On two open problems of the theory of permutable subgroups of finite groups

By BIN HU (Xuzhou), JIANHONG HUANG (Xuzhou) and ALEXANDER N. SKIBA (Gomel)

Abstract. Let $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set of all primes $\mathbb{P}$, $G$ a finite group and $\sigma(G) = \{\sigma_i | \sigma_i \cap \pi(G) \neq \emptyset\}$.

A set $\mathcal{H}$ of subgroups of $G$ is said to be a complete Hall $\sigma$-set of $G$ if every member $\neq 1$ of $\mathcal{H}$ is a Hall $\sigma_i$-subgroup of $G$ for some $\sigma_i \in \sigma$ and $\mathcal{H}$ contains exactly one Hall $\sigma_i$-subgroup of $G$ for every $\sigma_i \in \sigma(G)$; $G$ is said to be $\sigma$-full if $G$ possesses a complete Hall $\sigma$-set.

A subgroup $A$ of $G$ is said to be $\sigma$-permutable in $G$ if $G$ possesses a complete Hall $\sigma$-set and $A$ permutes with each Hall $\sigma_i$-subgroup $H$ of $G$, that is, $AH = HA$ for all $i \in I$.

We prove that if $G$ is $\sigma$-full, then the set $\mathcal{L}_{\sigma_{\text{per}}}(G)$, of all $\sigma$-permutable subgroups of $G$, forms a sublattice of the lattice of all subgroups of $G$. Also, answering to [9, Question 6.13], we describe the conditions under which the lattice $\mathcal{L}_{\sigma_{\text{per}}}(G)$ is distributive.

1. Introduction

Throughout this paper, $G$ always denotes a finite group. Moreover, we use $\mathcal{L}(G)$ to denote the lattice of all subgroups of $G$, and $\mathcal{L}_n(G)$ is the lattice of all normal subgroups of $G$. The symbol $\mathbb{P}$ denotes the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. As usual, $\pi(G)$ is the set of all primes dividing the order $|G|$ of $G$.

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The second author is the corresponding author.
The subgroups \( A \) and \( B \) of \( G \) are said to be *permutable* if \( AB = BA \). In this case they also say that \( A \) *permutes* with \( B \). If \( A \) permutes with all Sylow subgroups of \( G \), then \( A \) is called *\( S \)-permutable* in \( G \) [1]. Recall also that an element \( a \) of the lattice \( \mathcal{L} \) is called *meet-distributive* [8, p. 136] if \( a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \) for all \( b, c \in \mathcal{L} \).

In what follows, \( \sigma = \{ \sigma_i | i \in I \} \) is some partition of \( \mathbb{P} \), that is, \( \mathbb{P} = \bigcup_{i \in I} \sigma_i \) and \( \sigma_i \cap \sigma_j = \emptyset \) for all \( i \neq j \).

A set \( \mathcal{H} \) of subgroups of \( G \) is a *complete Hall \( \sigma \)-set* of \( G \) [9] if every member \( \neq 1 \) of \( \mathcal{H} \) is a Hall \( \sigma_i \)-subgroup of \( G \) for some \( \sigma_i \in \sigma \) and \( \mathcal{H} \) contains exactly one Hall \( \sigma_i \)-subgroup of \( G \) for every \( i \) such that \( \sigma_i \cap \pi(G) \neq \emptyset \); \( G \) is said to be *\( \sigma \)-full* [9] if \( G \) possesses a complete Hall \( \sigma \)-set.

Recall that a subgroup \( A \) of \( G \) is said to be *\( \sigma \)-permutable* in \( G \) [10] if \( G \) possesses a complete Hall \( \sigma \)-set \( \mathcal{H} \) such that \( AH^x = H^xA \) for all \( H \in \mathcal{H} \) and all \( x \in G \).

Our first observations are the following useful facts.

**Proposition 1.1.** Suppose that \( G \) is \( \sigma \)-full and \( A \) is a \( \sigma \)-permutable subgroup of \( G \). Then \( A \) permutes with all Hall \( \sigma_i \)-subgroups of \( G \) for all \( i \).

**Theorem A.** Suppose that \( G \) is \( \sigma \)-full. Then the set \( \mathcal{L}_{\sigma \per}(G) \), of all \( \sigma \)-permutable subgroups of \( G \), forms a sublattice of the lattice \( \mathcal{L}(G) \).

Note that Theorem A improves Theorem C in [10] and, in fact, gives an alternative proof for the following well-known result.

**Corollary 1.2** (Kegel in [5]). The set \( \mathcal{L}_S(G) \) of all \( S \)-permutable subgroups of \( G \) forms a sublattice of the lattice \( \mathcal{L}(G) \).

**Example 1.3.** (i) \( G \) is called *\( \sigma \)-nilpotent* [9] if \( G = H_1 \times \cdots \times H_t \), where \( \{H_1, \ldots, H_t\} \) is a complete Hall \( \sigma \)-set of \( G \). It is not difficult to show that \( G \) is \( \sigma \)-nilpotent if and only if every subgroup of \( G \) is \( \sigma \)-permutable in \( G \).

(ii) In the classical case when \( \sigma = \sigma^1 = \{\{2\}, \{3\}, \ldots\} \) (we use here the terminology in [11]), a subgroup \( A \) of \( G \) is \( \sigma^1 \)-permutable in \( G \) if and only if it is \( S \)-permutable in \( G \).

(iii) In the other classical case when \( \sigma = \sigma^\pi = \{\pi, \pi'\} \), a subgroup \( A \) of a \( \pi \)-separable group \( G \) is \( \sigma^\pi \)-permutable in \( G \) if and only if \( A \) permutes with all Hall \( \pi \)-subgroups and with all Hall \( \pi' \)-subgroups of \( G \).

(iv) In fact, in the theory of \( \pi \)-soluble groups \((\pi = \{p_1, \ldots, p_n\})\) we deal with the partition \( \sigma = \sigma^{1\pi} = \{\{p_1\}, \ldots, \{p_n\}, \pi'\} \) of \( \mathbb{P} \). In view of Proposition 1.1, a subgroup \( A \) of \( G \) is \( \sigma^{1\pi} \)-permutable in \( G \) if and only if \( G \) possesses a Hall
π′-subgroup $V$, and $A$ permutes with all conjugates of $V$ and with all Sylow $p$-subgroups of $G$ for all $p \in \pi$.

The conditions under which the lattice $L_{sn}(G)$ of all subnormal subgroups of $G$ is modular or distributive are known (see [8, Theorems 9.2.3, 9.2.4]). It is well-known also that the lattice $L_n(G)$ of all normal subgroups of $G$ is modular and this lattice is distributive if and only if in every factor group $G/R$, any two $G/R$-isomorphic normal subgroups coincide (see [7] and [8, Theorem 9.1.6]). Kegel proved [5] that the set $L_S(G)$ of all $S$-permutable subgroups of $G$ forms a sublattice of the lattice $L_{sn}(G)$. Since $L_n(G) \subseteq L_S(G) \subseteq L_{sn}(G)$, where both inclusions in general are strict, it seems natural to ask: Under what conditions the lattice $L_S(G)$ is modular or distributive?

Moreover, in view of Theorem A, it makes sense to consider the following generalQuestion 1.4 (see Questions 6.11 and 6.13 in [9]). Under what conditions the lattice $L_{\sigma\text{per}}(G)$ is modular or distributive?

Note that if $K \trianglelefteq H$ and $K, H \in L_{\sigma\text{per}}(G)$, where $L_{\sigma\text{per}}(G)$ is the set of all $\sigma$-permutable $\sigma_i$-subgroups of $G$, then $O^{\sigma_i}(G)$ normalizes both subgroups $K$ and $H$ [10, Lemma 3.1], and hence we can consider $O^{\sigma_i}(G)$ as a group of operators for $H/K$ (assuming, as usual, that $(hK)^a = h^a K$ for all $hK \in H/K$ and $a \in O^{\sigma_i}(G)$).

We do not know under which conditions on $G$ the lattice $L_{\sigma\text{per}}(G)$ is modular. Nevertheless, we give the full answer to the second part of Question 1.4.

Recall that $G^{\sigma_{nil}}$ denotes the $\sigma$-nilpotent residual of $G$, that is, the intersection of all normal subgroups $N$ of $G$ with $\sigma$-nilpotent quotient $G/N$.

Let $G$ be a $\sigma$-full group and $\mathcal{L} = L_{\sigma\text{per}}(G)$. Then we say that the lattice $\mathcal{L}$ satisfies the weak distributivity condition with respect to $\sigma$ (the $W\sigma D$-condition, in short) if the following hold: (i) every two members of $\mathcal{L}$ are permutable; (ii) the lattice $L_n(G)$ is distributive; (iii) $G/G^{\sigma_{nil}}$ is cyclic and $G^{\sigma_{nil}}$ is a meet-distributive element of $\mathcal{L}$.

**Theorem B.** Suppose that $G$ is $\sigma$-full. Let $\mathcal{L} = L_{\sigma\text{per}}(G)$. Then the following conditions are equivalent:

(i) The lattice $\mathcal{L}$ is distributive.

(ii) $\mathcal{L}$ satisfies the $W\sigma D$-condition and in every factor group $\bar{G} = G/R$, any two $O^{\sigma_i}(\bar{G})$-isomorphic sections $\bar{H}/\bar{K}$ and $\bar{L}/\bar{K}$, where $\bar{K}, \bar{H}, \bar{L} \in L_{\sigma_i\text{per}}(\bar{G})$ for some $i$, coincide.

(iii) $\mathcal{L}$ satisfies the $W\sigma D$-condition and in every factor group $\bar{G} = G/R$, any two
Each characteristic subgroup of $480$ Bin Hu, Jianhong Huang and Alexander N. Skiba and the following hold:

$L$ Then the lattice $\bar{L}$ to together with Proposition 1.1 and Theorem A are three of them.

$\bar{K}$, $\bar{A}$ and only if each characteristic subgroup of $L$ is distributive if and only if $\bar{L}$ is a prime, coincide.

In the case when $\sigma = \sigma^L$, we get from Theorem B the following result.

**Corollary 1.5.** Suppose that $G$ is $\pi$-separable, and let $L = L_{\sigma^L \text{per}}(G)$. Then the lattice $\mathcal{L}$ is distributive if and only if $\mathcal{L}$ satisfies the $W\sigma^1 D$-condition and the following hold:

1. In every factor group $\bar{G} = G/R$, any two $O^\sigma(\bar{G})$-isomorphic sections $\bar{H}/\bar{K}$ and $\bar{L}/\bar{K}$, where $\bar{K}, \bar{H}$, and $\bar{L}$ are $\sigma^\pi$-permutable $\pi$-subgroups of $\bar{G}$, coincide.

2. In every factor group $\bar{G} = G/R$, any two $O^\sigma(\bar{G})$-isomorphic sections $\bar{H}/\bar{K}$ and $\bar{L}/\bar{K}$, where $\bar{K}, \bar{H}$, and $\bar{L}$ are $\sigma^\pi$-permutable $\pi'$-subgroups of $\bar{G}$, coincide.

In the case when $\sigma = \sigma^1 \pi$, we get from Theorem B the following fact.

**Corollary 1.6.** Suppose that $G$ possesses a Hall $\pi'$-subgroup and let $L = L_{\sigma^1 \pi \text{per}}(G)$. Then the lattice $\mathcal{L}$ is distributive if and only if $\mathcal{L}$ satisfies the $W\sigma^1 D$-condition and the following hold:

1. In every factor group $\bar{G} = G/R$, any two $O^\circ(\bar{G})$-isomorphic sections $\bar{H}/\bar{K}$ and $\bar{L}/\bar{K}$, where $\bar{K}, \bar{H}, \bar{L} \in \mathcal{L}_{\pi S}(\bar{G})$ and $p \in \pi$, coincide;

2. In every factor group $\bar{G} = G/R$, any two $O^\circ(\bar{G})$-isomorphic sections $\bar{H}/\bar{K}$ and $\bar{L}/\bar{K}$, where $\bar{K}, \bar{H}$, and $\bar{L}$ are $\sigma^\pi$-permutable $\pi'$-subgroups of $\bar{G}$, coincide.

In this corollary, $\mathcal{L}_{\pi S}(\bar{G})$ denotes the set of all S-permutable $p$-subgroups of $\bar{G}$.

In the case when $\pi = \mathbb{P}$, we get from Corollary 1.6 the following

**Corollary 1.7** (see [12, Theorem A]). Let $L = L_\mathbb{S}(G)$. Then the lattice $\mathcal{L}$ is distributive if and only if $\mathcal{L}$ satisfies the $W\sigma^1 D$-condition and in every factor group $\bar{G} = G/R$, any two $O^\circ(\bar{G})$-isomorphic sections $\bar{H}/\bar{K}$ and $\bar{L}/\bar{K}$, where $\bar{K}, \bar{H}, \bar{L} \in \mathcal{L}_{\pi S}(\bar{G})$ and $p$ is a prime, coincide.

The proof of Theorem B consists of many steps, and the following result together with Proposition 1.1 and Theorem A are three of them.

**Proposition 1.8.** A $\sigma$-nilpotent subgroup $A$ of $G$ is $\sigma$-permutable in $G$ if and only if each characteristic subgroup of $A$ is $\sigma$-permutable in $G$.

**Corollary 1.9** (see [1, Theorem 1.2.17]). Let $A$ be a nilpotent subgroup of $G$. Then the following statements are equivalent:

1. $A$ is $S$-permutable in $G$.
2. Each Sylow subgroup of $A$ is $S$-permutable in $G$.
3. Each characteristic subgroup of $A$ is $S$-permutable in $G$. 
2. Proofs of Theorems A and Propositions 1.1 and 1.8

**Lemma 2.1** (see [3, A, Lemma 1.6]). Let $A$, $B$ and $H$ be subgroups of $G$. If $AH = HA$ and $BH = HB$, then $(A, B)H = H(A, B)$.

A subgroup $A$ of $G$ is called $\sigma$-subnormal in $G$ [10] if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_t = G$ such that either $A_{i-1} \leq A_i$ or $A_i/(A_{i-1})A_i$ is $\sigma$-primary for all $i = 1, \ldots, t$.

The importance of this concept is related to the following result.

**Lemma 2.2** (see [10, Theorem B]). Every $\sigma$-permutable subgroup $A$ of $G$ is $\sigma$-subnormal in $G$ and $A/A_G$ is $\sigma$-nilpotent.

**Proof of Theorem A.** In fact, in view of Lemmas 2.1 and 2.2, it is enough to show that if $A$ and $B$ are $\sigma$-subnormal subgroups of $G$ such that for a Hall $\sigma$-subgroup $H$ of $G$ we have $AH = HA$ and $BH = HB$, then $(A \cap B)H = H(A \cap B)$. Assume that this is false and let $G$ be a counterexample of minimal order. Then $G$ is not a $\sigma$-group, since otherwise we have $H = G$ and so $G = (A \cap B)H = H(A \cap B)$.

Let $E = AH \cap BH$. Then $A \cap E$ and $B \cap E$ are $\sigma$-subnormal subgroups of $E$ by [10, Lemma 2.6(1)]. Moreover, $AH \cap E = H(A \cap E) = (A \cap E)H$. Similarly, $(B \cap E)H = H(B \cap E)$. Hence the hypothesis holds for $(A \cap E, B \cap E, H, E)$. Assume that $E < G$. Then the choice of $G$ implies that $A \cap B = (A \cap E) \cap (B \cap E)$ is permutable with $H$. Hence $E = G$, so $G = AH = BH$. Thus $|G : A|$ and $|G : B|$ are $\sigma_i$-numbers. Hence we have $O^{\sigma_i}(A) = O^{\sigma_i}(G) = O^{\sigma_i}(B)$ by [10, Lemma 2.6(8)]. Therefore, since $G$ is not a $\sigma_i$-group, it follows that $V = A_G \cap B_G \neq 1$. Moreover, $A/V$ and $B/V$ are $\sigma$-subnormal subgroups of $G/V$ by [10, Lemma 2.6(4)]. Also, we have $(A/V)(HV/V) = AH/V = HA/V = (HV/V)(A/V)$ and $(B/V)(HV/V) = (HV/V)(B/V)$, where $HV/V$ is a Hall $\sigma$-subgroup of $G/V$. Hence the choice of $G$ implies that

\[
(A \cap B/V)(HV/V) = ((A/V) \cap (B/V))(HV/V) = (HV/V)((A/V) \cap (B/V)) = (HV/V)(A \cap B/V).
\]

But then $(A \cap B)H = (A \cap B)HV = HV(A \cap B) = H(A \cap B)$. This contradiction completes the proof of the result.

**Proposition 2.3.** Let $A$ be a $\sigma$-nilpotent $\sigma$-subnormal subgroup of $G$, and $B$ a characteristic subgroup of $A$. Let $H$ be a Hall $\sigma_i$-subgroup of $G$. If $AH = HA$, then $BH = HB$. 

Each characteristic subgroup of $A$ there is a complete Hall $\sigma$-set of $A$. Hence $B = (A_1 \cap B) \times \cdots \times (A_t \cap B)$, where $\{A_1 \cap B, \ldots, A_t \cap B\}$ is a complete Hall $\sigma$-set of $B$. We can assume without loss of generality that $A_k$ is a $\sigma_k$-subgroup of $A$ for all $k = 1, \ldots, t$.

It is clear that $A_i \cap B$ is characteristic in $A$ for all $i = 1,\ldots, t$. Therefore, if $A_i \cap B < B$, then $(A_i \cap B)H = H(A_i \cap B)$ by the choice of $G$ and so for some $j$, $j = 1$ say, we have $A_1 \cap B = B$, since otherwise we have

$$BH = ((A_1 \cap B) \times \cdots \times (A_t \cap B))H = H((A_1 \cap B) \times \cdots \times (A_t \cap B)) = HB.$$ 

Thus $B \leq A_1$. It is clear that $A_1$ is a $\sigma$-subnormal subgroup of $G$, so in the case when $i = 1$, we have $B \leq A_1 \leq H$ by [10, Lemma 2.6(7)]. But then $BH = H = HB$, a contradiction. Thus $i > 1$.

Now we show that $A_iH = HA_i$. First note that $A_i$ is $\sigma$-subnormal in $G$, so $A_i \leq H$ by [10, Lemma 2.6(7)]. Therefore $A = A_1 \times V \times A_t$, where $V = A_2 \cdots A_{t-1}A_{t+1} \cdots A_t$, and so

$$AH = HA = (A_1 \times V \times A_t)H = (A_1 \times V)H = H(A_1 \times V),$$ 

where $A_1 \times V$ is a $\sigma$-subnormal $\sigma'_t$-subgroup of $G$. Then $A_1 \times V$ is $\sigma$-subnormal in $(A_1 \times V)H$ by [10, Lemma 2.6(1)]. Hence $H \leq N_G(A_1 \times V)$ by [10, Lemma 2.6(8)]. Since $A_1$ is a characteristic subgroup of $A_1 \times V$, we have $H \leq N_G(A_1)$, and so $A_iH = HA_i$. But $B$ is a characteristic subgroup of $A_1$, since $B$ is characteristic in $A$ by hypothesis and $A = A_1 \times \cdots \times A_t$. Therefore $H \leq N_G(B)$, and so $BH = HB$, a contradiction. The proposition is proved. \qed

**Corollary 2.4.** Let $A$ be a $\sigma$-nilpotent subgroup of a $\sigma$-full group $G$. Then the following statements are equivalent:

(i) $A$ is $\sigma$-permutable in $G$.

(ii) Each Hall $\sigma_i$-subgroup of $A$ is $\sigma$-permutable in $G$ for all $i$.

(iii) Each characteristic subgroup of $A$ is $\sigma$-permutable in $G$.

**Proof.** By hypothesis, $A = A_1 \times \cdots \times A_t$, where $\{A_1, \ldots, A_t\}$ is a complete Hall $\sigma$-set of $A$. Then $A_i$ is characteristic in $A$ for all $i = 1, \ldots, t$. Therefore (ii), (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii), (iii) This follows from Proposition 2.3.

The corollary is proved. \qed
Proof of Proposition 1.8. This directly follows from Corollary 2.4. □

Now we are ready to prove Proposition 1.1.

**Proof of Proposition 1.1.** Assume that this proposition is false, and let $G$ be a counterexample with $|G| + |A|$ minimal. Then for some $i$ and some Hall $\sigma_i$-subgroup $H$ of $G$, we have $AH \neq HA$ but $A_1 H = HA_1$ for every $\sigma$-permutable subgroup $A_1$ of $G$ with $A_1 < A$. By hypothesis, $G$ possesses a complete Hall $\sigma$-set $\mathcal{H} = \{H_1, \ldots, H_t\}$ such that $AL^x = L^x A$ for all $L \in \mathcal{H}$ and all $x \in G$. We can assume without loss of generality that $H_k$ is a $\sigma_k$-group for all $k = 1, \ldots, t$. Let $V = H_i$.

First we show that $A_G = 1$. Indeed, assume that $R = A_G \neq 1$. Then $H_0 = \{H_1 R/R, \ldots, H_t R/R\}$ is a complete Hall $\sigma$-set of $G/R$ such that $AL^x/R = (A/R)(LR/R)^x R = (LR/R)^x (A/R) = L^x A/R$ for all $LR/R \in H_0$ and all $xR \in G/R$. On the other hand, $HR/R$ is a Hall $\sigma_i$-subgroup of $G/R$. Hence the choice of $G$ implies that $AH/R = (A/R)(HR/R) = (HR/R)(A/R) = HA/R$, where $\mathcal{A}$ is a complete Hall $\sigma$-set of $G/R$ by Lemma 2.2. Moreover, Lemma 2.2 implies also that $\mathcal{A} \leq N_{G}(A)$ by [10, Lemma 3.1]. Hence $V^G \leq N_{G}(A)$, and so $AH = HA$. This contradiction shows that $A \neq 1$.

The subgroups $A_1, \ldots, A_t$ are characteristic in $A$, so $A_i L^x = L^x A_i$ for all $L \in \mathcal{H}$ and all $x \in G$ by Proposition 1.8. Therefore, the minimality of $|G| + |A|$ implies that $A_i H = HA_i$ for all $i = 1, \ldots, t$, so $AH = HA$. This contradiction completes the proof of the result.

□

3. Proof of Theorem B

Now we use Proposition 1.1 to prove the following fact.

**Lemma 3.1.** Let $R \leq V$ be subgroups of a $\sigma$-full group $G$, where $R$ is normal in $G$. If $V/R$ is $\sigma$-permutable in $G/R$, then $V$ is $\sigma$-permutable in $G$. 


PROOF. Let $i \in I$ and $H$ be a Hall $\sigma_i$-subgroup of $G$. Then $HR/R$ is a Hall $\sigma_i$-subgroup of $G/R$, and so

$$VH/R = (V/R)(HR/R) = (HR/R)(V/R) = HV/R$$

by hypothesis and Proposition 1.1, hence $VH = HV$. The lemma is proved. \qed

Lemma 3.2 (see Lemma 5.2 in [6]). Let $\mathcal{L}$ be a modular sublattice of the lattice $\mathcal{L}(G)$, and $U, V, N \in \mathcal{L}$ with $N \trianglelefteq (U, V)$. If $U$ permutes both with $V \cap UN$ and $VN$, then $U$ permutes with $V$.

Proposition 3.3. Let $G$ be $\sigma$-full and $\mathcal{L} = \mathcal{L}_{\sigma_{\per}}(G)$. Then: (i) $\mathcal{L}$ is a sublattice of $\mathcal{L}_{\sigma_{\per}}(G)$ and (ii) if $\mathcal{L}$ is distributive, then $AB = BA$ for all $A, B \in \mathcal{L}$.

PROOF. (i) Let $A, B \in \mathcal{L}$. By hypothesis, for some Hall $\sigma_i$-subgroup $H$ of $G$ and for each $x \in G$, we have $H^x = AH^x = H^xA$, so $A \leq H \leq O_{\sigma_i}(G)$. Similarly, $B \leq O_{\sigma_i}(G)$. Thus $\langle A, B \rangle$ is a $\sigma_i$-subgroup of $G$ and this subgroup is $\sigma$-permutable in $G$ by Lemma 2.1. Finally, $A \cap B$ is also a $\sigma_i$-subgroup of $G$ and this subgroup is $\sigma$-permutable in $G$ by Theorem A. Thus we have (i).

(ii) Suppose that this assertion is false, and let $G$ be a counterexample with $|G| + |A| + |B|$ minimal. Thus $AB \neq BA$ but $A_1B_1 = B_1A_1$ for all $A_1, B_1 \in \mathcal{L}$ such that $A_1 \leq A$, $B_1 \leq B$ and either $A_1 \neq A$ or $B_1 \neq B$. Let $V = \langle A, B \rangle O_{\sigma_i}(G)$ and $R = \langle A, B \rangle \cap O_{\sigma_i}(G)$. Then $V$ is $\sigma$-subnormal in $G$.

(1) The group $V$ is $\sigma$-full and $\mathcal{L}_{\sigma_{\per}}(V)$ is a sublattice of $\mathcal{L}$.

First note that each Hall $\sigma_j$-subgroup of $G$ is contained in $V$ for all $j \neq i$, since $O_{\sigma_i}(G)$ is the subgroup of $G$ generated by all its $\sigma_i'$-elements. On the other hand, $H \cap V$ is a Hall $\sigma_i$-subgroup of $V$ for each Hall $\sigma_i$-subgroup $H$ of $G$ by [10, Lemma 2.6(7)], so $V$ is $\sigma$-full.

Now, let $H \in \mathcal{L}_{\sigma_{\per}}(V)$. Then $H \leq O_{\sigma_i}(V) \leq O_{\sigma_i}(G)$ by [10, Lemma 2.6(11)]. Therefore $H$ permutes with each Hall $\sigma_i$-subgroup of $G$. On the other hand, for every $j \neq i$ and for each Hall $\sigma_j$-subgroup $W$ of $G$, we have $HW = WH$ since $W \leq V$. Hence $H \in \mathcal{L}_{\sigma_{\per}}(G)$, which implies (1).

(2) $V = G$, so $\langle A, B \rangle \leq G$.

Claim (1) implies that the hypothesis holds for $\mathcal{L}_{\sigma_{\per}}(V)$, and so in the case when $V \neq G$, the choice of $G$ implies that $AB = BA$. Thus $G = \langle A, B \rangle O_{\sigma_i}(G)$. Therefore, since $O_{\sigma_i}(G) \leq N_G(\langle A, B \rangle)$ by [10, Lemma 3.1], $\langle A, B \rangle$ is normal in $G$. 

(3) \( R = 1 \).

Assume that \( R = \langle A, B \rangle \cap O^\sigma(G) \neq 1 \). First we show that \( BRA = \langle A, B \rangle R \). Indeed, let \( H/R \) be a \( \sigma \)-subgroup of \( G/R \). Then \( H \) is a \( \sigma \)-group since \( \langle A, B \rangle \leq O^\sigma(G) \). Moreover, Lemma 3.1 and [10, Lemma 2.8(2)] imply that \( H/R \) is \( \sigma \)-permutable in \( G/R \) if and only if \( H \) is \( \sigma \)-permutable in \( G \). Therefore the lattice \( L_{\sigma, \text{per}}(G/R) \) is isomorphic to the interval \([G/R] \) in the distributive lattice \( L \). Therefore, by the minimality of \( G \), \( (AR/R)(BR/R) = (BR/R)(AR/R) \), and so \( BRA = \langle A, B \rangle R \).

Now we show that \( BRA = BR \). Assume that this is false. Then \( A \cap BR < A \). But Theorem A implies that \( A \cap BR \) is \( \sigma \)-permutable in \( G \), so the minimality of \( |G| + |A| + |B| \) implies that \( B \) permutes with \( A \cap BR \). Also, \( B \) permutes with \( RA \) since \( B(AR) = \langle A, B \rangle R \), so \( AB = BA \) by Lemma 3.2, Part (i) and Theorem A. This contradiction shows that \( A \leq BR \), so \( BRA = BR \). But \( R \leq O^\sigma(G) \leq N_G(B) \) by [10, Lemma 3.1], hence \( B \) is normal in \( BR \), and since \( A \leq BR \), it follows that \( AB = BA \). This contradiction shows that we have (3).

Final contradiction. Claims (2) and (3) imply that \( G = \langle A, B \rangle O^\sigma(G) = \langle A, B \rangle \times O^\sigma(G) \), so every subgroup \( H \) of \( \langle A, B \rangle \) is \( O^\sigma(G) \)-invariant since \( \langle A, B \rangle \leq O^\sigma(G) \). It follows that every subgroup of \( \langle A, B \rangle \) is \( \sigma \)-permutable in \( G \). Hence \( L(\langle A, B \rangle) \) is a sublattice of the distributive lattice \( L \). Thus \( \langle A, B \rangle \) is cyclic by the Ore theorem [8, Theorem 1.2.3], so \( AB = BA \), a contradiction. The proposition is proved.

Corollary 3.4. If \( G \) is \( \sigma \)-full and the lattice \( L = L_{\sigma, \text{per}}(G) \) is distributive, then every two members \( A \) and \( B \) of \( L \) are permutable.

Proof. Suppose that this corollary is false, and let \( G \) be a counterexample with \( |G| + |A| + |B| \) minimal.

Let \( R \) be a minimal normal subgroup of \( G \). Then the lattice \( L_{\sigma, \text{per}}(G/R) \) is isomorphic to the interval \([G/R] \) in the distributive lattice \( L \) by Lemma 3.1 and [10, Lemma 2.8(2)]. Therefore [10, Lemma 2.8(2)] and the minimality of \( G \) imply that \( (AR/R)(BR/R) = (BR/R)(AR/R) \). It follows that \( RAB = \langle A, B \rangle R \) is a subgroup of \( G \), so \( A_G = 1 = B_G \). Hence, because of Lemma 2.2, \( A \) and \( B \) are \( \sigma \)-nilpotent. The minimality of \( |G| + |A| + |B| \) implies that for some \( i \) we have \( A, B \leq O^\sigma_G \), and so \( A, B \in L_{\sigma^i, \text{per}}(G) \). But \( L_{\sigma_i, \text{per}}(G) \) is a sublattice of the distributive lattice \( L_{\sigma, \text{per}}(G) \) by Proposition 3.3(i). Therefore, \( AB = BA \) by Proposition 3.3(iii), a contradiction. The corollary is proved.

Lemma 3.5 (see [4, p. 59]). A modular lattice \( L \) is distributive if and only if \( L \) has no distinct elements \( a, b \) and \( c \) such that \( a \lor b = a \lor c = b \lor c \) and \( a \land b = a \land c = b \land c \).
Lemma 3.6 (see [8, Theorem 1.6.2]). Let $G = A \times B$, $f : A \to B$ be an isomorphism and $C = \{aa^f | a \in A\}$. Then $G = AC = BC$ and $A \cap C = 1 = B \cap C$.

Proof of Theorem B. Let $D = G^{\alpha\sigma}$. (i) $\Rightarrow$ (ii) First note that every two members of $L$ are permutable by Corollary 3.4. Moreover, since the lattice $\mathcal{L}_n(G)$ is a sublattice of the lattice $\mathcal{L}$, it is distributive. Now note that since $G/D = G/G^{\alpha\sigma}$ is $\sigma$-nilpotent, every subgroup $E$ of $G$ satisfying $D \leq E \leq G$ is $\sigma$-permutable in $G$ by Lemma 3.1. Hence $\mathcal{L}(G/D) = \mathcal{L}_{\sigma\per}(G/D)$. In view of Lemma 3.1 and [10, Lemma 2.8(2)], the lattice $\mathcal{L}_{\sigma\per}(G/D)$ is isomorphic to the interval $[G/D]$ in lattice $\mathcal{L}$, so $\mathcal{L}_{\sigma\per}(G/D)$ is distributive. Hence $G/D$ is cyclic by the Ore theorem [8, Theorem 1.2.3]. It is clear also that $D$ is a meet-distributive element of $\mathcal{L}$. Thus the lattice $\mathcal{L}$ satisfies the $W\sigma D$-condition.

We show that in every factor group $\bar{G} = G/R$, any two $O^{\alpha\sigma}(\bar{G})$-isomorphic sections $H/K$ and $L/K$, where $K, H, L \in \mathcal{L}_{\sigma\per}(G)$, coincide. In view of Lemma 3.1 and [10, Lemma 2.8(2)], it is enough to consider the case when $\bar{G} = G$ and $\bar{K} = K, \bar{H} = H, \bar{L} = L \in \mathcal{L}_{\sigma\per}(G)$.

Suppose that $H \neq L$. Then $G$. Let $K < H_0 \leq H$, where $H_0$ covers $K$ in $L$, and let $L_0/K = (H_0/K)^f$, where $f : H/K \to L/K$ is an $O^{\alpha\sigma}(G)$-isomorphism. For $g \in O^{\alpha\sigma}(G)$ and $l_0K = (hK)^f \in L_0/K$, where $h \in H_0$, we have

$$(l_0K)^g = ((hK)^f)^g = ((h^gK)^f)^f = (h^gK)^f \in L_0/K,$$

where $l_0 \in H_0$, since $H_0$ is $O^{\alpha\sigma}(G)$-invariant by [10, Lemma 3.1]. Hence $(l_0K)^g \in L_0/K$. It follows that $L_0$ is $O^{\alpha\sigma}(G)$-invariant, and so $L_0$ covers $K$ in $L$, since the inverse map $f^{-1} : L/K \to H/K$ is an $O^{\alpha\sigma}(G)$-isomorphism too.

First assume that $H_0 \neq L_0$, and let $E_0/K = \{hK(hK)^f | h \in H_0/K\}$. Then $(H_0/K)(L_0/K) = (H_0/K) \times (L_0/K)$. Indeed, if $H_0 \neq H_0$ for some $x \in L_0$, then (i) and the fact that $H_0$ and $L_0$ cover $K$ in $L$ would imply that $\{K; H_0; H_0; L_0; H_0L_0\}$ would be a diamond in the distributive lattice $\mathcal{L}$, contradicting Lemma 3.5. Hence, by Lemma 3.6, $E_0/K$ is a subgroup of $(H_0/K) \times (L_0/K)$, and we have

$$(H_0/K) \times (L_0/K) = (H_0/K) \times (E_0/K) = (L_0/K) \times (E_0/K).$$

Note that if $g \in O^{\alpha\sigma}(G)$ and $hK(hK)^f \in E_0/K$, then

$$(hK(hK)^f)^g = (hK)^g((hK)^f)^g = (h^gK)^f \in E_0/K,$$

since $f_{H_0/K}$ is an $O^{\alpha\sigma}(G)$-isomorphism from $H_0/K$ onto $L_0/K = (H_0/K)^f$. Hence $E_0/K$ is $O^{\alpha\sigma}(G)$-invariant, so $O^{\alpha\sigma}(G) \leq N_G(E_0)$. Therefore, $H_0, L_0$ and $E_0$
are distinct elements of $\mathcal{L}$ such that $H_0 \cap L_0 = H_0 \cap E_0 = L_0 \cap E_0 = K$ and $H_0L_0 = H_0E_0 = L_0E_0$, which is impossible by Lemma 3.5, since $H_0L_0$ is a $\sigma$-permutable subgroup of $G$. Therefore $H_0 = L_0$. Now $f$ induces an $O^\sigma(G)$-isomorphism $f': H/H_0 \to L/H_0$, and an obvious induction yields that $H = L$.

Hence the implication (i) $\Rightarrow$ (ii) holds.

(ii) $\Rightarrow$ (iii) This implication is evident.

(iii) $\Rightarrow$ (i) Suppose that this is false, and let $G$ be a counterexample of minimal order.

First note that if $A, B, C \in \mathcal{L}_{\sigma \text{per}}(G)$ and $A \leq C$, then

$$C \cap \langle A, B \rangle = C \cap AB = A(C \cap B) = \langle A, C \cap B \rangle$$

by hypothesis, so the lattice $\mathcal{L}_{\sigma \text{per}}(G)$ is modular. Hence, by Lemma 3.5, there are distinct $\sigma$-permutable subgroups $A, B$ and $C$ of $G$ such that for some $\sigma$-permutable subgroups $E$ and $T$ of $G$, we have $E = A \cap B = A \cap C = B \cap C$ and $T = AB = AC = BC$.

(1) The lattice $\mathcal{L}_{\sigma \text{per}}(G/R)$ is distributive for each non-identity normal subgroup $R$ of $G$.

In view of the choice of $G$, it is enough to show that the hypothesis holds for $G = G/R$.

Let $\bar{K}, \bar{H} \in \mathcal{L}_{\sigma \text{per}}(\bar{G})$. Then $K, H \in \mathcal{L}$ by Lemma 3.1, and so $KH = HK$ by hypothesis, which implies that $\langle K/R \rangle(H/R) = (H/R)(K/R)$. It is clear also that the lattice $\mathcal{L}_n(\bar{G})$ is isomorphic to some sublattice of the lattice $\mathcal{L}_n(G)$, so $\mathcal{L}_n(\bar{G})$ is distributive.

In view of [2, Proposition 2.2.8] and [10, Corollary 2.4 and Lemma 2.5], we have $\bar{G}^{\sigma R} = \bar{G}^{\sigma R}/R = DR/R$. Thus $G/\bar{G}^{\sigma R} = (G/R)/(DR/R) \cong G/DR \cong (G/D)/(DR/D)$ is cyclic, since $G/D$ is cyclic by hypothesis.

By hypothesis we have also that $D \cap \langle K, H \rangle = D \cap KH = \langle D \cap K, D \cap H \rangle = (D \cap K)(D \cap H)$, since $D \cap K$ and $D \cap H$ are $\sigma$-permutable in $G$ by Theorem A, so

$$\bar{G}^{\sigma R} \cap \langle \bar{K}, \bar{H} \rangle = (DR \cap K/H)/R = R(D \cap KH)/R = ((D \cap K)R/R)((D \cap H)R/R)$$

$$= ((DR/R) \cap (K/R))((DR/R) \cap (H/R)) \cong (\bar{G}^{\sigma R} \cap K, \bar{G}^{\sigma R} \cap H).$$

Hence $\bar{G}^{\sigma R}$ is a meet-distributive element of $\mathcal{L}_{\sigma \text{per}}(\bar{G})$. Thus the lattice $\mathcal{L}_{\sigma \text{per}}(\bar{G})$ satisfies the $W\sigma D$-condition.

Finally, let $\bar{N}$ be any normal subgroup of $\bar{G}$. Let $\bar{G} = \bar{G}/\bar{N}$, and let $\hat{H}/\hat{K} = (\bar{H}/\bar{N})/(\bar{K}/\bar{N})$ and $\hat{L}/\hat{K} = (\bar{L}/\bar{N})/(\bar{K}/\bar{N})$ be $O^\sigma(\bar{G})$-isomorphic sections, where $\bar{K}, \bar{H}, \bar{L} \in \mathcal{L}_{\sigma \text{per}}(\bar{G})$ and the subgroups $\hat{H}$ and $\hat{L}$ cover $\bar{K}$ in $\mathcal{L}_{\sigma \text{per}}(\bar{G})$. Then we have $(H/N)/(K/N)$ and $(L/N)/(K/N)$ are $O^\sigma(G/N)$-isomorphic sections,
where \( K/N, H/N, L/N \in \mathcal{L}_{\sigma, \text{per}}(G/N) \) and the subgroups \( H/N \) and \( L/N \) cover \( K/N \) in \( \mathcal{L}_{\sigma, \text{per}}(G/N) \). But then \( H/N = L/N \) by hypothesis, which implies that \( \bar{H}/\bar{K} = \bar{L}/\bar{K} \). Therefore the hypothesis holds for \( G/R \), so we have (1).

(2) \( E_G = 1 \).

In view of Lemma 3.1, this follows from Claim (1), Lemma 3.5 and the choice of \( G \).

(3) \( A_G B_G \cap A_G C_G \cap B_G C_G = 1 \).

Since \( A \cap B = E \), we have \( B_G \cap A_G \leq E_G = 1 \) by Claim (2). Similarly, \( B_G \cap C_G = 1 \) and \( A_G \cap C_G = 1 \). Therefore,

\[
(A_G B_G \cap A_G C_G) \cap B_G C_G
\]

\[
= A_G (B_G \cap A_G C_G) \cap B_G C_G = A_G (B_G \cap A_G) (B_G \cap C_G) \cap B_G C_G
\]

\[
= A_G \cap B_G C_G = (A_G \cap B_G) \cap (A_G \cap C_G) = 1
\]

by hypothesis.

(4) The subgroup \( T \) is \( \sigma \)-nilpotent.

Note that

\[
\]

where

\[
AA_G B_G/A_G B_G \cong A/A \cap A_G B_G = A/A_G (A \cap B_G) \cong (A/A_G)/(A_G (A \cap B_G)/A_G)
\]

and

\[
B A_G B_G/A_G B_G \cong (B/B_G)/(B_G (B \cap A_G)/B_G)
\]

are \( \sigma \)-nilpotent by Lemma 2.2. We know that the subgroups \( AA_G B_G/A_G B_G \) and \( B A_G B_G/A_G B_G \) are \( \sigma \)-subnormal in \( G/A_G B_G \) by Lemma 2.2. Hence \( T/A_G B_G \)

is \( \sigma \)-nilpotent by [10, Lemma 2.6(11)]. Similarly, \( T/A_G C_G \) and \( T/C_G B_G \) are \( \sigma \)-nilpotent. Hence from Claim (3) it follows that \( T \cong T/(A_G B_G \cap A_G C_G \cap B_G C_G) \)

is \( \sigma \)-nilpotent by Corollary 2.4 and [10, Lemma 2.5].

(5) For some \( i \), there are distinct \( \sigma_i \)-subgroups \( A_i, B_i, C_i \in \mathcal{L} \) such that \( H_i = A_i B_i = A_i C_i = B_i C_i \) and \( K_i = A_i \cap B_i = A_i \cap C_i = B_i \cap C_i \) are \( \sigma_i \)-permutable subgroups of \( G \).

Let \( \sigma_i \in \sigma(T) \), that is, \( \sigma_i \cap \pi(T) \neq \emptyset \). Then, by Claim (4), \( H_i = O_{\sigma_i}(T) \) is the Hall \( \sigma_i \)-subgroup of \( T \) and \( A_i = O_{\sigma_i}(A) \), \( B_i = O_{\sigma_i}(B) \) and \( C_i = O_{\sigma_i}(C) \) are the Hall \( \sigma_i \)-subgroups of \( A, B \) and \( C \), respectively. Hence \( H_i = A_i B_i = A_i C_i = B_i C_i \).

Moreover, \( A_i, B_i \) and \( C_i \) are \( \sigma \)-permutable in \( G \) by Proposition 1.8. It is clear also that \( K_i = A_i \cap B_i = A_i \cap C_i = B_i \cap C_i = O_{\sigma_i}(E) \).
Suppose that $A_i = B_i$. Then $H_i = A_iB_i = A_i = B_i = K_i \leq C_i \leq H_i$. Hence $A_i = B_i = C_i$. Therefore, since $A \neq B \neq C$ and $A \neq C$, there is $\sigma_i \in \sigma(T)$ such that $A_i \neq B_i \neq C_i$ and $A_i \neq C_i$. Finally, $H_i$ and $K_i$ are evidently $\sigma$-permutable subgroups of $G$, so we have (5).

(6) There are distinct $\sigma_i$-subgroups $A_0, B_0, C_0 \in \mathcal{L}$ such that $H_0 = A_0B_0 = A_0C_0 = B_0C_0$ and $K_0 = A_0 \cap B_0 = A_0 \cap C_0 = B_0 \cap C_0$ are $\sigma$-permutable subgroups of $G$ and $A_0, B_0, C_0$ are normal subgroups of $O^{\sigma_i}(G)$.

Let $A_0 = A_i \cap D$, $B_0 = B_i \cap D$ and $C_0 = C_i \cap D$. Then $A_0$, $B_0$ and $C_0$ are $\sigma$-permutable $\sigma_i$-subgroups of $G$ by Claim (5) and Theorem A. Moreover, Claim (5) implies that $K_0 = A_0 \cap B_0 = A_i \cap B_i \cap D = A_i \cap C_i \cap D = A_0 \cap C_0 = B_i \cap C_i \cap D = B_0 \cap C_0$.

Since $D$ is a meet-distributive element of $\mathcal{L}$ by hypothesis,

$$H_0 = D \cap A_iB_i = (D \cap A_i)(D \cap B_i) = A_0B_0 = A_0C_0 = D \cap A_iC_i = D \cap B_iC_i = B_0C_0.$$

Now we show that $A_0, B_0, C_0$ are distinct elements of $\mathcal{L}$. First note that

$$|H_i : K_i| = |A_i : K_i||B_i : K_i| = |A_i : K_i||C_i : K_i| = |B_i : K_i||C_i : K_i|,$$

so $|A_i : K_i| = |B_i : K_i| = |C_i : K_i|$. Hence $|A_i| = |B_i| = |C_i|$. Suppose that $A_0 = B_0$. Then

$$D \cap H_i = D \cap A_iB_i = (D \cap A_i)(D \cap B_i) = A_0B_0 = A_0 = B_0 = D \cap K_i.$$

Hence $K_i D \cap H_i = K_i(D \cap H_i) = K_i$ is normal in $H_i$ and $DH_i/DK_i \simeq H_i/(H_i \cap K_iD) = H_i/K_i(H_i \cap D) = H_i/K_i$ is cyclic, since $G/D$ is cyclic by hypothesis. On the other hand, $H_i/K_i = (A_i/K_i)(B_i/K_i)$, where $|A_i/K_i| = |B_i/K_i|$, so $A_i/K_i = B_i/K_i = 1$, which implies that $A_i = B_i$. This contradiction shows that $A_0 \neq B_0$. Similarly, $A_0 \neq C_0$ and $B_0 \neq C_0$. Finally, $A_0, B_0, C_0$ are normal subgroups of $O^{\sigma_i}(G)$ by [10, Lemma 3.1]. Finally, Claim (5) and Theorem A imply that $K_0$ and $H_0$ are $\sigma$-permutable in $G$.

(7) $A_0/K_0$ and $B_0/K_0$ are $O^{\sigma_i}(G)$-isomorphic.

From Claim (6) we get that

$$H_0/K_0 = (A_0/K_0) \times (B_0/K_0) = (A_0/K_0) \times (C_0/K_0) = (B_0/K_0) \times (C_0/K_0).$$
Therefore,

\[ A_0/K_0 \simeq ((A_0/K_0) \times (C_0/K_0))/(C_0/K_0) = (H_0/K_0)/(C_0/K_0) \]

and

\[ B_0/K_0 \simeq ((B_0/K_0) \times (C_0/K_0))/(C_0/K_0) = (H_0/K_0)/(C_0/K_0) \]

are \( O^{\sigma_i}(G) \)-isomorphisms by [10, Lemma 3.1]. Hence we have (7).

**Final contradiction.** Let \( f : A_0/K_0 \to B_0/K_0 \) be an \( O^{\sigma_i}(G) \)-isomorphism. Let \( K_0 < X \leq A_0 \), where \( X \) covers \( K_0 \) in \( L \). Then \( X/K_0 \) is a chief factor of \( O^{\sigma_i}(G) \) by [10, Lemma 3.1], so \( L/K_0 = f(X/K_0) \) is also a chief factor of \( O^{\sigma_i}(G) \). Hence \( L \) covers \( K_0 \) in \( L \). Now \( f \) induces an \( O^{\sigma_i}(G) \)-isomorphism from \( X/K_0 \) onto \( L/K_0 \), and so \( L = X \) by hypothesis. Hence \( K_0 < A_0 \cap B_0 \), contrary to (6).

The implication is proved.

The theorem is proved. □

**References**

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BIN HU
SCHOOL OF MATHEMATICS AND STATISTICS
JIANGSU NORMAL UNIVERSITY
XUZHOU, 221116
P. R. CHINA
E-mail: hubin118@126.com

JIANHONG HUANG
SCHOOL OF MATHEMATICS AND STATISTICS
JIANGSU NORMAL UNIVERSITY
XUZHOU, 221116
P. R. CHINA
E-mail: jhh320@126.com

ALEXANDER N. SKIBA
DEPARTMENT OF MATHEMATICS AND
TECHNOLOGIES OF PROGRAMMING
FRANCISK SKORINA GOMEL STATE UNIVERSITY
GOMEL, 246019
BELARUS
E-mail: alexander.skiba49@gmail.com

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