On weak quasicontractions in $b$-metric spaces

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Abstract. Recently, weak quasicontractions have been studied by Bessenyei [2]. The aim of the present paper is to establish fixed point results for weak quasicontractions involving comparison function in $b$-metric spaces. As applications of our theorems, we deduce certain well-known results as corollaries.

1. Introduction and preliminaries

Banach contraction mapping principle is a simple and powerful result with a wide range of applications, including iterative methods for solving linear, nonlinear, differential, integral, and difference equations.

**Theorem 1.1.** Let $(X, d)$ be a complete metric space. Let $T$ be a contraction mapping on $X$, that is, one for which exists $\lambda \in [0, 1)$ satisfying

$$d(Tx, Ty) \leq \lambda d(x, y)$$

for all $x, y \in X$. Then $T$ has a unique fixed point $x \in X$.

Because of its significance and simplicity, various authors have established numerous interesting extensions and generalizations of the Banach contraction principle (see, for example, the monographs of Rus [20], Kirk and Shazad [16]). In 2016, Bessenyei [2] rediscovered Theorems of Hegedűs and Szilágyi [9], and those of Walter [21], introduced weak quasicontraction involving comparison functions and also proved a theorem that generalizes the results obtained by Ćirić [7], Browder [4] and Matkowski [17].

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The orbit and the double orbit induced by $T$ are defined in the next way:

$$O(x) := \{T^n(x) | n \in \mathbb{N} \cup \{0\} \}; \quad O(x, y) := O(x) \cup O(y),$$

where $T^{n+1} = T \circ T^n$ and $T^0 = \text{id}$.

We mention some properties of comparison functions $\varphi : [0, \infty) \rightarrow [0, \infty)$ which are being used in metric fixed point theory:

- $(P_1)$ $\varphi$ is increasing.
- $(P_2)$ $\varphi$ is upper semi-continuous.
- $(P_3)$ $\varphi(0) = 0$.
- $(P_4)$ $\varphi(t) < t$, for all $t > 0$.
- $(P_5)$ $\lim_{n \rightarrow \infty} \varphi^n(t) = 0$, for each $t > 0$.

**Lemma 1.2** (Bessenyei, [2]). $(P_1) + (P_2) + (P_3) + (P_4) \Rightarrow (P_5)$.

**Lemma 1.3** (Matkowski, [18]). $(P_1) + (P_3) \Rightarrow (P_4)$.

**Definition 1.4** ([2]). Let $(X,d)$ be a metric space. A mapping $T : X \rightarrow X$ is called a weak quasicontraction with comparison function $\varphi$ (or briefly: a weak $\varphi$-quasicontraction) if it induces bounded orbits, and for all $x, y \in X$,

$$d(Tx, Ty) \leq \varphi(\text{diam}O(x,y)). \quad (1.2)$$

The main result of Bessenyei [2] is the below-mentioned fixed point theorem for weak quasicontractions defined on complete metric spaces.

**Theorem 1.5** ([2]). Let $(X,d)$ be a complete metric space, and $T : X \rightarrow X$ a weak quasicontraction with comparison function $\varphi$ which meets the conditions $(P_1), (P_2), (P_3)$ and $(P_4)$. Then $T$ has a unique fixed point. Moreover, the sequence of iterates at any point converges to this fixed point.

**Remark 1.6.** We note that condition $(1.1)$ implies $\text{diam}O(x) \leq \frac{d(x,Tx)}{1-\lambda}$, so if we put $\varphi(t) = \lambda t$, $t \geq 0$, in Theorem 1.5, we obtain the Banach fixed point theorem.

Bakhtin [1] and Czerwik [6] defined the notion of $b$-metric spaces and proved some fixed point theorems for single-valued and multi-valued mappings in $b$-metric spaces. Successively, this notion has been reintroduced by Khamsi [14], Khamsi and Hussain [15], and Hussain et al. [11], [12] with the name of metric-type space.

**Definition 1.7.** Let $X$ be a nonempty set, and let $s \geq 1$ be a given real number. A function $d : X \times X \rightarrow [0, \infty)$ is said to be a $b$-metric if for all $x, y, z \in X$ the following conditions are satisfied:
(1) \( d(x, y) = 0 \) if and only if \( x = y \);
(2) \( d(x, y) = d(y, x) \);
(3) \( d(x, z) \leq s[d(x, y) + d(y, z)] \).

A triplet \((X, d, s)\) is called a \(b\)-metric space with coefficient \(s\).

Note that the class of metric spaces is a proper subset of the class of \(b\)-metric spaces with coefficient \(s \geq 1\). Fixed point theory in \(b\)-metric spaces was studied by many authors (see [8], [10], [13], [16], [19]). Note also that in a \(b\)-metric space, distance function \(d\) need not be continuous, i.e., there exists a \(b\)-metric space \((X, d, s)\) and sequences \(\{x_n\}, \{y_n\}\) in \(X\) such that \(\lim_{n \to \infty} x_n = x\) and \(\lim_{n \to \infty} y_n = y\), but \(\lim_{n \to \infty} d(x_n, y_n) \neq d(x, y)\).

One of the main results of [6] Czerwik is the following;

**Theorem 1.8** ([6, Theorem 1]). Let \((X, d, s)\) be a complete \(b\)-metric space, and suppose \(T : X \to X\) satisfies

\[
d(T(x), T(y)) \leq \varphi(d(x, y)),
\]

for each \(x, y \in X\), where mapping \(\varphi : [0, \infty) \to [0, \infty)\) satisfies conditions \((P_1)\) and \((P_5)\). Then \(T\) has a unique fixed point \(x^* \in X\), and \(\lim_{n \to \infty} T^n(x) = x^*\) for each \(x \in X\).

**Remark 1.9.** Note that Theorem 1.8 is a direct consequence of the main result of [3]. The recent paper [5] gives excellent overview of possible generalizations of metric spaces.

**Remark 1.10.** We note that due to Lemma 1.3, condition (1.3) implies that \(T\) is a continuous mapping. Also, condition (1.3) implies that \(T\) induces bounded orbits (see [2]).

The fact that \(d\) is not continuous in the \(b\)-metric space leads to the introduction of strong \(b\)-metric space.

**Definition 1.11** ([16, Definition 12.7]). Let \(X\) be a nonempty set, and \(s \geq 1\) be a given real number. A function \(d : X \times X \to [0, \infty)\) is said to be a strong \(b\)-metric if for all \(x, y, z \in X\) the following conditions are satisfied:

(1) \(d(x, y) = 0\) if and only if \(x = y\);
(2) \(d(x, y) = d(y, x)\);
(3) \(d(x, z) \leq s(d(x, y) + d(y, z))\).

A triplet \((X, d, s)\), is called a strong \(b\)-metric space with coefficient \(s\).
Remark 1.12. The distance function $d$ in a strong $b$-metric space is continuous (see [16, Proposition 12.3]).

The aim of this paper is to obtain Theorem 1.5 in $b$-metric spaces using weak quasicontraction involving comparison function $\varphi$. As consequences, we derive certain known results as corollaries.

2. Main result

The proof of the next lemma is a straightforward adaptation of the reasoning from [2].

Lemma 2.1. Let $(X, d, s)$ be a complete $b$-metric space, and $T : X \to X$ a weak $\varphi$-quasicontraction where $\varphi$ satisfies conditions $(P_1)$ and $(P_5)$. Then there exists $x^* \in X$ such that $\lim_{n \to \infty} T^n(x) = x^*$ for each $x \in X$.

Proof. The boundedness of orbits implies the following:

\[
\text{diam} O(Tx, Ty) = \sup_{k, l \in \mathbb{N}} \{d(T^k x, T^l y), d(T^k x, T^l x), d(T^k y, T^l y)\}. \tag{2.1}
\]

From condition (1.2), we obtain

\[
d(T^k x, T^l y) \leq \varphi(\text{diam} O(T^{k-1} x, T^{l-1} y)) \leq \varphi(\text{diam} O(x, y)).
\]

Similarly, we have

\[
d(T^k x, T^l x) \leq \varphi(\text{diam} O(T^{k-1} x, T^{l-1} x)) \leq \varphi(\text{diam} O(x)) \leq \varphi(\text{diam} O(x, y)),
\]

which implies

\[
d(T^k y, T^l y) \leq \varphi(\text{diam} O(x, y)). \tag{2.2}
\]

According to the foregoing, we conclude that

\[
\text{diam} O(Tx, Ty) \leq \varphi(\text{diam} O(x, y)). \tag{2.3}
\]

From inequality (2.3), we have the following:

\[
\text{diam} O(T^2 x, T^2 y) \leq \varphi(\text{diam} O(Tx, Ty)) \leq \varphi(\varphi(\text{diam} O(x, y))) = \varphi^2(\text{diam} O(x, y)).
\]

Using induction, we obtain

\[
\text{diam} O(T^n x, T^n y) \leq \varphi^n(\text{diam} O(x, y)). \tag{2.4}
\]
Choose $x \in X$, we show that $\{T^n x\}$ is a Cauchy sequence. The boundedness of orbits and condition $(P_5)$ imply that for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}(\text{diam } O(x)) < \frac{\epsilon}{2s}$. Therefore, using inequality (2.4), we obtain, for all $n \geq n_0$,
\[
d(T^{n_0} x, T^n x) \leq \varphi^{n_0}(\text{diam } O(x, T^{n-n_0} x)) \leq \varphi^{n_0}(\text{diam } O(x)) \leq \frac{\epsilon}{2s}.
\]
Using inequality (3) in Definition 1.7, we conclude that for all $m, n \geq n_0$,
\[
d(T^m x, T^n x) \leq s[d(T^m x, T^{n_0} x) + d(T^{n_0} x, T^n x)] \leq s \left[\frac{\epsilon}{2s} + \frac{\epsilon}{2s}\right] = \epsilon.
\]
So, $\{T^n x\}$ is a Cauchy sequence, and hence there exists $x^* \in X$ such that $\lim_{n \to \infty} T^n x = x^*$.

**Theorem 2.2.** Let $(X, d, s)$ be a complete $b$-metric space, and $T : X \to X$ a weak $\varphi$-quasicontraction such that function $\varphi$ satisfies conditions $(P_1)$ and $(P_5)$. Then $\lim_{n \to \infty} T^n x = x^*$ for each $x \in X$, and $x^*$ is a unique fixed point of $T$, provided one of the following conditions is satisfied:

(i) $T$ is continuous at $x^*$;

(ii) $d$ is continuous.

**Proof.** Lemma 2.1 implies that $\lim_{n \to \infty} T^n x = x^*$ for each $x \in X$. Let us prove that $x^*$ is a unique fixed point of $T$.

(i) Suppose that $T$ is continuous at $x^*$.

Then we have
\[
x^* = \lim_{n \to \infty} T^{n+1} x = T \lim_{n \to \infty} T^{n+1} x = Tx^*.
\]

(ii) Let $d$ be continuous. If $x^*$ is not a fixed point of $T$, then $\text{diam } O(x^*) > 0$. The methods of [2] provide $n_0 \in \mathbb{N}$ such that $\text{diam } O(x^*) = d(x^*, T^{n_0} x^*)$ holds. By Lemma 1.3, we obtain that
\[
\varphi(\text{diam } O(x^*)) < \text{diam } O(x^*).
\]
Since $d$ is continuous, we obtain
\[
\text{diam } O(x^*) = d(x^*, T^{n_0} x^*) = \lim_{n \to \infty} d(T^{n+n_0} x^*, T^{n_0} x^*) \leq \varphi^{n_0}(\text{diam } O(T^n x^*, x^*)) \leq \varphi(\text{diam } O(x^*)) < \text{diam } O(x^*).
\]
Consequently, $\text{diam } O(x^*) = 0$. Thus $x^*$ is the fixed point of the mapping $T$.

For uniqueness, let $y^*$ be another fixed point of $T$. Since,
\[
d(x^*, y^*) = d(Tx^*, Ty^*) \leq \varphi(\text{diam } O(x^*, y^*)) < \text{diam } O(x^*, y^*) = d(x^*, y^*),
\]
which implies $T$ has exactly one fixed point. □
Remark 2.3. Since \( d(x, y) \leq \text{diam} O(x, y) \), for all \( x, y \in X \) and \( \varphi \), satisfies condition \( (P_4) \) (see Lemma 1.3), so from Theorem 2.2 we obtain Theorem 1.8.

Corollary 2.4. Let \((X, d, s)\) be a complete \( b \)-metric space, and let mapping \( T : X \rightarrow X \) induce bounded orbits. Suppose that for all \( x, y \in X \),
\[
    d(Tx, Ty) \leq \text{diam} O(x, y) - \psi(\text{diam} O(x, y)),
\]

where function \( \psi : [0, \infty) \rightarrow [0, \infty) \) satisfies the below conditions:
(a) \( \psi \) is a decreasing,
(b) \( id - \psi \) satisfy condition \( (P_5) \).

Then there exists \( x^* \in X \) such that \( \lim_{n \to \infty} T^n(x) = x^* \) for each \( x \in X \), and \( x^* \) is a unique fixed point of \( T \), provided one of the following conditions is satisfied:
(i) \( T \) is continuous at \( x^* \in X \),
(ii) \( d \) is continuous.

3. Some applications

In this section, we present certain consequences of Theorem 2.2 in \( b \)-metric spaces.

Lemma 3.1. Let \((X, d, s)\) be a complete \( b \)-metric space, and \( T : X \rightarrow X \) a map such that for all \( x, y \in X \) and some \( \lambda \in [0, 1) \), we have
\[
    d(Tx, Ty) \leq \lambda d(x, y).
\]

Then \( T \) induces bounded orbits.

Proof. Since \( \lim_{n \to \infty} \lambda^n = 0 \), there exists a natural number \( n_0 \) such that
\[
    0 < \lambda^{n_0} \cdot s^2 < 1.
\]
Let \( O_n(x) = \{x, Tx, \ldots, T^n x\} \). Then, we conclude that \( \text{diam} O_n(x) = d(T^k x, T^l x) \) for some \( k, l \in \{1, 2, \ldots, n\} \), or \( \text{diam} O_n(x) = d(x, T^k x) \) for some \( k \in \{1, 2, \ldots, n\} \).

If \( \text{diam} O_n(x) = d(T^k x, T^l x) \), we obtain (where it is understood that \( T^0 x = x \))
\[
    d(T^k x, T^l x) \leq \lambda d(T^{k-1} x, T^{l-1} x) < d(T^{k-1} x, T^{l-1} x) \leq \text{diam} O_n(x).
\]
Therefore, we conclude that \( d(x, T^k x) = \text{diam} O_n(x) \) for some \( k \in \{1, 2, \ldots, n\} \).

Applying inequality (3) in Definition 1.7, we obtain
\[ d(x, T^k x) \leq s[d(x, T^{n_0} x) + d(T^{n_0} x, T^k x)] \]
\[ \leq s[d(x, T^{n_0} x) + s(d(T^{n_0} x, T^{n_0 + k} x) + d(T^{n_0 + k} x, T^k x))] \]
\[ \leq sd(x, T^{n_0} x) + s^2[\lambda^{n_0} d(x, T^k x) + \lambda^k d(T^{n_0} x, x)] \]
\[ \leq (s + s^2)d(x, T^{n_0} x) + s^2\lambda^{n_0} d(x, T^k x)]. \]

Therefore, we get
\[
\text{diam } O_n(x) \leq s + s^2 \frac{1}{1 - \lambda^{n_0} s^2} d(x, T^{n_0} x). \tag{3.3}
\]

Since \( \text{diam } O(x) = \sup \{\text{diam } O_n(x) : n \in \mathbb{N}\} \), we obtain that \( T \) induces bounds orbits.

**Theorem 3.2** (The Banach contraction principle in \( b \)-metric spaces, Dung [8, Theorem 2.1]). Let \((X, d, s)\) be a complete \( b \)-metric space, and let \( T : X \to X \) be a continuous map such that for all \( x, y \in X \) and some \( \lambda \in [0, 1) \),

\[ d(Tx, Ty) \leq \lambda d(x, y). \tag{3.4} \]

Then \( T \) has a unique fixed point \( x^* \), and \( \lim_{n \to \infty} T^nx = x^* \) for all \( x \in X \).

**Proof.** If we put \( \varphi(t) = \lambda t \), then the assertion follows from Theorem 2.2 and Lemma 3.1.

**Theorem 3.3.** Let \((X, d, s)\) be a complete \( b \)-metric space, and let \( T : X \to X \) be a quasicontraction inducing bounded orbits, i.e., there exists \( \lambda \in [0, 1) \) such that

\[ d(Tx, Ty) \leq \lambda \text{diam } O(x, y), \tag{3.5} \]

for all \( x, y \in X \). Then there exists \( x^* \in X \) such that \( \lim_{n \to \infty} T^n(x) = x^* \) for each \( x \in X \), and \( T \) has a unique fixed point \( x^* \), provided one of the following conditions are satisfied:

(i) \( T \) is continuous at \( x^* \in X \);

(ii) \( d \) is continuous.

**Proof.** The proof follows from Theorem 2.2, if we put \( \varphi(t) = \lambda t \).

**Remark 3.4.** (1) If \( T \) is a quasicontraction on \( b \)-metric space \((X, d, s)\) with \( \lambda \in [0, \frac{1}{2}) \), then similar to the proof of Lemma 3.1, there exists some \( k \in \{1, 2, \ldots, n\} \), such that \( d(x, T^k x) = \text{diam } O_n(x) \). Since,

\[ \text{diam } O_n(x) \leq s[d(x, Tx) + \text{diam } O_{n-1}(Tx)] \leq s[d(x, Tx) + \lambda \text{diam } O_n(x)], \]
which implies
\[ \text{diam } O(x) \leq \frac{s}{1 - \lambda s} d(x, Tx). \] (3.6)

(2) If \((X, d, s)\) is a strong \(b\)-metric space, then
\[ \text{diam } O_n(x) \leq sd(x, Tx) + \text{diam } O_{n-1}(Tx) \leq sd(x, Tx) + \lambda \text{diam } O_n(x), \]
which implies
\[ \text{diam } O(x) \leq \frac{s}{1 - \lambda} d(x, Tx). \] (3.7)

From Lemma 2.1, Theorem 3.3 and Remark 3.4, we obtain the following quasi-contraction principle in \(b\)-metric and strong \(b\)-metric spaces.

**Corollary 3.5** (Version of the fixed point theorem of Ćirić in \(b\)-metric spaces). Let \((X, d, s)\) be a complete \(b\)-metric space and let \(T : X \to X\) be a map such that for all \(x, y \in X\) and some \(\lambda \in [0, 1/s)\),
\[ d(Tx, Ty) \leq \lambda \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s} \right\}. \]

Then \(T\) has a unique fixed point \(x^*\).

**Proof.** From Lemma 2.1 and Remark 3.4, there exists \(x^* \in X\) such that \(\lim_{n \to \infty} T^n(x) = x^*\) for each \(x \in X\). Let us show that \(x^*\) is a fixed point. We have the following:
\[ d(x^*, Tx^*) \leq sd(x^*, T^{n+1}x) + \lambda \max \left\{ d(T^n x, x^*), d(T^n x, T^{n+1} x), \frac{d(x^*, T^{n+1} x)}{2s}, \frac{d(T^n x, Tx^*)}{2s} \right\}. \]

Since,
\[ \frac{d(x^*, T^{n+1} x)}{2s} \leq \frac{d(x^*, T^n x) + d(T^n x, T^{n+1} x)}{2} \leq \max \{ d(x^*, T^n x), d(T^n x, T^{n+1} x) \}, \]
and
\[ \frac{d(T^n x, Tx^*)}{2s} \leq \frac{d(T^n x, x^*) + d(x^*, Tx^*)}{2} \leq \max \{ d(x^*, T^n x), d(x^*, Tx^*) \}, \]
so we obtain
\[ d(x^*, Tx^*) \leq s d(x^*, T^{n+1}x) + s \lambda \max\{d(T^n x, x^*), d(T^{n+1} x, x^*), d(x^*, Tx^*)\}. \]

Since \( \lim_{n \to \infty} T^n x = x^* \) and \( \lim_{n \to \infty} d(T^n x, T^{n+1} x) = 0 \), this shows that \( (1 - \lambda s)d(x^*, Tx^*) = 0 \), which implies that \( x^* \) is a fixed point of \( T \). The uniqueness follows from the quasi-contractivity of \( T \). □

**Corollary 3.6** (Version of the fixed point theorem of Ćirić in strong \( b \)-metric spaces). Let \( (X, d, s) \) be a complete strong \( b \)-metric space, and let \( T : X \to X \) be a map such that for all \( x, y \in X \) and some \( \lambda \in [0, 1) \),
\[ d(Tx, Ty) \leq \lambda \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \]

Then \( T \) has a unique fixed point \( x^* \).

**Proof.** The continuity of \( d \) follows directly from Theorem 3.3 and Remark 3.4. □

**Remark 3.7.**

(i) The conclusion of Corollary 3.6 does not hold in the setting of \( b \)-metric spaces for \( \lambda \in [0, 1) \) (see [8, Example 2.6]).

(ii) Corollary 3.5 improves the result of Jovanović *et al.* ([13, Corollary 3.12]).

(ii) Corollary 3.6 improves the results of Dung (see [8, Corollaries 2.4 and 2.5]).

**References**


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