9731)</td>
<td>8594564351</td>
<td>(74)</td>
<td>(74)</td>
</tr>
<tr>
<td>43</td>
<td>$431 \times P18$</td>
<td>43</td>
<td>215</td>
<td>(22)</td>
<td>(22)</td>
</tr>
<tr>
<td>59</td>
<td>$14425532687 \times P18$</td>
<td>(3953)</td>
<td>(118)</td>
<td>(106)</td>
<td>(10679)</td>
</tr>
<tr>
<td>61</td>
<td>$603901 \times P24$</td>
<td>201300</td>
<td>(12)</td>
<td>150975</td>
<td>(145)</td>
</tr>
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<td>223</td>
<td>$P26 \times P81$</td>
<td>(446)</td>
<td>(12)</td>
<td>(6)</td>
<td>(446)</td>
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</tbody>
</table>

Table 5. The residual orders of $2, 3$ modulo $p$ for $p = (5^3 - 1)/4$.

<table>
<thead>
<tr>
<th>$e_3$</th>
<th>$o_p(2)$</th>
<th>$o_p(3)$</th>
</tr>
</thead>
<tbody>
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<td>3</td>
<td>5</td>
<td>30</td>
</tr>
<tr>
<td>7</td>
<td>6510</td>
<td>6510</td>
</tr>
<tr>
<td>11</td>
<td>1220703</td>
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<td>13</td>
<td>61035156</td>
<td>1211015</td>
</tr>
<tr>
<td>47</td>
<td>(94)</td>
<td>(6)</td>
</tr>
<tr>
<td>127</td>
<td>(18)</td>
<td>(18)</td>
</tr>
<tr>
<td>149</td>
<td>(10)</td>
<td>(6)</td>
</tr>
<tr>
<td>181</td>
<td>(12)</td>
<td>(15)</td>
</tr>
</tbody>
</table>

Table 6. The residual orders of $2, 3$ modulo $p, q$ for $pq = (5^3 - 1)/4$, $p < q$.

<table>
<thead>
<tr>
<th>$e_3$</th>
<th>$(5^3 - 1)/4 = pq$</th>
<th>$o_p(2)$</th>
<th>$o_q(2)$</th>
<th>$o_p(3)$</th>
<th>$o_q(3)$</th>
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<tbody>
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<td>2</td>
<td>$2 \times 3$</td>
<td>N/A</td>
<td>2</td>
<td>1</td>
<td>N/A</td>
</tr>
<tr>
<td>5</td>
<td>$11 \times 71$</td>
<td>10</td>
<td>35</td>
<td>5</td>
<td>35</td>
</tr>
<tr>
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<td>$409 \times 466344409$</td>
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<td>3429003</td>
<td>204</td>
<td>116586102</td>
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<tr>
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<td>$8971 \times P12$</td>
<td>8970</td>
<td>(8)</td>
<td>8970</td>
<td>2306995565</td>
</tr>
<tr>
<td>31</td>
<td>$1861 \times P18$</td>
<td>1860</td>
<td>(15)</td>
<td>310</td>
<td>(6)</td>
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<tr>
<td>41</td>
<td>$2238236249 \times P19$</td>
<td>279779531</td>
<td>(8)</td>
<td>(8)</td>
<td>(8)</td>
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<td>43</td>
<td>$1644512641 \times P21$</td>
<td>(8)</td>
<td>(15)</td>
<td>(8)</td>
<td>(10)</td>
</tr>
<tr>
<td>59</td>
<td>$P17 \times P25$</td>
<td>(12)</td>
<td>(9)</td>
<td>(6)</td>
<td>(118)</td>
</tr>
<tr>
<td>71</td>
<td>$569 \times P47$</td>
<td>284</td>
<td>(142)</td>
<td>568</td>
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</tbody>
</table>
On some simultaneous equations

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