Characterization of the Euler gamma function
with the aid of an arbitrary mean

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Abstract. We prove that a continuous function \( f : (0, \infty) \rightarrow (0, \infty) \) satisfying the functional equation
\[
f(x + 1) = x f(x), \quad x > 0, \quad f(1) = 1,
\]
is the Euler gamma function iff for some \( a > 0 \) and a strict and continuous mean \( M : (a, \infty)^2 \rightarrow (a, \infty) \), the following inequality holds:
\[
f(M(x, y)) f \left( \frac{xy}{M(x, y)} \right) \leq f(x) f(y), \quad x, y \in (a, \infty).
\]

Taking for \( M \) the geometric mean \( G(x, y) = \sqrt{xy} \), we obtain the result of [2] generalizing the classical BOHR–MOLLERUP theorem [1]. For \( M = A \), where \( A(x, y) = \frac{x + y}{2} \) is the arithmetic mean, the assumed inequality reduces to \( f(A(x, y)) f(H(x, y)) \leq f(x) f(y) \) for all \( x, y > a \), where \( H \) is the harmonic mean, and the result gives a new characterization of the gamma function, involving the arithmetic and harmonic means.

1. Introduction

According to the celebrated result of BOHR and MOLLERUP [1], the Euler gamma function \( \Gamma \) is the only function \( f : (0, \infty) \rightarrow (0, \infty) \) satisfying the functional equation
\[
f(x + 1) = x f(x) \quad \text{for all } x > 0, \quad f(1) = 1,
\]
and such that \( \log \circ f \) is convex.

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This theorem has been improved in [2], where it is shown that $\Gamma$ is the only function $f : (0, \infty) \to (0, \infty)$ satisfying (1) and such that $\log \circ f \circ \exp$ is convex in a vicinity of $\infty$. Interpreting this result, note that [2], for a positive real function $f$ defined in an interval $I \subset (0, \infty)$ and continuous at least at one point, the function $\log \circ f \circ \exp$ is convex in the interval $\log (I)$, iff $f$ is Jensen geometrically convex in $I$, i.e., iff

$$f \left( G(x, y) \right) \leq G \left( f(x) f(y) \right), \quad x, y \in I,$$

where $G(x, y) := \sqrt{xy}$ is the geometric mean.

In the present paper, we show that any two-variable strict and continuous mean defined in a vicinity of $\infty$ can be used in a characterization of the Euler gamma function. The main results imply that: a continuous function $f : (0, \infty) \to (0, \infty)$ satisfying (1) is the Euler gamma function iff there is an $a > 0$ and a strict continuous mean $M : (a, \infty)^2 \to (a, \infty)$ such that, for all $x, y > a$,

$$f \left( M(x, y) \right) f \left( \frac{xy}{M(x, y)} \right) \leq f(x) f(y).$$

Taking $M = G$, one gets the result of [2]; taking $M = A$, where $A$ is the arithmetic mean, we obtain the following characterization of the gamma function: a continuous function $f : (0, \infty) \to (0, \infty)$ satisfying (1) is the Euler gamma function, if and only if there is an $a > 0$ such that

$$f \left( A(x, y) \right) f \left( H(x, y) \right) \leq f(x) f(y), \quad x, y \in (a, \infty),$$

where $H$ is the harmonic mean.

2. Auxiliary results

Let $I \subset \mathbb{R}$ be an interval. A function $M : I \times I \to I$ is called a bivariable mean in $I$ if

$$\min (x, y) \leq M(x, y) \leq \max (x, y), \quad x, y \in I,$$

and the mean $M$ is called strict, if these inequalities are sharp for all distinct $x, y \in I$.

We shall need the following:
Lemma 1 ([5, Theorem 1]). Let $M : I^2 \to I$ and $N : I^2 \to I$ be continuous means in $I$. If, for all $x, y \in I$, $x \neq y$,

$$\max (M(x, y), N(x, y)) - \min (M(x, y), N(x, y)) < \max (x, y) - \min (x, y),$$

then there exists a unique mean $K : I^2 \to I$ that is invariant with respect to the mean-type mapping $(M, N) : I^2 \to I^2$, i.e., such that

$$K (M(x, y), N(x, y)) = K (x, y), \quad x, y \in I;$$

moreover, the sequence $((M, N)^n)_{n \in \mathbb{N}}$ of iterates of the mapping $(M, N)$ converges pointwise in $I^2$ and

$$\lim_{n \to \infty} (M, N)^n (x, y) = (K(x, y), K(x, y)), \quad (x, y) \in I^2.$$

Remark 1. It is obvious that condition (3) is satisfied if one of the means $M$ or $N$ is strict.

From Lemma 1, we obtain

Lemma 2. Let $I \subseteq (0, \infty)$ be an interval, $M : I^2 \to I$ be a (strict) mean, and let $N : I^2 \to I$ be given by

$$N (x, y) := \frac{xy}{M(x, y)}, \quad x, y \in I.$$

Then

(i) the function $N$ is a (strict) mean in $I$;

(ii) the geometric mean $G$ is invariant with respect the mean-type mapping $(M, N)$, i.e., $G \circ (M, N) = G$;

(iii) for every $n \in \mathbb{N}$, the mapping $(M, N)^n$, the $n$-th iterate of $(M, N)$, is a mean-type mapping;

(iv) if $M$ is a continuous and strict mean, then the sequence $((M, N)^n)_{n \in \mathbb{N}}$ of iterates of $(M, N)$ converges pointwise in $I^2$ and

$$\lim_{n \to \infty} (M, N)^n (x, y) = (\sqrt{x}, \sqrt{y}), \quad (x, y) \in I^2.$$

Proof. To prove (i), it is enough to observe that condition (2) is equivalent to

$$\min (x, y) \leq \frac{xy}{M(x, y)} \leq \max (x, y), \quad x, y \in I,$$

and these inequalities are sharp iff so are inequalities (2).
To verify (ii), note that for all \( x, y \in I \), we have

\[
G \circ (M(x, y), N(x, y)) = \sqrt{M(x, y) \frac{xy}{M(x, y)}} = \sqrt{xy} = G(x, y).
\]

Part (iii) is an obvious consequence of the definition of mean. Part (iv) follows from (ii) and Lemma 1. \( \square \)

Remark 2. In [4], the mean \( N \) such that \( G \circ (M, N) = G \) is referred to as the complementary to \( M \) with respect to \( M \).

Let us also quote Lemma 3 ([2, Corollary 1]). If a function \( f : (0, \infty) \to (0, \infty) \) satisfying (1) is continuous at a point (or bounded above at a point) and there is \( a > 0 \) such that \( f \) is Jensen geometrically convex in the interval \((a, \infty)\), i.e.,

\[
f(G(x, y)) \leq G(f(x), f(y)), \quad x, y > a,
\]

then \( f = \Gamma \).

In view of the Bernstein–Doetsch theorem, if \( f \) is bounded from above in a neighborhood of a point ([3, Kuczma, p. 145]), the function \( \log \circ f \circ \exp \) is Jensen convex in the interval \( \log (I) \) iff

\[
f(x^ty^{1-t}) \leq [f(x)]^t [f(y)]^{1-t} \quad \text{for all } x, y \in I, \quad t \in (0, 1),
\]

that is, iff \( f \) is convex with respect to the family of weighted geometric means \( I \), and referred to as the geometric convexity of \( f \) ([2]).

3. Main results

**Theorem 1.** Let a function \( f : (0, \infty) \to (0, \infty) \) be continuous and such that

\[
f(x + 1) = xf(x), \quad x > 0, \quad f(1) = 1.
\]

If there is an \( a > 0 \) and a strict continuous mean \( M : (a, \infty)^2 \to (a, \infty) \) such that

\[
f(M(x, y)) f\left(\frac{xy}{M(x, y)}\right) \leq f(x) f(y), \quad x, y \in (a, \infty),
\]

then \( f \) is the Euler gamma function.
Proof. In view of Lemma 2, the function $N : (a, \infty)^2 \to (a, \infty)$ defined by
\[
N(x, y) := \frac{xy}{M(x, y)}, \quad x, y \in (a, \infty),
\]
is a continuous and strict mean. Since
\[
M(x, y)N(x, y) = xy, \quad x, y \in (a, \infty),
\]
taking the square root of both sides, we get
\[
G \circ (M, N) = G,
\]
where $G : (0, \infty)^2 \to (0, \infty)$, $G(x, y) = \sqrt{xy}$, $x, y > 0$, is the geometric mean.

Thus the geometric mean $G$ is invariant with respect to the mean-type mapping $(M, N)$ : $(a, \infty)^2 \to (a, \infty)^2$.

For every $n \in \mathbb{N}$, denote by $(M_n, N_n)$ the $n$-th iterate $(M, N)^n$ of the mean-type mapping $(M, N)$.

From (4), we have
\[
f(M(x, y))f(N(x, y)) \leq f(x)f(y), \quad x, y \in (a, \infty).
\]
Replacing here $x$ by $M(x, y)$ and $y$ by $N(x, y)$, we have
\[
f(M(M(x, y), N(x, y)))f(N(M(x, y), N(x, y))) \leq f(M(x, y))f(N(x, y))
\]
for all $x, y \in (a, \infty)$, that is,
\[
f(M_2((x, y)))f(N_2((x, y))) \leq f(M(x, y))f(N(x, y)) \leq f(x)f(y)
\]
for $x, y \in (a, \infty)$, whence
\[
f(M_2((x, y)))f(N_2((x, y))) \leq f(x)f(y), \quad x, y \in (a, \infty).
\]
Similarly, applying the induction, we obtain
\[
f(M_n(x, y))f(N_n(x, y)) \leq f(x)f(y), \quad n \in \mathbb{N}, x, y \in (a, \infty).
\]
The invariance of $G$ with respect to the mean-type mapping $(M, N)$ and Lemma 2 imply that
\[
\lim_{n \to \infty} M_n(x, y) = G(x, y) = \lim_{n \to \infty} N_n(x, y), \quad x, y \in (a, \infty).
\]
Therefore, letting $n \to \infty$ in (6), and making use of the continuity of $f$, we obtain
\[
[f(G(x, y))]^2 \leq f(x)f(y), \quad x, y \in (a, \infty),
\]
or, equivalently,
\[
f(G(x, y)) \leq G(f(x)f(y)), \quad x, y \in (a, \infty),
\]
which proves that $f$ is Jensen geometrically convex in $(a, \infty)$. Applying Lemma 3, we conclude that $f = \Gamma$. \qed
Theorem 2. For every $a > a_0 := 1.462$ and every strict mean $M : (a, \infty)^2 \rightarrow (a, \infty)$, the Euler gamma function satisfies the inequality
\[
\Gamma(M(x,y)) \Gamma \left( \frac{xy}{M(x,y)} \right) \leq \Gamma(x) \Gamma(y), \quad x, y \in (a, \infty).
\]

Proof. Since $\Gamma$ is logarithmically convex and increasing in $(a_0, \infty)$, where $a_0 = 1.462$, it is geometrically convex in every interval $(a, \infty)$ with $a > a_0$. Thus, for an arbitrarily fixed $a > a_0$, and for all $x, y \in (a, \infty)$ and $t \in (0, 1)$, we have
\[
\Gamma \left( x^t y^{1-t} \right) \leq \left[ \Gamma(x) \right]^t \left[ \Gamma(y) \right]^{1-t}. \tag{7}
\]
Let $M : (a, \infty)^2 \rightarrow (a, \infty)$ be a strict mean. Taking arbitrary $x, y \in (a, \infty)$, $x \neq y$, and putting
\[
t = t(x, y) := \frac{\log M(x, y) - \log y}{\log x - \log y},
\]
we have
\[
0 < t(x, y) < 1, \quad M(x, y) = x^t(x,y) y^{1-t(x,y)}
\]
and, by (5),
\[
N(x,y) = x^{1-t(x,y)} y^{t(x,y)}.
\]
Hence, applying (7), we get
\[
\Gamma(M(x,y)) = \Gamma \left( x^{t(x,y)} y^{1-t(x,y)} \right) \leq \left[ \Gamma(x) \right]^{t(x,y)} \left[ \Gamma(y) \right]^{1-t(x,y)}
\]
and
\[
\Gamma(N(x,y)) = \Gamma \left( x^{1-t(x,y)} y^{t(x,y)} \right) \leq \left[ \Gamma(x) \right]^{1-t(x,y)} \left[ \Gamma(y) \right]^{t(x,y)},
\]
whence, multiplying the respective sides of these inequalities, we obtain
\[
\Gamma(M(x,y)) \Gamma(N(x,y)) \leq \left( \Gamma(x) \right)^{t(x,y)} \left[ \Gamma(y) \right]^{1-t(x,y)} \left( \Gamma(x) \right)^{1-t(x,y)} \left[ \Gamma(y) \right]^{t(x,y)}
\]
\[
= \Gamma(x) \Gamma(y),
\]
that is, for all $x, y \in (a, \infty)$, $x \neq y$,
\[
\Gamma(M(x,y)) \Gamma \left( \frac{xy}{M(x,y)} \right) \leq \Gamma(x) \Gamma(y).
\]
Since this inequality is obvious for all $x, y \in (a, \infty)$ such that $x = y$, the proof is completed. □
Remark 3. Theorem 1 with \( M = G \) reduces to the main result of [2].

Indeed, taking \( M = G \) in inequality (4), we get \( [f(G(x,y))]^2 \leq f(x)f(y) \)
or, equivalently, \( f(G(x,y)) \leq G(f(x)f(y)) \) for all \( x, y \in (a, \infty) \), which means
that \( f \) is Jensen geometrically convex.

Finally note that the arithmetic and harmonic means can be used for a character-
ization of the gamma function. Namely, we have the following:

**Theorem 3.** A continuous function \( f : (0, \infty) \to (0, \infty) \) satisfying the functional equation
\[
f(x + 1) = xf(x), \quad x > 0, \quad f(1) = 1,
\]
is the Euler gamma function \( \Gamma \), if and only if, there is an \( a > 0 \) such that
\[
f(A(x,y))f(H(x,y)) \leq f(x)f(y), \quad x, y \in (a, \infty),
\]
where \( A(x,y) = \frac{x+y}{2} \) is the arithmetic mean and \( H(x,y) = \frac{2xy}{x+y} \) is the harmonic mean.

**Proof.** Define \( M : (0, \infty)^2 \to (0, \infty) \) by
\[
M(x,y) := A(x,y) = \frac{x+y}{2}, \quad x, y > 0.
\]
Then
\[
\frac{xy}{M(x,y)} = \frac{2xy}{x+y} = H(x,y), \quad x, y > 0,
\]
therefore the result is a consequence of Theorem 1 and Theorem 2. \( \square \)

**References**

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