Finite groups in which the cores of every non-normal subgroups are trivial

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Abstract. In this paper, we characterize the finite groups $G$ in which $\bigcap_{g \in G} H^g = 1$ or $H$ for every subgroup $H$ in $G$.

1. Introduction

All groups considered in this paper are finite. Recall that for a subgroup $H$ of a group $G$, the subgroup $H_G = \bigcap_{g \in G} H^g$ is called the core of $H$. A group is called a Dedekind group if its every subgroup is normal. And a subgroup $H$ of a group $G$ is called a $TI$-subgroup if $H \cap H^x = 1$ or $H$ for all $x \in G$.

If a subgroup $H$ of a group $G$ satisfies $H_G = 1$ or $H$, then we call it a $CT$-subgroup for convenience. It is obvious that every subgroup of a Dedekind group is a $CT$-subgroup.

For a group $G$, its $TI$-subgroup is a $CT$-subgroup. But a $CT$-subgroup may not be a $TI$-subgroup.

Example A. Let $G = S_{\{1,2,3,4\}}$ and $\langle (12), (123) \rangle = M \leq G$. Then $M$ is a $CT$-subgroup. Since $M \cap M^{(14)} = \{2, 3, 1\}$, we see $M$ is not a $TI$-subgroup.

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Example B. Let $G = \text{Aut}(A_6) = PTL(2,9)$. For any subgroup $H$ of $G$, $H$ is a $CT$-subgroup. But there exists a subgroup which is not a $TI$-subgroup.

In fact, $G$ has six normal subgroups which are isomorphic to $1, A_6, S_6, M_{10}, PGL(2,9), G$. $\text{Soc}(G) \cong A_6$ and $G/\text{Soc}(G) \cong C_2 \times C_2$, and then the number of maximal subgroups containing $A_6$ is 3. So the subgroups containing $A_6$ are only $S_6, M_{10}, PGL(2,9)$ and $G$. Therefore, for any subgroup $H \leq G$, if $A_6 \leq H$, then $H \leq G$, which implies that $H_G = H$. If $A_6 \not\leq H$, then $H_G = 1$. So every subgroup of $G$ is a $CT$-subgroup. And there exists a subgroup $M \cong S_5$ which is not a $TI$-subgroup.

A topic of some interest is the classification of groups, in which certain subgroups are assumed to be $TI$-subgroups. In [7], Walls described the groups all of whose subgroups are $TI$-subgroups. In [5], Li classified the non-nilpotent groups whose second maximal subgroups are $TI$-subgroups. In [3] and [6], Guo, Li and Flavell classified the groups whose abelian subgroups are $TI$-subgroups.

What we are interested in is the structure of groups $G$ in which some subgroups are $CT$-subgroups. For convenience, we give two notations:

(*) for every subgroup $H$ in $G$, $H$ is a $CT$-subgroup.

(**) for every abelian subgroup $H$ in $G$, $H$ is a $CT$-subgroup.

In this paper, we have the following results:

**Theorem 1.** Let $G$ be a finite group. If for every subgroup $H$ in $G$, $H_G = \cap_{g \in G} H^g = 1$ or $H$, then one of the following conditions is true:

1. $G$ is a Dedekind group.
2. $G$ is of prime power order, $Z(G)$ is cyclic, and $G'$ is of prime order when $Z(G) \neq 1$.
3. $G$ is a primitive group when $Z(G) = 1$.

The terminology and notation in this paper are standard. If $G$ is a finite $p$-group, then $\Omega_1(G) = \langle g \in G | g^p = 1 \rangle$.

**2. Preliminaries**

In this section, we give some basic facts, which are useful for the later use.

**Definition 2 ([2]).** A finite group $G$ is called primitive if it has a maximal subgroup $M$ such that $M_G = 1$. 
Lemma 3 ([1]). Let $G$ be a finite group. Then $G$ is Dedekindian if and only if $G$ is abelian or $G \cong Q_8 \times A \times B$, where $A$ is an elementary abelian 2-group, $B$ is abelian, and $B$ is of odd order.

Lemma 4 ([4]). Assume that $\pi'$-group $H$ acts on an abelian $\pi$-group $G$. Then $G = C_G(H) \times [G, H]$.

Lemma 5. Let $G$ be a non-Dedekind group. If $G$ satisfies (*), then $G$ has exactly one minimal normal subgroup.

Proof. For any minimal subgroup $N$ of $G$, we see $G/N$ is a Dedekind group by the hypothesis. If there are two different minimal normal subgroups $N_1$ and $N_2$, then $N_1 \cap N_2 = 1$. Hence

$$G \cong G/(N_1 \cap N_2) \leq (G/N_1 \times G/N_2) \leq Q_8 \times Q_8 \times C_2^k \times A = T \times A = L.$$  
For any subgroup $H \not\subseteq G$, then $H \not\subseteq L$. Assume that $H = H_1 \times H_2$, where $H_1 \leq T, H_2 \leq A$. It is easy to see that $H_1 \not\subseteq T$. Since $\Omega_1(T) = Z(T)$, there exists an element $x \in H_1 \leq H$ such that $o(x) = 4$. Therefore, $x^2 \in Z(T), \langle x^2 \rangle \leq L$. Then $\langle x^2 \rangle \leq G$, which implies that $H_G \neq 1$, a contradiction. □

3. Main results

Firstly, we characterize finite groups with non-trivial center which satisfy condition (**).

Lemma 6. Let $G$ be a non-Dedekind group. If $G$ satisfies (**), then $G$ is of prime power order, $Z(G)$ is cyclic, and $G'$ is of prime order.

Proof. We claim that $Z(G)$ has exactly one minimal subgroup. If not, then there exist two distinct minimal subgroups $N, M$ in $Z(G)$. For any element $a \in G$, we have $\langle a \rangle \cap Z(G) \leq \langle a \rangle_G$. Since $G$ satisfies (**), $\langle a \rangle_G = 1$ or $\langle a \rangle$. If $\langle a \rangle \cap Z(G) > 1$, then $\langle a \rangle_G = \langle a \rangle$, and so $\langle a \rangle \leq G$. If $\langle a \rangle \cap Z(G) = 1$, then $\langle a \rangle N \leq G$ by $N \leq (\langle a \rangle N)_G$. Similarly, we see $\langle a \rangle M \leq G$. Then $\langle a \rangle = \langle a \rangle N \cap \langle a \rangle M \leq G$. Thus $G$ is a Dedekind group, a contradiction. Hence $Z(G)$ has exactly one minimal subgroup, and then $Z(G)$ is cyclic.

Assume that $N$ is the unique minimal subgroup of $Z(G)$ and $|N| = p$. For any cyclic subgroup $H$ in $G$, we see $1 \neq N \leq (HN)_G$, and so $HN \leq G$. Thus $G/N$ is a Dedekind group.

If $G/N$ is non-abelian, then

$$G/N = A \times \langle \bar{a}, \bar{b} | \bar{a}^4 = 1, \bar{a}^2 = \bar{b}^2, [\bar{a}, \bar{b}] = \bar{a}^2 \rangle,$$
and $G = NA(a, b)$, where $A$ is an abelian group. If $p = 2$ and $a^4 = 1$, then $[a^2, b] = [a, b]^2 = a^4 = 1$ and $[a^2, x] = [a, x]^2 = 1$, for any element $x \in A$. Hence $a^2 \in Z(G)$, and then $(a^2) = N$ by the above paragraph, a contradiction. If $p = 2$ and $a^4 \neq 1$, then, assuming that $a^2 = b^2n$ ($n \in N$), $a^4 = [a^2, b] = [b^2n, b] = 1$, a contradiction. When $p > 2$, it is easy to see that $[a^{2p}, b] = a^{4p} = 1$ and $[a^{2p}, g] = [a, g]^{2p} = 1$, for any element $g \in A$. Then $a^{2p} \in Z(G)$, hence $Z(G)$ is not of prime power order, a contradiction.

So $G/N$ is abelian. Assume that $G/N = \times_{i=1}^k \langle \bar{y}_i \rangle$, where $|\langle \bar{y}_i \rangle| = q_i^{m_i}$, $q_i$ is a prime and $m_i$ is an integer, for all $i \in \{1, 2, \ldots, k\}$. We see $[\bar{y}_i, \bar{y}_j] = 1$, for any $i, j \in \{1, 2, \ldots, k\}$ by $|N| = p$. Noting $G = \langle N, y_1, y_2, \ldots, y_k \rangle$, $y_i^p \in Z(G)$ for any $i \in \{1, 2, \ldots, k\}$. If $y_i^p \neq 1$, then $N \leq \langle y_i^p \rangle$, and so $p|q_i^{m_i}$. Thus $p = q_i$, for any $i \in \{1, 2, \ldots, k\}$. So $G/N$ is a $p$-group.

Since $G$ is a non-Dedekind group, $G' \neq 1$, and then $G' = N$ is of order $p$. The proof is complete. \hfill \Box

**Lemma 7.** Let $G$ be a $p$-group. If $Z(G)$ is cyclic and $|G'| = p$, then $G$ satisfies condition (*).

**Proof.** Let $H$ be a subgroup of $G$. If $H_G \neq 1$, then $H_G \cap Z(G) \neq 1$, and therefore $G' \leq H_G \cap Z(G)$. Then $G' \leq H_G \leq H$ and $H \unlhd G$, which implies $H_G = H$. Hence $G$ satisfies condition (*). \hfill \Box

Now we obtained the following theorem:

**Theorem 8.** Let $G$ be a non-Dedekind group, and $Z(G) \neq 1$. Then the following conclusions are equivalent:

1. $G$ satisfies condition (*);
2. $G$ satisfies condition (**);
3. $G$ is of prime power order, $Z(G)$ is cyclic, and $G'$ is of prime order.

In fact, (*) is not equivalent to (**) when $Z(G) = 1$.

**Example C.** The Dihedral group $G = \langle a, b | a^2 = 1, b^9 = 1, b^3 = b^{-1} \rangle$. Then $G$ does not satisfy (*), but $G$ satisfies (**).

It is easy to see that $Z(G) = 1$. $\text{Syl}_1(G) = \langle b \rangle \triangleleft G$, and then $G$ has exactly one subgroup of order 3, which is $\langle b^3 \rangle$. The subgroup of order 2 is in $\{\langle ab^i \rangle | i = 0, 1, 2, 3, 4, 5, 6, 7, 8 \}$. For any subgroup $H$ of order 6, we see that $\langle b^3 \rangle \leq H$, and there exists $i \in \{0, \ldots, 8\}$ such that $\langle ab^i \rangle \leq H$. Since $[ab^i, b^3] \neq 1$, $H$ is non-abelian.

For any abelian subgroup $K \leq G$, then $|K| = 1, 2, 3$ or 9. If $|K| = 1, 3, 9$, then $K \unlhd G$, $K_G = K$. If $|K| = 2$, then $K_G = 1$. So $G$ satisfies (**).
Considering subgroup $H = \langle a, b^3 \rangle$, $H \nsubseteq G$, by $[b, a] = b^2$, and $H_G = \langle b^3 \rangle$. Hence $G$ does not satisfy condition (*).

Secondly, we study finite groups $G$ with trivial center which satisfy condition (*).

**Theorem 9.** Let $G$ be a finite non-Dedekind group, and $Z(G) = 1$. Then $G$ satisfies condition (*) if and only if $G$ is a primitive group, $\text{Soc}(G)$ is a minimal normal subgroup, and $G/\text{Soc}(G)$ is Dedekindian.

**Proof.** Suppose that $G$ satisfies condition (*). By Lemma 5, we see that $\text{Soc}(G)$ is the unique minimal normal subgroup, and $G/\text{Soc}(G)$ is Dedekindian by condition (*).

If $\text{Soc}(G)$ is abelian, then we may assume that the unique normal subgroup $\text{Soc}(G) = N \cong C_p^k$. Then there exists a subgroup $P \in \text{Syl}_p(G)$ such that $N \leq P$. Hence $P \unlhd G$ by (*). Let $G = P \rtimes H$. If $H \unlhd G$, then $G = P \times H$ and $1 \neq Z(P) \leq Z(G)$, which contradicts to $Z(G) = 1$. So $H \nsubseteq G$. Since $G/N = (P/N) \times (HN/N)$, $[P, H] \leq N$. By $Z(P) \text{char} P \leq G$, we see $N \leq Z(P)$. It follows that $C_N(H) = 1$ from $Z(G) = 1$. Using Lemma 4, $N = C_N(H)[N, H] = [N, H]$. Thus $[P, H] = N$. Using Lemma 4 again, we see that $P = C_P(H)[P, H] = C_P(H)N$. Since $N$ is the unique minimal normal subgroup, we see that $N \leq Z(P)$ and $[C_P(H), N] = 1$. Hence $C_P(H) \leq P$. Noting that $G = PH$, we have $C_P(H) \leq G$. Therefore, $N \leq C_P(H)$ or $C_P(H) = 1$. If $N \leq C_P(H)$, then $N \leq Z(G)$, a contradiction. So $C_P(H) = 1$. Then $P = C_P(H)[P, H] = [P, H] = N$. So $N \nsubseteq \Phi(G)$, $\Phi(G) = 1$, and therefore there exists a maximal subgroup $M$ of $G$ such that $M_G = 1$ by (*). Thus $G$ is a primitive group. If $\text{Soc}(G)$ is not abelian, then $\Phi(G) = 1$, because $\Phi(G)$ is nilpotent. So $G$ is a primitive group.

Conversely, if $H \leq G$ and $H_G \neq 1$, then $\text{Soc}(G) \leq H$. Since $G/\text{Soc}(G)$ is Dedekindian, we see that $H \unlhd G$, $H_G = H$. So $G$ satisfies condition (*). The proof is complete.

Since a soluble primitive group has exactly one minimal normal subgroup, we get the following result.

**Corollary 10.** Let $G$ be a finite non-Dedekind soluble group, and $Z(G) = 1$. Then $G$ satisfies condition (*) if and only if $G$ is a primitive group and $G/\text{Soc}(G)$ is Dedekindian.

However, for a non-soluble group $G$, the above conclusion does not hold.

**Example D.** $G = D \times D$, where $D$ is a non-abelian simple group.

It is easy to see that $G$ is primitive and $G/\text{Soc}(G) = 1$. But $G$ does not satisfy condition (*).
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References