Commutativity of torsion and normal Jacobi operators on real hypersurfaces in the complex quadric

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Abstract. On a real hypersurface in the complex quadric we can consider the Levi-Civita connection and, for any non-zero real constant $k$, the $k$-th generalized Tanaka–Webster connection. Associated to this connection we can define a differential operator whose difference with the Lie derivative is the torsion operator of the $k$-th generalized Tanaka–Webster connection. We prove the non-existence of real hypersurfaces in the complex quadric for which the torsion operators commute with the normal Jacobi operator of the real hypersurface.

1. Introduction

The complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ is a compact Hermitian symmetric space of rank 2. It is also a complex hypersurface in the complex projective space $\mathbb{CP}^{m+1}$ (see [5], [6], [8]). The space $Q^m$ is equipped with two geometric structures: a Kaehler structure $J$ and a parallel circle subbundle $\mathfrak{A}$ of the endomorphism bundle $\text{End}(TQ^m)$, which consists of all the real structures on the tangent space of $Q^m$. For any $A \in \mathfrak{A}$ the following relations hold: $A^2 = I$ and $AJ = -JA$. A nonzero tangent vector $W$ at a point of $Q^m$ is called singular if it is tangent to more than one maximal flat in $Q^m$. There are two types of singular tangent vectors for $Q^m$: $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic vectors.

Real hypersurfaces $M$ are immersed submanifolds of real co-dimension 1 in a Hermitian manifold. Since $Q^m$ is a compact Hermitian symmetric space with
rank 2, it is interesting to study real hypersurfaces $M$ in $Q^m$. The Kaehler structure $J$ of $Q^m$ induces on $M$ an almost contact metric structure $(\phi, \xi, \eta, g)$, where $\phi$ is the structure tensor field, $\xi$ is the Reeb vector field, $\eta$ is a 1-form and $g$ is the induced Riemannian metric of $Q^m$.

The study of real hypersurfaces $M$ in $Q^m$ is initiated by Berndt and Suh in [1]. In this paper the geometric properties of real hypersurfaces $M$ in complex quadric $Q^m$, which are tubes of radius $r$, $0 < r < \pi/2$, around the totally geodesic $CP^k$ in $Q^m$, when $m = 2k$ or tubes of radius $r$, $0 < r < \pi/2\sqrt{2}$, around the totally geodesic $Q^{m-1}$ in $Q^m$, are presented. The condition of isometric Reeb flow is equivalent to the commuting condition of the shape operator $S$ with the structure tensor $\phi$ of $M$. The classification of such real hypersurfaces in $Q^m$ is obtained in [2].

Given a Riemannian manifold $(\tilde{M}, \tilde{g})$, Jacobi fields along geodesics satisfy a differential equation which results in the notion of Jacobi operator. That is, if $\tilde{R}$ is the Riemannian curvature tensor of $\tilde{M}$, and $X$ is a tangent vector field on $\tilde{M}$, then the Jacobi operator with respect to $X$ at a point $p \in \tilde{M}$ is given by

$$(\tilde{R}_X Y)(p) = (\tilde{R}(Y,Y)X)(p),$$

and becomes a self adjoint endomorphism of the tangent bundle $T\tilde{M}$ of $\tilde{M}$, i.e., $\tilde{R}_X \in \text{End}(T_p\tilde{M})$. In the case of real hypersurfaces $M$ in $Q^m$, we can consider the normal Jacobi operator $\tilde{R}_N$, where $\tilde{R}$ is the Riemannian curvature tensor of $Q^m$ and $N$ is the unit normal vector field on the real hypersurface $M$.

As $M$ has an almost contact metric structure, for any non-zero real constant $k$, we can define the so called $k$-th generalized Tanaka–Webster connection $\tilde{\nabla}^{(k)}$ on $M$ by

$$\tilde{\nabla}^{(k)}_XY = \nabla_X Y + g(\phi SY, Y)\xi - \eta(Y)\phi SY - k\eta(X)\phi Y$$

for any $X, Y$ tangent to $M$, where $\nabla$ is the Levi-Civita connection on $M$, and $S$ denotes the shape operator on $M$ associated to $N$ (see [3]). Let us call $F^{(k)}_X Y = g(\phi SY, Y)\xi - \eta(Y)\phi SY - k\eta(X)\phi Y$, for any $X, Y$ tangent to $M$. $F^{(k)}_X$ is called the $k$-th Cho operator on $M$ associated to $X$. Notice that if $X \in \mathcal{C}$, the maximal holomorphic distribution on $M$, given by all the vector fields orthogonal to $\xi$, the associated Cho operator does not depend on $k$ and we will denote it simply by $F_X$. Then, given a symmetric tensor field $L$ of type $(1,1)$ on $M$, $\nabla_X L = \tilde{\nabla}^{(k)}_X L$ for a tangent vector field $X$ on $M$ if and only if $F^{(k)}_X L = LF^{(k)}_X$, that is, the eigenspaces of $L$ are preserved by $F^{(k)}_X$. If $L = \tilde{R}_N$, in [4] we proved

**Theorem 1.1.** There do not exist real hypersurfaces $M$ in $Q^m$, $m \geq 3$, such that $\nabla\tilde{R}_N = \tilde{\nabla}^{(k)}\tilde{R}_N$, for any non-zero real constant $k$. 
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The torsion of the \( k \)-th generalized Tanaka–Webster connection is given by
\[
T^{(k)}(X,Y) = F_X^{(k)}Y - F_Y^{(k)}X
\]
for any \( X,Y \) tangent to \( M \). For any \( X \) tangent to \( M \), we define the torsion operator associated to \( X \) by \( T^X_{(k)}Y = T^{(k)}(X,Y) \) for any \( Y \) tangent to \( M \).

Let \( \mathcal{L} \) denote the Lie derivative on \( M \). Associated to the \( k \)-th generalized Tanaka–Webster connection, we can define the differential operator of first order \( \mathcal{L}^{(k)} \) by
\[
\mathcal{L}^{(k)}X Y = \nabla^{(k)}_X Y - \nabla^{(k)}_Y X = \mathcal{L}_X Y + T^X_{(k)}Y
\]
for any \( X,Y \) tangent to \( M \).

Then for a symmetric tensor of type (1,1) on \( M \),
\[
\mathcal{L}^{(k)}X Y = \frac{1}{2} [\mathcal{L} \bar{R}_N X Y] \quad \text{for any tangent vector field } X \text{ on } M
\]
if and only if \( T^X_{(k)} = \mathcal{L}_X \bar{R}_N \).

In this paper we study real hypersurfaces \( M \) in \( Q^m \) such that the Lie derivative and the differential operator \( \mathcal{L}^{(k)} \) associated to the \( k \)-th generalized Tanaka–Webster connection coincide when we apply them to the normal Jacobi operator \( \bar{R}_N \), that is
\[
\mathcal{L} \bar{R}_N = \mathcal{L}^{(k)} \bar{R}_N
\]
for some non-zero real constant \( k \). We will prove the following

**Theorem 1.2.** There do not exist real hypersurfaces \( M \) in \( Q^m \), \( m \geq 3 \), such that \( \mathcal{L} \bar{R}_N = \mathcal{L}^{(k)} \bar{R}_N \), for any non-zero real constant \( k \).

2. The space \( Q^m \)

The complex projective space \( \mathbb{C}P^{m+1} \) is considered as the Hermitian symmetric space of the special unitary group \( SU_{m+2} \), namely
\[
\mathbb{C}P^{m+1} = SU_{m+2}/SU(m+1U_1).
\]
The symbol \( o = [0,...,0,1] \) in \( \mathbb{C}P^{m+1} \) is the fixed point of the action of the stabilizer \( SU_{m+1}U_1 \). The action of the special orthogonal group \( SO_{m+2} \subset SU_{m+2} \) on \( CP^{m+1} \) is of cohomogeneity one. A totally geodesic real projective space \( \mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1} \) is an orbit containing point \( o \). The second singular orbit of this action is the complex quadric \( Q^m = SO_{m+2}/SO_mSO_2 \). It is a homogeneous model, which interprets geometrically the complex quadric \( Q^m \) as the Grassmann manifold \( G^2_2(\mathbb{R}^{m+2}) \) of oriented 2-planes in \( \mathbb{R}^{m+2} \). Thus, the complex quadric \( Q^m \) is considered as a Hermitian space of rank 2. For \( m = 1 \), the complex quadric \( Q^2 \) is isometric to a sphere \( S^2 \) of constant curvature. For \( m = 2 \), the complex quadric \( Q^2 \) is isometric to the Riemannian product of two 2-spheres with constant curvature. Therefore, we assume the dimension of complex quadric \( Q^m \) to be greater than or equal to 3.
Moreover, the complex quadric $Q^m$ is the complex hypersurface in $\mathbb{C}P^{m+1}$ defined by the homogeneous quadric equation $z_1^2 + \cdots + z_{m+2}^2 = 0$, where $z_i$, $i = 1, \ldots, m+2$, are homogeneous coordinates on $\mathbb{C}P^{m+1}$. The Kähler structure of complex projective space $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure $(J,g)$ on $Q^m$, where $g$ is a Riemannian metric with maximal holomorphic sectional curvature 4 induced by the Fubini Study metric of $\mathbb{C}P^{m+1}$.

Consider the Riemannian fibration $\pi : S^{2m+3} \subset \mathbb{C}^{m+2} \to \mathbb{C}P^{m+1}$, $z \to [z]$. Then $\mathbb{C}^{m+2} \subset [z]$ is the horizontal space of $\pi$ at $z \in S^{2m+3}$. Then at each $[z]$ in $Q^m$ the tangent space $T_{[z]}Q^m$ can be identified canonically with the orthogonal complement of $\mathbb{C}^{m+2} \subset ([z] \oplus [\bar{z}])$ of $[z] \oplus [\bar{z}]$ in $\mathbb{C}^{m+2}$. Thus $\pi_*|_z \bar{z}$ is a unit normal vector of $Q^m$ in $\mathbb{C}P^{m+1}$ at the point $[z]$.

The shape operator $A_z$ of $Q^m$ with respect to the unit normal vector $\bar{z}$ is given by
\[ A_z \pi_*|_z w = \pi_*|_z \bar{w}, \]
for all $w \in T_{[z]}Q^m$. The shape operator $A_z$ is a complex conjugation restricted to $T_{[z]}Q^m$. The complex vector space $T_{[z]}Q^m$ is decomposed into
\[ T_{[z]}Q^m = V(A_z) \oplus JV(A_z), \]
where $V(A_z) = \mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the $(+1)$-eigenspace of $A_z$, i.e., $A_z X = X$, and $JV(A_z) = i\mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the $(-1)$-eigenspace of $A_z$, i.e., $A_z JX = -JX$ for any $X \in V(A_z)$. Geometrically, it means that $A_z$ defines a real structure on the complex vector space $T_{[z]}Q^m$, which is an antilinear involution. The set of all such shape operators $A_z$ defines a parallel circle subbundle $\mathfrak{Q}$ of the endomorphism bundle $\text{End}(TQ^m)$, which consists of all the real structures on the tangent space of $Q^m$. For any $A \in \mathfrak{Q}$ the following relations hold:
\[ A^2 = I \quad \text{and} \quad AJ = -JA. \]

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ yields that the Riemannian curvature tensor $R$ of $Q^m$ is given by
\[
\hat{R}(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY,
\]
where $J$ is the complex structure, $g$ is the Riemannian metric and $A$ is a real structure in $\mathfrak{Q}$.

A nonzero tangent vector $W \in T_{[z]}Q^m$ is called singular if it is tangent to more than one maximal flat in $Q^m$. There are two types of singular tangent vectors for $Q^m$:
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1. **A-principal.** In this case, there exists a real structure $A \in \mathfrak{A}$ such that $W \in V(A)$.

2. **A-isotropic.** In this case, there exists a real structure $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$.

For every unit vector field $W$ tangent to $Q^n$, there is a complex conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that

$$ W = \cos(t)X + \sin(t)JY, $$

for some $t \in [0, \pi/4]$. The singular vectors correspond to the values $t = 0$ and $t = \pi/4$.

3. **Real hypersurfaces in $Q^n$**

Let $M$ be a real hypersurface in $Q^n$ and $N$ a unit normal vector field of $M$. Any vector field $X$ tangent to $M$ satisfies the relation

$$ JX = \phi X + \eta(X)N. $$

The tangential component of the above relation defines on $M$ a skew-symmetric tensor field of type (1,1) $\phi$, named the structure tensor. The structure vector field $\xi$ is defined by $\xi = -NJ$ and is called the Reeb vector field. The 1-form $\eta$ is given by $\eta(X) = g(X, \xi)$ for any vector field $X$ tangent to $M$. So, on $M$ an almost contact metric structure $(\phi, \xi, \eta, g)$ is defined. The elements of the almost contact structure satisfy the following relations:

$$ \phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) $$

for all tangent vectors $X, Y$ to $M$. Relation (3.2) implies

$$ \phi \xi = 0. $$

The tangent bundle $TM$ of $M$ splits orthogonally into

$$ TM = \mathcal{C} \oplus \mathcal{F}, $$

where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of $TM$ and $\mathcal{F} = \mathbb{R}\xi$. The structure tensor field $\phi$ restricted to $\mathcal{C}$ coincides with the complex structure $J$. 

The shape operator of a real hypersurface $M$ in $Q^m$ is denoted by $S$. The real hypersurface is called Hopf hypersurface if the Reeb vector field is an eigenvector of the shape operator, i.e.,
\[ S\xi = \alpha \xi, \tag{3.3} \]
where $\alpha = g(S\xi, \xi)$ is the Reeb function.

At each point $[z] \in M$, we choose a real structure $A \in \mathfrak{A}[z]$ such that
\[ N[z] = \cos(t)Z_1 + \sin(t)JZ_2, \quad AN[z] = \cos(t)Z_1 - \sin(t)JZ_2, \tag{3.4} \]
where $Z_1, Z_2$ are orthonormal vectors in $V(A)$ and $0 \leq t \leq \frac{\pi}{4}$. Moreover, the above relations due to $\xi = -JN$ imply
\[ \xi[z] = -\cos(t)JZ_1 + \sin(t)Z_2, \quad A\xi[z] = \cos(t)JZ_1 + \sin(t)Z_2. \tag{3.5} \]

So, we have $g(AN[z], \xi[z]) = 0$.

The Codazzi equation of $M$ is given by
\[ g((\nabla X S)Y - (\nabla Y S)X, Z) = \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) \\
+ g(X, AN)g(AY, Z) - g(Y, AN)g(AX, Z) \\
+ g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z) \tag{3.6} \]
for any $X, Y, Z$ tangent to $M$.

The normal Jacobi operator of a real hypersurface in $Q^m$ is calculated by the Gauss equation for $Y = Z = N$ and, because of (3.4), is given by
\[ \bar{R}_N(X) = X + 3\eta(X)\xi + \cos(2t)AX - g(AX, N)AN - g(AX, \xi)A\xi, \tag{3.7} \]
for any $X \in TM$, where $g(AN, N) = \cos(2t) = -g(A\xi, \xi)$. Let us suppose that
\[ (\mathcal{L}_X \bar{R}_N)Y = (\mathcal{L}_X^{(k)} \bar{R}_N)Y \text{ for any } X, Y \text{ tangent to } M. \]
This yields $F_X^{(k)} \bar{R}_N Y - F_Y^{(k)} X = 0$, for any $X, Y$ tangent to $M$. That is
\[ g(\phi SX, \bar{R}_N Y)\xi - \eta(\bar{R}_N Y)\phi SX \tag{-\eta(\bar{R}_N Y)\phi SX} - k\eta(X)\phi \bar{R}_N Y - g(\phi S\bar{R}_N Y, X)\xi \\
+ \eta(X)\phi S\bar{R}_N Y + k\eta(\bar{R}_N Y)\phi X - g(\phi SX, Y)\bar{R}_N \xi + \eta(Y)\bar{R}_N \phi SX \\
+ k\eta(X)\bar{R}_N \phi Y + g(\phi SY, X)\bar{R}_N \xi - \eta(X)\bar{R}_N \phi SY - k\eta(Y)\bar{R}_N \phi Y = 0. \tag{3.8} \]
If we take $X = \xi$ in (3.8), we obtain
\[
g(\phi S\xi, \bar{R}_N Y)\xi - \eta(\bar{R}_N Y)\phi S\xi - k\phi \bar{R}_N Y + \phi S \bar{R}_N Y
- g(\phi S\xi, Y)\bar{R}_N \xi + \eta(Y)\bar{R}_N \phi S\xi + k\bar{R}_N \phi Y - \bar{R}_N \phi SY = 0 \tag{3.9}
\]
for any $Y$ tangent to $M$. Taking $X \in \mathcal{C}$ in (3.8), we get
\[
g((\phi S + S\phi)X, \bar{R}_N Y)\xi - \eta(\bar{R}_N Y)\phi SX + k\eta(\bar{R}_N Y)\phi X
- g((\phi S + S\phi)X, Y)\bar{R}_N \xi + \eta(Y)\bar{R}_N \phi SX - k\eta(Y)\bar{R}_N \phi X = 0 \tag{3.10}
\]
for any $X \in \mathcal{C}$, $Y$ tangent to $M$.

We finish this section with the following Proposition, which concerns Hopf hypersurfaces in $Q^m$ whose shape operator commutes with the structure tensor, see [2].

**Proposition 3.1.** The following statements hold for a tube $M$ of radius $r$, $0 < r < \pi/2$ around the totally geodesic $\mathbb{C}P^k$ in $Q^m$, $m = 2k$:

1. $M$ is a Hopf hypersurface.
2. The normal bundle of $M$ consists of $\mathfrak{A}$-isotropic singular tangent vectors of $Q^m$.
3. $M$ has four distinct principal curvatures, unless $m = 2$, in which case $M$ has two distinct principal curvatures.
4. The shape operator commutes with the structure tensor field $\phi$, i.e., $S\phi = \phi S$.
5. $M$ is a homogeneous hypersurface.

And see also [7]:

**Proposition 3.2.** Let $M$ be a Hopf hypersurface in $Q^m$ such that the normal vector field $N$ is $\mathfrak{A}$-principal everywhere. Then $\alpha = g(S\xi, \xi)$ is constant, and if $X \in \mathcal{C}$ is a principal curvature vector of $M$ with principal curvature $\lambda$, then $2\lambda \neq \alpha$, and $\phi X$ is a principal curvature vector of $M$ with principal curvature $\frac{\alpha + \lambda}{2\alpha}$.

4. **Proof of Theorem 1.2. The case of Hopf real hypersurfaces**

All the following calculations take place at an arbitrary point $[z] \in M$, but we can omit the subscript $[z]$ from the vector fields and other objects for the sake of brevity.

Let us suppose that $M$ is Hopf at $[z]$, i.e., that $S\xi = \alpha \xi$ holds. We will first prove the following:
Lemma 4.1. Let $M$ be a Hopf real hypersurface in $Q^m$, $m \geq 3$. If $\mathcal{L} \bar{R}_N = \mathcal{L}^{(k)} \bar{R}_N$ for some non-zero real constant $k$, then $N$ is either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.

Proof. As $M$ is Hopf, (3.9) becomes

$$-k\phi \bar{R}_N Y + \phi S \bar{R}_N Y + k\bar{R}_N \phi Y - \bar{R}_N \phi SY = 0 \quad (4.1)$$

for any $Y$ tangent to $M$. If, in particular, $Y = \xi$, we get

$$-k\phi \bar{R}_N \xi + \phi S \bar{R}_N \xi = 0. \quad (4.2)$$

If in (3.10) we take $Y = \xi$, we have

$$g((\phi S + S\phi)X, \bar{R}_N \xi) - \eta(\bar{R}_N \xi) \phi S X + k\eta(\bar{R}_N \xi) \phi X + \bar{R}_N \phi S X - k\bar{R}_N \phi X = 0 \quad (4.3)$$

for any $X \in \mathfrak{C}$.

Taking the scalar product of both sides of (4.3) by $\xi$ gives $2g(\phi SX, \bar{R}_N \xi) + g(S\phi X, \bar{R}_N \xi) - k\eta(\phi X, \bar{R}_N \xi) = 0$ for any $X \in \mathfrak{C}$. From (4.2) we obtain

$$g(\bar{R}_N \xi, \phi SX) = 0, \text{ for any } X \in \mathfrak{C}. \text{ As } \bar{R}_N \xi = 4\xi + 2\cos(2t)A\xi, \text{ it follows that}$$

$$2\cos(2t)g(A\xi, \phi SX) = 0 \quad (4.4)$$

for any $X \in \mathfrak{C}$. From (4.4), if $\cos(2t) = 0$, $N$ is $\mathfrak{A}$-isotropic. If $\cos(2t) \neq 0$, $g(A\xi, \phi SX) = 0$ for any $X \in \mathfrak{C}$. In this case, from (3.10), if $X \in \mathfrak{C}$ satisfies $SX = \lambda X$, where $\lambda \neq k$, then $g(A\xi, X) = 0$.

Therefore, if in $\mathfrak{C}$ $k$ does not appear as an eigenvalue of $S$ or $k$ is the unique eigenvalue of $S$, $g(A\xi, X) = 0$ for any $X \in \mathfrak{C}$ and $N$ is $\mathfrak{A}$-principal. If the unique eigenvalue of $S$ in $\mathfrak{C}$ is $k$, $\phi S = S\phi$ and $N$ should be $\mathfrak{A}$-isotropic, which is a contradiction. Therefore, if in $\mathfrak{C}$ $k$ does not appear as an eigenvalue of $S$, $N$ must be $\mathfrak{A}$-principal.

Thus we must suppose there exists $X \in \mathfrak{C}$ such that $SX = kX$, and therefore $g(AN, X) = 0$, and there exists $Z \in \mathfrak{C}$ such that $SZ = \lambda Z$, $\lambda \neq k$, and then $g(A\xi, Z) = 0$. Moreover, we must suppose there exists $W \in \mathfrak{C}$ such that $\eta(\bar{R}_N W) = g(A\xi, W) \neq 0$. If not, $N$ should be $\mathfrak{A}$-principal.

Let $X \in \mathfrak{C}$ such that $SX = kX$. From (3.10) we have $g((\phi S + S\phi)X, \bar{R}_N Y)\xi - g((\phi S + S\phi)X, Y)\bar{R}_N \xi = 0$ for any $Y$ tangent to $M$. Its scalar product with $W$ gives $g((\phi S + S\phi)X, Y) = 0$ for any $Y$ tangent to $M$, that is, $\phi SX = -S\phi X = k\phi X$. Therefore $S\phi X = -k\phi X$. Again from (3.10) we have $g((\phi S + S\phi)X, \bar{R}_N Y)\xi - \eta(\bar{R}_N Y)\phi S \phi X - k\eta(\bar{R}_N Y)X - g((\phi S + S\phi)X, Y)\bar{R}_N \xi + \eta(Y)\bar{R}_N \phi S \phi X + k\eta(Y)\bar{R}_N X = 0$. But $\phi S \phi X + S\phi^2 X = 0$. Therefore, it follows that $-2k\eta(\bar{R}_N Y)X + 2k\eta(Y)\bar{R}_N X = 0$. If $Y = \xi$, we have $-\eta(\bar{R}_N \xi)X + \bar{R}_N X = 0$. Its scalar product with $\xi$ implies $\eta(\bar{R}_N X) = 0$. As for any $Z \in \mathfrak{C}$ such that $SZ = \lambda Z$, $\lambda \neq k$, we have $g(A\xi, Z) = 0$, we arrive to a contradiction and we have finished the proof.
Lemma 4.2. There do not exist Hopf real hypersurfaces in $Q^m$, $m \geq 3$, such that $\mathcal{L}\bar{R}_N = \mathcal{L}^{(k)}\bar{R}_N$ for a non-zero real constant $k$ if $N$ is $\mathfrak{A}$-isotropic.

Proof. If $N$ is $\mathfrak{A}$-isotropic, $\bar{R}_N\xi = 4\xi$. Let $X \in \mathfrak{c}$ be a unit vector field such that $SX = \lambda X$. Introducing it in (4.3), we have $-4\lambda \phi X + 4\phi X + \lambda \bar{R}_N\phi X = 0$. That is, $(k - \lambda)\bar{R}_N\phi X = 4(k - \lambda)\phi X$. There are two possibilities, either $\lambda = k$ or if $\lambda \neq k$, $\bar{R}_N\phi X = 4\phi X$.

In the second case, $4\phi X = \phi X - ((\phi X, AN)AN - g(\phi X, A\xi)A\xi$. Its scalar product with $\phi X$ gives $3 = -g(\phi X, AN)^2 - g(\phi X, A\xi)^2$, which is impossible.

Therefore $SX = kX$ for any $X \in \mathfrak{c}$. Take $X, Y \in \mathfrak{c}$ in (3.10). This yields $g((\phi S + S\phi)X, \bar{R}_N Y)\xi - 4g((\phi S + S\phi)X, Y)\xi = 0$. That is, $2k g(\phi X, \bar{R}_N Y)\xi - 8k g(\phi X, Y)\xi = 0$ for any $X, Y \in \mathfrak{c}$. Therefore $g(\phi X, \bar{R}_N Y) = 4g(\phi X, Y)$. Taking $Y = \phi X$, we arrive to the same contradiction, finishing the proof.

Lemma 4.3. There do not exist Hopf real hypersurfaces in $Q^m$, $m \geq 3$, such that $\mathcal{L}\bar{R}_N = \mathcal{L}^{(k)}\bar{R}_N$ for some non-zero real constant $k$ if $N$ is $\mathfrak{A}$-principal.

Proof. As we suppose $N$ is $\mathfrak{A}$-principal, we can write $AN = N$, $A\xi = -\xi$ and $\bar{R}_N\xi = 2\xi$. We also know that $\alpha$ is constant, and that if $X \in \mathfrak{c}$ satisfies $SX = \lambda X$, then $S\phi X = \mu \phi X$, with $\mu = \frac{\alpha \lambda^2 + 2}{2}$.

Let $\{E_1, ..., E_{2m-2}\}$ be an orthonormal basis of eigenvectors of $S$ in $\mathfrak{c}$ such that $SE_i = \lambda_i E_i$, $i = 1, ..., 2m - 2$. For any $X \in \mathfrak{c}$, $\bar{R}_N X = X + \lambda X$. As there exists $Y \in \mathfrak{c}$ such that $AY = -Y$, for such a vector field, $\bar{R}_N Y = 0$. For such a $Y$ and $X \in \mathfrak{c}$, (3.10) yields $g((\phi S + S\phi)X, Y) = 0$. Therefore $(\lambda_i + \mu_i)g(\phi E_i, Y) = 0$, for any $i = 1, ..., 2m - 2$. As $\{\phi E_1, ..., \phi E_{2m-2}\}$ is also an orthonormal basis of $\mathfrak{c}$, there exists $j \in \{1, ..., 2m - 2\}$ such that $g(\phi E_j, Y) \neq 0$. Therefore $\lambda_j + \mu_j = 0$.

From the Codazzi equation,
\[
g((\nabla_{E_i} S)\phi E_j - (\nabla_{\phi E_j} S)E_i, \xi) = -2g(\phi E_j, \phi E_i) = -2
\]
\[
= g(\nabla_{E_j}(-\lambda_j \phi E_i) - S\nabla_{E_j} \phi E_i - \nabla_{\phi E_j} (\lambda_j E_i) + S\nabla_{\phi E_j} E_i, \xi)
\]
\[
\lambda_j g(\phi E_j, \phi SE_i) + \alpha g(E_j, \phi SE_i) + \lambda_j g(E_j, \phi SE_i) - \alpha g(E_j, \phi SE_i)
\]
\[
= \lambda_j^2 + \alpha \lambda_j + \lambda_j^2 - \alpha \lambda_j = 2\lambda_j^2,
\]
which is impossible and finishes the proof.

The proof of Theorem 1.2 for Hopf real hypersurfaces follows from the Lemmas above.
5. Proof of Theorem 1.2. The case of non-Hopf real hypersurfaces

If $M$ is not Hopf at $z$, we write $S\xi = \alpha \xi + \beta U$, where $U$ is a unit vector in $\mathcal{C}$, and $\beta$ is a nonzero number. Let us call $\mathcal{C}_U = \{X \in \mathcal{C} | g(X, U) = g(X, \phi U) = 0\}$.

We will prove the following:

**Lemma 5.1.** Let $M$ be a non-Hopf real hypersurface in $Q^m$, $m \geq 3$, such that $\mathcal{L} \bar{R}_N = \mathcal{L}^{(k)} \bar{R}_N$, for some non-zero real constant $k$. Then $N$ is either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.

**Proof.** As $M$ is non-Hopf, (3.9) becomes

\[
\beta g(\phi U, \bar{R}_N Y)\xi - \beta \eta(\bar{R}_N Y)\phi U - k\phi \bar{R}_N Y + \phi S \bar{R}_N Y - \beta g(\phi U, Y)\bar{R}_N \xi + \beta \eta(Y)\bar{R}_N \phi U + k\bar{R}_N \phi Y - \bar{R}_N \phi SY = 0
\]

(5.1)

for any $Y$ tangent to $M$. Taking $Y = \xi$ in (5.1), we get $\beta g(\phi U, \bar{R}_N \xi)\xi - \beta \eta(\bar{R}_N \xi)\phi U - k\phi \bar{R}_N \xi + \phi S \bar{R}_N \xi = 0$. Its scalar product with $\xi$ gives

\[
g(\phi U, \bar{R}_N \xi) = 0,
\]

(5.2)

that is, $2 \cos(2t) g(A \phi U, \xi) = 0$. Therefore, if $\cos(2t) = 0$, $N$ is $\mathfrak{A}$-isotropic. Thus we suppose $\cos(2t) \neq 0$, and then

\[
g(A \phi U, \xi) = 0.
\]

(5.3)

From (5.2) the above expression becomes

\[
-\beta \eta(\bar{R}_N \xi)\phi U - k\phi \bar{R}_N \xi + \phi S \bar{R}_N \xi = 0.
\]

(5.4)

Its scalar product with $\xi$, bearing in mind (5.2), yields

\[
g(\bar{R}_N \xi, S \phi U) = 0,
\]

(5.5)

and its scalar product with $X \in \mathcal{C}_U$ implies

\[
kg(\bar{R}_N \xi, \phi X) - g(\bar{R}_N \xi, S \phi X) = 0
\]

(5.6)

for any $X \in \mathcal{C}_U$.

The scalar product of (3.10) and $\phi U$ yields

\[
- \eta(\bar{R}_N Y)g(SX, U) + k\eta(\bar{R}_N Y)g(X, U) + \eta(Y)g(\bar{R}_N \phi SX, \phi U) - k\eta(Y)g(\bar{R}_N \phi X, \phi U) = 0
\]
for any $X \in \mathcal{C}$, $Y$ tangent to $M$. If also $Y \in \mathcal{C}$, we get $-\eta(\bar{R}_N Y)(g(SX, U) - kg(SX, U)) = 0$ for any $X, Y \in \mathcal{C}$. Therefore, if for any $Y \in \mathcal{C}$, $g(Y, \bar{R}_N \xi) = 0 = 2 \cos(2t)g(Y, A\xi)$, as $\cos(2t) \neq 0$, we obtain $g(Y, A\xi) = 0$ for any $Y \in \mathcal{C}$, and therefore $N$ is $A$-principal. Let us suppose now that there exists $Z \in \mathcal{C}$ such that $\eta(\bar{R}_N Z) \neq 0$, and that for any $X \in \mathcal{C}$, $g(SU, X) = kg(U, X)$. That is, 

$$SU = g(SU, \xi)\xi + g(SU, U) = \beta \xi + kU. \quad (5.7)$$

Taking $X = U$ in $(3.10)$, we get $g((\phi S + S\phi)U, \bar{R}_N Y)\xi - g((\phi S + S\phi)U, Y)\bar{R}_N \xi = 0$, for any $Y$ tangent to $M$. Its scalar product with $Z$ yields $g((\phi S + S\phi)U, Y) = 0$ for any $Y$ tangent to $M$. Therefore $S\phi U = -\phi SU$, that is, 

$$S\phi U = -k\phi U. \quad (5.8)$$

Take $X = \phi U$ in $(3.10)$. Then $-2k\eta(\bar{R}_N Y) - g((\phi S + S\phi)\phi U, Y)\eta(\bar{R}_N U) = 0$, that is, 

$$-2k\eta(\bar{R}_N Y) - g(\phi S\phi U, Y)\eta(\bar{R}_N U) + g(SU, Y)\eta(\bar{R}_N U) = 0$$

for any $Y \in \mathcal{C}$. This implies $-2k\eta(\bar{R}_N Y) = 0$ for any $Y \in \mathcal{C}$, which contradicts the existence of $Z$ and finishes the proof. \(\square\)

From Lemma 5.1, $N$ is either $A$-isotropic or $A$-principal. Suppose first that $N$ is $A$-isotropic. Then $\bar{R}_N \xi = 4\xi$. Moreover, $\bar{R}_N \phi U = \phi U - g(A\phi U, N)AN - g(A\phi U, \xi)A\xi$. If $(5.1)$ is satisfied, we have $\beta g(\phi U, \bar{R}_N \phi U)\xi - k\phi \bar{R}_N \phi U + \phi S\bar{R}_N \phi U - \beta \bar{R}_N \xi - k\bar{R}_N U - \bar{R}_N \phi SU = 0$. Its scalar product with $\xi$ yields $\beta g(\phi U, \bar{R}_N \phi U) - 4\beta = 0$. That is, $4 = g(\bar{R}_N \phi U, \phi U) = 1 - g(A\phi U, N)^2 - g(A\phi U, \xi)^2$, which is impossible.

Let us suppose now $N$ is $A$-principal. In this case, $AN = N$, $A\xi = -\xi$, $\bar{R}_N \xi = 2\xi$, and for any $X \in \mathcal{C}$, $\bar{R}_N X = X + AX$.

Take $Y = \phi U$ in $(5.1)$. We obtain 

$$\beta g(\phi U, \bar{R}_N \phi U)\xi - k\phi \bar{R}_N \phi U + \phi S\bar{R}_N \phi U - \beta \bar{R}_N \xi - k\bar{R}_N U - \bar{R}_N \phi SU = 0. \quad (5.9)$$

Its scalar product with $\xi$ gives $g(\phi U, \bar{R}_N \phi U) = 2 = 1 + g(A\phi U, \phi U)$. This yields $g(A\phi U, \phi U) = 0$, which implies $A\phi U = \phi U$. Therefore $\bar{R}_N \phi U = 2\phi U$. As $A\phi U = \phi U$, we have $AJU = JU = -JAU$. This gives $AU = -U$ and $\bar{R}_N U = 0$. Thus $(5.9)$ becomes $2kU + 2\phi SU - \bar{R}_N \phi SU = 0$. Its scalar product with $U$ implies $2k - 2g(S\phi U, \phi U) = 0$. That is, 

$$g(S\phi U, \phi U) = k. \quad (5.10)$$
Taking $Y = U$ in (5.1), we have $k\bar{R}_N\phi U - \bar{R}_N\phi SU = 0$. Its scalar product with $\phi U$ gives $2k - 2g(\phi SU, \phi U) = 0 = 2k - 2g(SU, U)$. Then
\begin{equation}
 g(SU, U) = k. \tag{5.11}
\end{equation}

Taking $Y = U$ in (3.10), we have $-g((\phi S + S\phi)X, U)\bar{R}_N\xi = 0$, that is, $-2g((\phi S + S\phi)X, U)\xi = 0$ for any $X \in \mathcal{C}$. If $X = \phi U$, we obtain $g(\phi S\phi U, U) + g(S\phi^2 U, U) = 0 = -g(S\phi U, \phi U) - g(SU, U) = -2k$, which is impossible, finishing the proof of Theorem 1.2.

\textbf{References}


