Geodesics and geodesic circles in a geodesically convex surface: a sub-mixing property
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Abstract. Let $M$ be an orientable finitely connected and geodesically convex Finsler surface with genus $g \geq 1$. We prove that if all geodesics in $M$ are reversible, then for any number $\varepsilon > 0$ and for any points $p, q \in M$, there exists a number $R > 0$ such that any geodesic circle with center $p$ and radius $t$ meets the $\varepsilon$-ball with center $q$ for any $t > R$. Most of the proofs do not use the reversibility assumption for geodesics.

1. Introduction

In a connected smooth Finsler manifold $M$, a geodesic is by definition a solution of the Euler–Lagrange equation of the length function of piecewise smooth curves. A geodesic sphere $S_M(p, t)$ with center $p$ and radius $t$ is by definition the set of all terminal points of geodesics emanating from $p$ with length $t$. A distance sphere $S^d_M(p, t)$ with center $p$ and radius $t$ is by definition the set of all terminal points of minimal geodesics emanating from $p$ with length $t$. Obviously, $S^d_M(p, t) \subset S_M(p, t)$ and, generally, $S^d_M(p, t) \neq S_M(p, t)$.

Light has the nature behaving like both a particle and a wave. Mathematically, the geodesics describe the trajectories as the behavior of the particles, and the geodesic spheres describe the behavior of wave fronts which spread according to Huygens’ principle. The indicatrix of a Finsler metric describes the shape of an infinitesimal wave front (cf. [25, (1.17)]).

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These facts suggest that the geometry of geodesics in a Finsler manifold is useful to study the global behavior of wave fronts. In fact, the triangle inequality of the intrinsic distance $d(\cdot, \cdot)$ induced by the Finsler metric is almost equivalent to Huygens’ principle when we consider the geodesic spheres as wave fronts. Namely, for any point $p_1$ in a minimal geodesic from $p$ to $q$, $S^d_M(p_1, d(p_1, q))$ is inscribed in $S^d_M(p, d(p, q))$ at the point $q$ if and only if, for any point $q_1$ in $S^d_M(p_1, d(p_1, q))$, we have $d(p, q_1) \leq d(p, p_1) + d(p_1, q_1)$, where the equality holds if $q_1 = q$.

We have studied the behavior of geodesics and the asymptotic behavior of geodesic circles in a 2-torus equipped with a Riemannian metric in [18], and have proved $\varepsilon$-density of geodesic circles with sufficiently large radii, based on the following facts: the universal covering space of a torus is topologically a plane $\mathbb{R}^2$, which has a nice property such as the Jordan curve theorem, and the covering transformation group is isomorphic to $\mathbb{Z}^2$, where $\mathbb{Z}$ is the set of all integers.

In this paper, we study the same properties for surfaces with Finsler metrics and with genus $\geq 1$. We make their special covering spaces in which Jordan curve theorem is true and on which $\mathbb{Z}^2$ acts, and work in them.

The length of a minimal geodesic from $p$ to $q$ equals the distance from $p$ to $q$. However, a shortest curve may not be a geodesic if, for example, an interior point of it touches the boundary of $M$ and it is not smooth at the point of contact. We say that $M$ is geodesically convex if there exists a minimal geodesic from $p$ to $q$ in $M$ for any points $p, q \in M$. All forward complete Finsler manifolds without boundary are geodesically convex because of the Hopf–Rinow theorem (cf. [25]).

We say that a geodesic $c : [a, b] \to M$ is reversible if the reverse curve $c^{-1} : [a, b] \to M, c^{-1}(t) = c(a + b - t)$, is a geodesic as a point set. If $F$ is reversible, i.e., $F(x, y) = F(x, -y)$ for all $y \in T_xM$, then all geodesics are reversible. Here $T_xM$ denotes the tangent space of $M$ at $x$. It is well known that all geodesics are reversible in $(M, F)$ if $F = \alpha + \beta$ is a Randers metric, where $\alpha$ is the norm induced by a Riemannian metric and $\beta$ is a closed 1-form with $\|\beta\|_\alpha < 1$ (cf. [10], [16], [25]).

We say that a surface $M$ is finitely connected if there exist a compact surface $S$, with or without boundary, and finitely many points $p_1, \ldots, p_k \in S$ such that $M$ is homeomorphic to $S \setminus \{p_1, \ldots, p_k\}$ (cf. [26, p. 41]).

The main theorem and corollaries are the following.

**Theorem 1.1.** Let $(M, F)$ be an orientable finitely connected and geodesically convex smooth Finsler surface with genus $g \geq 1$. Assume that all geodesics are reversible. Then, for any number $\varepsilon > 0$ and any points $p, q \in M$, there exists a number $R > 0$ such that the geodesic circle $S_M(p, t)$ with center $p$ and radius $t$ meets the $\varepsilon$-ball $B(q, \varepsilon)$ with center $q$ for any $t > R$. 
Here $S_M(p,t) \cap B(q,\varepsilon)$ consists of many subarcs of $S_M(p,t)$ (see Figure 1), although we do not count the number of those subarcs (see Corollary 1.5 below). In the process of the proof of Theorem 1.1, we know the movement of subarcs of $S_M(p,t)$ in $B(q,\varepsilon)$: There exists a geodesic circle $S_M(p,t_0)$ passing through $q$ while a subarc of $S_M(p,t)$ gets into $B(q,\varepsilon)$ and leaves.

**Remark 1.2.** If a Finsler surface $(M,F)$ is not orientable, then $M$ has an orientable double covering surface $M_1 \to M$. If a Finsler metric $F_1$ of $M_1$ is the pullback of $F$ by $\pi$, i.e., $F_1 = \pi^* F$, then the image of any geodesic in $(M_1,F_1)$ by $\pi$ is a geodesic in $(M,F)$ with the same length. Hence, $(M_1,F_1)$ satisfies the assumption of Theorem 1.1. Assume that the genus of $M_1$ is greater than or equal to one. Choosing $p_1, q_1 \in M_1$ such that $\pi(p_1) = p$ and $\pi(q_1) = q$, and applying Theorem 1.1 to $p_1$ and $q_1$ in $M_1$, we have the same conclusion for $p$ and $q$ through the projection $\pi$.

The following corollary is a direct consequence, but the situation may often arise when $(M,F)$ is complete.

**Corollary 1.3.** Let $(M,F)$ be an orientable Finsler surface such that all geodesics are reversible and $p, q \in M$. Assume that there exists a surface $M_1$ embedded in $M$ containing $p$ and $q$ and with genus $g \geq 1$ such that $M_1$ is finitely connected and geodesically convex. Then for any $\varepsilon > 0$, there exists a number $R > 0$ satisfying the same property in Theorem 1.1.

Let $E : \Omega_{p,q} \to \mathbb{R}$ be the energy function on the path space $\Omega_{p,q}$ from $p$ to $q$. The critical points of $E$ are geodesics from $p$ to $q$ in $M$ (cf. [22], [25]).

**Corollary 1.4.** Let $(M,F)$ be as in Theorem 1.1. For any number $\varepsilon > 0$ and any points $p, q \in M$, there exists a number $R > 0$ such that the set of critical values of $E$ is $\varepsilon$-dense in $[R, \infty)$. 
To clarify the role of the geodesic reversibility assumption, we study geodesics and geodesic circles under more general settings. Hereafter, let \((M, F)\) be an orientable finitely connected and geodesically convex smooth Finsler surface with genus \(g \geq 1\). From the assumption on the genus of \(M\), there exist \(g\) simple closed curves \(c_1, \ldots, c_g\) in \(M\) such that they are disjoint and \(M \setminus \cup_{i=1}^{g} c_i\) is connected. We assume that \(g - 1\) curves of them, say \(c_1, \ldots, c_{g-1}\), are reversible geodesics. This assumption is automatically satisfied if all geodesics are reversible in the Finsler surface. We study geodesics in \(M \setminus \cup_{i=1}^{g} c_i\) and the asymptotic behavior of geodesic circles. Namely, we develop geometry of geodesics in \(M\), using no geodesic which intersects \(c_1, \ldots, c_{g-1}\). The set \(M \setminus \cup_{i=1}^{g} c_i\) and its covering space \(N\) are geodesically convex when \(c_1, \ldots, c_{g-1}\) are reversible geodesics. In fact, if the distance \(d_1(p, q)\) from a point \(p\) to a point \(q\) in \(M \setminus \cup_{i=1}^{g} c_i\) is defined as the infimum of the lengths of all piecewise smooth curves from \(p\) to \(q\) in \(M \setminus \cup_{i=1}^{g} c_i\), then \(M \setminus \cup_{i=1}^{g} c_i\) is geodesically convex with respect to \(d_1\). The distances of \(M\) and \(M \setminus \cup_{i=1}^{g} c_i\) induced by \(F\) are different, but a geodesic in \(M \setminus \cup_{i=1}^{g} c_i\) remains a geodesic in \(M\). We make use of those geodesics which do not intersect \(c_1, \ldots, c_{g-1}\) in \(M\) to obtain the properties mentioned in Theorem 1.1.

The following corollary is a rough estimate of the number of critical points and sufficiently large critical values of \(E\). In the following corollary we use the phrase ‘a pencil of geodesics’, which is a set of geodesics converging to or narrowly diverging from a point.

**Corollary 1.5.** Let \((M, F)\) be as mentioned above. Let \(n\) be any positive integer. For any number \(\epsilon > 0\) and any points \(p, q \in M \setminus \cup_{i=1}^{g} c_i\), there exists a number \(R > 0\) such that at least \(n\) pencils of geodesics emanating from \(p\) with length \(t\) intersect the \(\epsilon\)-ball with center \(q\) for any \(t > R\) where the sequences of the lifts of these \(n\) pencils of geodesics into \(N\) converge to rays with different slopes as \(t \to \infty\).

Here, the covering space \(N\) of \(M \setminus \cup_{i=1}^{g} c_i\) is defined in Section 2, and the notion of slopes for rays is defined in Subsection 4.2. We work in the covering space \(N\) where the covering transformation group \(\Phi\) is isomorphic to \(\mathbb{Z}^2\). Such a covering space can be constructed because \(M \setminus \cup_{i=1}^{g} c_i\) is considered to be a subset in a 2-torus. Hence we find and use many analogous results on the behavior of geodesics on 2-tori. Working in \(N\), we prove Theorem 2.3, which is sufficient for Theorem 1.1.

The geodesics on 2-tori of revolution embedded in the Euclid space \(E^3\) have been studied by BLISS\[4\] and KIMBALL\[20\]. Recently, GRAVESEN, MARKVORSEN, SINCLAIR and TANAKA \[9\] have studied the cut locus in a 2-torus of revolution.
Innami [14] has studied geodesics in a 2-torus having poles. Morse [23] and Hedlund [11] studied the geodesics on arbitrary Riemannian tori whose lifts into the universal covering space are straight lines. Busemann and Pedersen [6] have determined how the straight lines behave in the universal covering planes of 2-tori with one-parameter groups of motions. Their methods are unified by Bangert [3] with those of Mather [21] and Aubry-Le Daeron [2] to study a monotone twist map of the annulus and the discrete Frenkel-Kontrova model (cf. [19]). The method of finding straight lines by displacement functions can be applied in more general situations. Indeed, in [3], we can see the complete classification of straight lines in the universal covering plane of an arbitrary 2-torus, as an application. Recently, Schröder [24] has generalized those results for non-symmetric distance cases. We modify the methods in [6] to have analogous results for studying the asymptotic behavior of geodesic circles. In the light of the classification of straight lines, we can study the limit circles which are the level sets of Busemann functions.

Let $G^t : SX \to SX$ be the geodesic flow of a unit tangent bundle $SX$ of a complete Finsler manifold $X$ without boundary. It follows from Poincaré’s recurrence theorem that for almost all $y \in SX$, there exists a sequence of numbers $t_n$ such that $t_n \to \infty$ and $G^{t_n}(y) \to y$ as $n \to \infty$ if the volume of $X$ is finite. We can estimate the averages of the return time for almost all $y \in SX$ by using the Birkhoff ergodic theorem (cf. [1]). In comparison with these results, Theorem 1.1 states that some terminal points of geodesics emanating from $p$ and with length $t > R$ always exist near $q$. An event occurs at a point $p$, its influence spreads according to Huygens’ principle, and after the time $R$, at the point $q$, it is affected every time less than $\varepsilon$.

We say that $G^t$ is topologically mixing if for any two open sets $U$ and $V$ of the unit tangent bundle $SX$, there exists a number $R > 0$ such that $G^t(U) \cap V \neq \emptyset$ for all $t$ with $|t| > R$. Eberlein [8] has proved that the geodesic flow $G^t$ is topologically mixing on $SX$ if the Riemannian manifold $X$ is a compact visibility manifold of non-positive curvature. We are interested in the existence of wave fronts more than the directions of trajectories, so it is important to study the asymptotic behavior of geodesic spheres related to the property of topological mixing in the underlying manifold, since the geodesic circles spread according to Huygens’ principle.

We say that the geodesic flow $G^t$ is topologically sub-mixing if for any open sets $U$ and $V$ of $X$, there exists a number $R > 0$ such that geodesic circles $\rho(G^t(S_qX)) = \exp_q(tS_qX)$ intersect $V$ for some point $q \in U$ and for all $t > R$, i.e., $G^t(\rho^{-1}(U)) \cap \rho^{-1}(V) \neq \emptyset$, where $\rho : SX \to X$ is the natural projection.
and \( \exp_q : T_q X \to X \) is the exponential map at \( q \). Here we note that \( S_X(q,t) = \exp_q(tS_q X) \) is the geodesic sphere with center \( q \) and radius \( t \). The geodesic flow of a flat \( n \)-torus, \( n \geq 2 \), is topologically sub-mixing, but not mixing. Sierpinski (in 1906) (cf. [12]) has estimated the asymptotic difference between the area \( \pi t^2 \) of the circle \( S(t) \) with radius \( t \) and the number \( N(t) \) of lattice points contained in \( S(t) \) in the Euclidean plane, proving that \( |\pi t^2 - N(t)| \leq O(t^{2/3}) \), which means that \( N(t + \epsilon) - N(t) = \pi (t + \epsilon)^2 - \pi t^2 + O(t^{2/3}) = 2\pi \epsilon t + O(t^{2/3}) \to \infty \) as \( t \to \infty \).

We find the similar estimate for a flat \( n \)-torus \( T^n \) in [7], where the error term is \( O(t^\alpha) \), \( 0 \leq \alpha < n - 1 \). These properties prove the topological sub-mixing property of \( T^n \). In the previous paper [15], we have investigated the asymptotic behavior of geodesic circles in a 2-torus of revolution and proved that the geodesic flow of a 2-torus of revolution is topologically sub-mixing. In [18], we have proved the geodesic flow of any 2-torus is topologically sub-mixing. Theorem 1.1 states that the sub-mixing property of a geodesic flow is true for a much wider class of surfaces.

2. A surface cut along simple closed geodesics

2.1. Cutting and opening \( M \) along simple closed geodesics. We cut and open \( M \) along simple closed geodesics \( c_1, \ldots, c_{g-1} \).

![Figure 2. The case of genus 2.](image)

We construct an orientable finitely connected Finsler surface \((M_0, F_0)\) with boundary and with genus one (see Figure 2), satisfying the following properties: There exists a map \( I : M_0 \to M \) such that

1. the interior \( \text{Int}(M_0) \) of \( M_0 \) is isometric to \( M \setminus \cup_{i=1}^{g-1} c_i \), i.e., the restriction \( I : \text{Int}(M_0) \to M \setminus \cup_{i=1}^{g-1} c_i \) is a diffeomorphism such that \( F_0(x,y) = F(I(x), dI(x)(y)) \) for all \( x \in \text{Int}(M_0) \) and \( y \in T_x M_0 \).
(2) for any point \( x \in \cup_{i=1}^{g-1} c_i \), \( I^{-1}(x) \) consists of exactly two points in the boundary \( \partial M_0 \) of \( M_0 \). When those two points are identified, the quotient space \( M_0/I \) is naturally thought as \( M \) by the quotient map \( M_0/I \to M \).

The boundary \( \partial M_0 \) of \( M_0 \) consists of \( I^{-1}(\cup_{i=1}^{g-1} c_i) \) and \( I^{-1}(\partial M) \) (the original boundary of \( M \)) as surfaces. Hence \( \partial M_0 \) has at least \( 2(g-1) \) connected components. The set \( \{p_1,\ldots,p_k\} \) is contained in the boundary \( \partial M \) of \( M = S \setminus \{p_1,\ldots,p_k\} \) as a topological sub-space in a certain compact surface \( S \), but we think no point in \( \partial M_0 \) is sent to \( p_1,\ldots,p_k \) by \( I \).

The finitely connected Finsler surface \( M_0 \) with genus one defined as above is geodesically convex, since \( M \) is geodesically convex and all \( c_i, i = 1,\ldots,g-1, \) are reversible geodesics. The boundary of \( M_0 \) is not empty if \( g > 1 \).

Notice that \( c \) is a geodesic in the interior \( \text{Int}(M_0) \) of \( M_0 \) if and only if \( I(c) \) is a geodesic in \( M \setminus \cup_{i=1}^{g-1} c_i \). It should be noted that there exists no geodesic touching \( \partial M_0 \) any place other than its endpoints, because all curves \( c_i \) and reverse curves \( c_i^{-1} \) are geodesics for \( i = 1,\ldots,g-1 \), and the geodesic is uniquely determined from the initial condition. Namely, any geodesic \( c \) whose end points are in \( \text{Int}(M_0) \) satisfies that \( I(c) \subset M \setminus \cup_{i=1}^{g-1} c_i \). We define a distance \( d_0(p,q) \) for \( p,q \in M_0 \) as usual; \( d_0(p,q) \) is the infimum of the lengths of piecewise smooth curves from \( p \) to \( q \) in \( M_0 \). Then a shortest curve from \( p \) to \( q \) in \( \text{Int}(M_0) \) is a minimal geodesic \( c \) in \( \text{Int}(M_0) \), i.e., \( d_0(p,q) \) is the length of \( c \).

![Minimal geodesics in \( M_0 \) and \( M \).](image_url)

Figure 3. Minimal geodesics in \( M_0 \) and \( M \).

Obviously, \( d_0(p,q) \geq d(I(p),I(q)) \) for any \( p,q \in M_0 \), where \( d(\cdot,\cdot) \) is the distance on \( M \) induced by \( F \). In fact, if a minimal geodesic \( \gamma \) from \( I(p) \) to \( I(q) \) in \( M \) crosses \( c_i \) for some \( i = 1,\ldots,g-1 \), then the minimal geodesics from \( p \) to \( q \) in \( M_0 \) are longer than \( \gamma \) (see Figure 3).

### 2.2. Another geodesically convex surface \( M_0 \) with genus one

Assume in this subsection that all geodesics in \( M \) are reversible. Since \( M \setminus \cup_{i=1}^{g-1} c_i \) is
a geodesically convex set, we have $g - 1$ simple closed geodesics $d_i$, $i = 1, \ldots, g - 1$, such that

1. $c_i \cap d_i$ is a single point for each $i = 1, \ldots, g - 1$;
2. $c_i \cup d_i$, $i = 1, \ldots, g - 1$, are mutually disjoint.

Then there exists an open neighborhood $U_i$ of $c_i \cup d_i$ such that $U_i \smallsetminus c_i \cup d_i$ is homeomorphic to an open cylinder $S^1 \times (0, 1)$ for each $i = 1, \ldots, g - 1$. As was seen before, we make an orientable finitely connected and geodesically convex Finsler surface $(p_0, F_0)$, $M_0 = M \smallsetminus \bigcup_{i=1}^{g-1} c_i \cup d_i$, with boundary and with genus 1, such that there exists a map $I : M_0 \to M$ satisfying the same property as above.

In this construction, the boundary $\partial M_0$ is the union of $I^{-1}(\bigcup_{i=1}^{g-1} c_i \cup d_i)$ and $I^{-1}(\partial M)$. Each connected component of $I^{-1}(\bigcup_{i=1}^{g-1} c_i \cup d_i)$ is a broken geodesic, but not a (smooth) geodesic.

2.3. The geodesic circles in $M_0$. Let $M_1$ be a finitely connected and geodesically convex Finsler surface. For a point $p \in M_1$ and a unit vector $v \in S_p M_1$, let $\gamma_v : (-a(v), b(v)) \to M_1$ be a unit speed geodesic such that $\gamma_v(0) = v$, possibly $a(v) = \infty, b(v) = \infty$. Here the interval $(-a(v), b(v))$ is supposed to be maximal, i.e., there exists no proper extension of $\gamma_v$ in $M_1$. If $M_1$ is complete and without boundary, then $a(v) = \infty$ and $b(v) = \infty$ for all $v \in SM_1$.

For a number $t \in (-\infty, \infty)$, let $SM_1(t)$ be the set of all $v \in SM_1$ such that $t \in (-a(v), b(v))$. Then a map $G^t : SM_1(t) \to SM_1$ is defined by $G^t(v) = \gamma_v(t)$ for any $v \in SM_1(t)$. Let $p : SM_1 \to M_1$ be a natural projection of the unit tangent bundle of $M_1$. Then $S_{M_1}(p, t) := \rho(G^t(S_p M_1 \cap SM_1(t))) = \{\gamma_v| v \in S_p M_1(t)\}$ is called a geodesic circle with center $p$ and radius $t$ in $M_1$ for any point $p$ and any $t > 0$. If $M_1$ is complete and without boundary, then $SM_1(t) = SM_1$ for all $t \in (-\infty, \infty)$ and $G^t$ is called the geodesic flow on $SM_1$ (cf. [1]).

Hereafter, let $M_0$ denote a surface constructed in Subsection 2.1 from $M$. We emphasize that $I(S_{M_0}(p, t)) \subset SM(I(p), t)$ for all $t > 0$ and any $p \in M_0$. This inclusion relation is not true for the distance spheres, i.e., $I(S^d_{M_0}(p, t)) \not\subset S^d_{M}(I(p), t)$, in general. Here $S^d_{M_0}(p, t) := \{q \in M_0 | d_{M_0}(p, q) = t\}$.

2.4. A covering surface and its transformation group. We make a surface $S_0$ from $S \supset M$ in the same way as $M_0$ from $M$. Then we think $M_0 = S_0 \smallsetminus \{p_1, \ldots, p_k\}$. Let $k'$ be the number of the connected components of the boundary $\partial M$ of $M$. Recall that the genus of $M_0$ is one, $\partial M_0$ has $2(g - 1) + k'$ connected components and $k$ points $p_1, \ldots, p_k$ are removed. If $2(g - 1) + k'$ disks $K_i$, $i = 1, \ldots, 2(g - 1) + k'$, are glued along the boundary $\partial M_0$ and $k$ points
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$p_j$, $j = 1, \ldots, k$, are plugged up at the original location in $S_0$, then this operation turns $S_0$ into a 2-torus topologically. Hence, its universal covering surface is topologically a plane $\mathbb{R}^2$, and the covering transformation group $\Phi$ is isomorphic to $\mathbb{Z}^2$. We define a surface $N$ by

$$N = \mathbb{R}^2 \setminus \Phi((\bigcup_{i=1}^{2(g-1)+k'} \text{Int}(\tilde{K}_i) \cup \{	ilde{p}_1, \ldots, \tilde{p}_k\})),$$

where $\tilde{K}_i$ (resp., $\tilde{p}_j$) is a lift of $K_i$ (resp., $p_j$) into $\mathbb{R}^2$ for each $i = 1, \ldots, 2(g-1)+k'$ (resp., $j = 1, \ldots, k$). Then $N$ is a covering surface of $M_0$ with a natural covering map $\pi : N \to M_0$.  

**Lemma 2.1** (Jordan curve theorem). Let $C$ be a simple closed curve in $\text{Int}(N)$. Then $N \setminus C$ consists of two connected components.

**Proof.** Since $C \subset N \subset \mathbb{R}^2$, the Jordan curve theorem for $\mathbb{R}^2$ implies that $\mathbb{R}^2 \setminus C$ consists of two connected components $X_1$ and $X_2$. Then $X_1 \cap N$ and $X_2 \cap N$ are the connected components of $N \setminus C$. \qed

If we define a Finsler metric $\tilde{F}$ on $N$ by $\tilde{F}(x, y) = F_0(\pi(x), d\pi(y))$ for any $x \in N$ and any $y \in T_xN$, then $\Phi$ acts on $N$ as an isometry group isomorphic to $\mathbb{Z}^2$ such that $M_0 = N/\Phi$. From the definitions of geodesic circles and distance circles, we have the following lemma.

**Lemma 2.2.** Let $(M, F)$ be as mentioned above. Then there exist an isometric surface $I : \text{Int}(M_0) \to M \setminus \cup_{i=1}^{g-1} e_i$ and its covering surface $\pi : N \to M_0$ such that $M_0 = N/\Phi$, where $\Phi$ is a covering transformation group isomorphic to $\mathbb{Z}^2$, satisfying that

$$I(\pi(S_{M_0}^t(p, t))) \subset I(S_M(\pi(p), t)) = I(\pi(S_N(p, t))) \subset S_M(I(\pi(p)), t) = \rho(G_I(S_{M(\pi(p))}M \cap SM(t)))$$

for any $p \in N$ and any $t > 0$.

From Lemma 2.2, it suffices to prove Theorem 2.3 in order to obtain Theorem 1.1.

**Theorem 2.3.** Let $N$ be a covering surface of $M_0$ constructed as above. Let $p, q \in N$. Given $\varepsilon > 0$, there exists a number $R > 0$ such that $S_{M_0}^t(p, t) \cap \Phi(B(q, \varepsilon)) \neq \emptyset$, equivalently, $S_{M_0}^t(p, t) \cap \Phi(B(q, \varepsilon)) \neq \emptyset$ for all $t > R$.

Thanks to Lemmas 2.1 and 2.2, the process of the proof for Theorem 2.3 is the same as in [18], although $N$ is not homeomorphic to a plane and the distance is not symmetric. However, from the next section up to Proof of Theorem 1.1 in Section 7, we progress the study parallel to ones in [6] and [18]. It makes this paper self-contained. The arguments here include some improvements.
3. Axial straight lines

Let $N$ be a surface constructed in Section 2. Then $N$ is topologically a plane, from which many open disks and points are removed, and geometrically a geodesically convex Finsler surface on which the isometry group $\Phi$ isomorphic to $\mathbb{Z}^2$ acts so that $M_0 = N/\Phi$. Therefore, $N$ has many properties which a plane has: A simple closed curve and a simple curve diverging both directions as a curve in $\mathbb{R}^2$ divides $N$ into two connected components, etc. Let $d(\cdot, \cdot)$ be the distance on $N$ induced by $F = \pi^* F_0$. Since we do not assume that $F$ is reversible, we have $d(p, q) \neq d(q, p)$ for points $p, q \in N$, in general. For $\tau \in \Phi$, let $d_\tau : N \to \mathbb{R}$ denote the displacement function of $\tau$ which is defined by $d_\tau(p) = d(p, \tau(p))$ for all $p \in N$. We say that a minimal geodesic $T$ is a straight line in $N$ if $T$ is unbounded in both directions as a curve in $\mathbb{R}^2$ where $N$ lies. Note that if $M_0$ contains two unbounded tubes, then there exists a minimal geodesic defined on $\mathbb{R}$ in $N$ which is not a straight line. There exists a parametrization $\gamma : (-\infty, \infty) \to N$ of $T$ such that $d(\gamma(s), \gamma(t)) = t - s$ for any $s, t \in (-\infty, \infty)$ with $s < t$. The unboundedness of $T$ in $\mathbb{R}^2$ implies that any half part of $T$ does not stay in a fundamental domain in $N$ over $S_0 = M_0 \cup \{p_1, \ldots, p_k\}$ and is not contained in any tube of $N$. The reverse curve $T^{-1}$ of $T$ may not be a geodesic. We have seen in [16], [17] and [18] what phenomenon happens on geodesics in Finsler 2-tori, in comparison with the case of a Riemannian surface.

**Proposition 3.1** (cf. [6]). Let $\tau \in \Phi$, $\tau \neq id$. Then $d_\tau$ takes a positive minimum. If $p \in N$ is a minimum point of $d_\tau$, then there exists a unique straight line $\gamma : (-\infty, \infty) \to N$ such that $\gamma(0) = p$ and $\gamma(s) = \gamma(t + c)$ for all $s \in (-\infty, \infty)$, where $c = \min d_\tau > 0$.

We call a straight line $\gamma$ as in Proposition 3.1 an axis of $\tau$.

**Remark 3.2.** It follows that $t^{-1}(\gamma(t)) = \gamma(t - c)$ for all $t \in (-\infty, \infty)$ for an axis $\gamma$ of $\tau$. However, the reverse curve $\gamma^{-1}(t) = \gamma(-t)$ for $t \in (-\infty, \infty)$ is neither axis of $\tau^{-1}$ nor geodesic, in general (cf. [16]).

Proposition 3.1 is certified by the following Lemmas 3.3 to 3.5.

**Lemma 3.3.** For all $\tau \in \Phi$, $\tau \neq id.$, the displacement function $d_\tau$ takes the positive minimum on $N$. The set of all minimum points of $d_\tau$ is contained in $\text{Int}(N)$ and invariant under $\Phi$.

**Proof.** Since $\Phi$ is abelian, we have $d_\tau(\sigma(q)) = d_\tau(q)$ for all $q \in N$ and all $\sigma \in \Phi$. Hence, the set of all minimum points of $d_\tau$ is invariant under $\Phi$. 

Let $\Phi$ be a covering transformation group and $\tau \in \Phi$, we have $m := \inf\{d_{\tau}(q) \mid q \in N\} > 0$. We prove that the set of minimum points of $d_{\tau}$ is not empty, and if $d_{\tau}(q) = \min d_{\tau}$ for a point $q \in N$, then $q \in \text{Int}(N)$. Let $q_j \in N$ be a sequence of points in a fundamental domain $\tilde{M}_0$ for $M_0$ such that $d_{\tau}(q_j)$ converges to $m$ as $j \to \infty$. We suppose for an indirect proof that $q_j$ converges to a point $q \in \partial N$ or $q = \tilde{p}_i \in \mathbb{R}^2$ for some $i = 1, \ldots, k$, where $\pi(\tilde{p}_i) = p_i \in S_0$.

In case $q = \tilde{p}_i$, $M_0$ is bounded around $p_i$ with respect to the distance $d_0$, since $q_j$ and $\tau(q_j)$ belong to different fundamental domains. This is not the case when $M$ is geodesically complete. Then the minimal geodesics $T(q_j, \tau(q_j))$ from $q_j$ to $\tau(q_j)$ in $N$ satisfy $T(q_j, \tau(q_j)) \cap \{q_j, \tau(q_j)\} \subset \text{Int}(N)$, since $N$ is geodesically convex and any connected component of $\partial N$ cannot contain both $q_j$ and $\tau(q_j)$. In particular, the midpoint $r_j \in T(q_j, \tau(q_j))$ is contained in $\text{Int}(N)$. We assume that $r_j$ converges to a point $r$ as well. Then we have $r \in \text{Int}(N)$, because $r$ is an interior point of a minimal geodesic. Further, $T(q, \tau(q)) \cup T(q, \tau(q))$ is the union of minimal geodesics broken at $\tau(q)$. Since $N$ is geodesically convex and $r \in \text{Int}(N)$, a minimal geodesic $T(r, \tau(r))$ is contained in $\text{Int}(N)$. Hence, we have that $d_{\tau}(r) < d(r, \tau(q)) + d(\tau(q), \tau(r)) = d(r, \tau(q)) + d(q, r) = \lim_{j \to \infty} d_{\tau}(q_j) = m$, a contradiction. Therefore, we have $q \in \text{Int}(N)$.

**Lemma 3.4.** Let $\tau \in \Phi$, $\tau \neq id$. If $p \in N$ is a minimum point of $d_{\tau}$, then

$$T_{\tau}(p) := \bigcup_{n=-\infty}^{\infty} T(\tau^n(p), \tau^{n+1}(p))$$

is a unique $\tau$-invariant and simple geodesic through $p$ in $N$.

**Proof.** We first prove that $T_{\tau}(p)$ is a geodesic in $N$. Let $q \in T(p, \tau(p))$ be a point between $p$ and $\tau(p)$, i.e., $q \in T(p, \tau(p)) \setminus \{p, \tau(p)\}$. We then have

$$d(p, \tau(p)) \leq d(q, \tau(q)) \leq d(q, \tau(p)) + d(\tau(p), \tau(q))$$

$$= d(p, q) + d(q, \tau(p)) = d(p, \tau(p)).$$

Therefore, we have $d(p, \tau(p)) = d(q, \tau(q)) = d(q, \tau(p)) + d(\tau(p), \tau(q))$, meaning that $T(p, \tau(p))$ and $T(\tau(p), \tau^2(p))$ are smoothly joined at $\tau(p)$ to make a geodesic segment $T(p, \tau(p)) \cup T(\tau(p), \tau^2(p))$ in $N$. In particular, we note that there exists a unique minimal geodesic segment $T(p, \tau(p))$ from $p$ to $\tau(p)$, because $\tau$ preserves the orientation of $N$. In fact, if there exist two minimal geodesics $T_1$ and $T_2$ from $p$ to $\tau(p)$, then both $T_1 \cup \tau(T_1)$ and $T_2 \cup \tau(T_2)$ are smooth geodesics having the same end points $p$ and $\tau^2(p)$ and crossing at $\tau(p)$. However, two simple closed curves $T_1 \cup T_2^{-1}$ and $\tau(T_1 \cup T_2^{-1}) = \tau(T_1) \cup \tau(T_2)^{-1}$ have different orientations,
a contradiction. From the uniqueness of the minimal geodesic from \( p \) to \( \tau(p) \), the
joined geodesics \( T_\tau(p) \) form a unique \( \tau \)-invariant geodesic passing through \( p \).
Since \( \{\tau^n(p) \mid n \in \mathbb{Z}\} \) is unbounded, \( T_\tau(p) \) is not a closed geodesic in \( N \).
We next prove that \( T_\tau(p) \) is simple. Suppose for an indirect proof that
\( \tau^n(T(p, \tau(p))) \cap \tau^m(T(p, \tau(p))) \neq \emptyset \) for some integers \( n \) and \( m \), \( n \neq m \). Since
\( T_\tau(p) \) is not a closed geodesic, \( \tau^n(T(p, \tau(p))) \cap \tau^m(T(p, \tau(p))) \) consists of a single
point \( q \). However, it is impossible, because \( \tau^n(T(p, \tau(p))) \) and \( \tau^m(T(p, \tau(p))) \)
contain a sub-segment of \( T(q, \tau(q)) \) in common. \( \square \)

The straightness of \( T_\tau(p) \) in \( N \) can be proved by the same way as in [6]. We then use Lemma 2.1 (Jordan curve theorem) for \( N \).

**Lemma 3.5.** Let \( \tau \in \Phi, \tau \neq id \). If \( p \in N \) is a minimum point of \( d_\tau \), then
\( T_\tau(p) \) is a straight line in \( N \) invariant under \( \tau \).

**Proof.** Suppose for an indirect proof that \( T_\tau(p) \) is not minimal in \( N \). There exists a minimum integer \( k \) such that \( T_\tau(p)^k := \bigcup_{n=0}^{k-1} T(\tau^n(p), \tau^{n+1}(p)) \) is not
a minimal geodesic segment in \( N \). We then have \( k \geq 2 \) and \( d(p, \tau^k(p)) < k \min d_\tau \).

Since \( T_\tau(p)^k \) is not minimal, a minimal geodesic \( T(p, \tau^k(p)) \) from \( p \) to \( \tau^k(p) \)
is different from \( T_\tau(p)^k \). In fact, \( T(p, \tau^k(p)) \cap T_\tau(p)^k = \{p, \tau^k(p)\} \), because both
\( T(p, \tau^{k-1}(p)) \) and \( T(\tau^{k-1}(p), \tau^k(p)) \) are minimal. Since \( \tau \) is an orientation
preserving isometry of \( N \) and \( T_\tau(p) \) is invariant under \( \tau \), we see from Lemma 2.1 that
\( T(p, \tau^k(p)) \) intersects \( \tau(T(p, \tau^k(p))) = T(\tau(p), \tau^{k+1}(p)) \) at one point \( q \). Further,
we have \( T(\tau(p), \tau^{k+1}(p)) \cap T(\tau^2(p), \tau^{k+2}(p)) \ni \tau(q) \).

Since

\[
\begin{align*}
k \min d_\tau &> d(p, \tau^k(p)) = d(\tau(p), \tau^{k+1}(p)) \\
&= d(\tau(p), q) + d(q, \tau(q)) + d(\tau(q), \tau^{k+1}(p)) \\
&= d(\tau(p), q) + d(q, \tau(q)) + d(q, \tau^k(p)) \\
&\geq d(\tau(p), \tau^k(p)) + d(q, \tau(q)) = (k-1) \min d_\tau + d(q, \tau(q)),
\end{align*}
\]

we have \( \min d_\tau > d(q, \tau(q)) \), a contradiction. \( \square \)

Any point in \( T_\tau(p) \) is a minimum point of \( d_\tau \). Hence the parametrization
\( \gamma : (-\infty, \infty) \to N \) of \( T_\tau(p) \) satisfies the property \( \tau(\gamma(t)) = \gamma(t+c) \) as in Proposition 3.1. There are some phenomena which do not happen in the case of reversible
geodesics.
The following are true:

1. Let \( \tau \in \Phi \), \( \tau \neq id \). If \( p, q \in N \) are minimum points of \( d_{\tau} \), then either \( T_{\tau}(p) = T_{\tau}(q) \) or \( T_{\tau}(p) \cap T_{\tau}(q) = \emptyset \) is true. Further, \( \tau^{-1}(T_{\tau}(p)) = T_{\tau}(p) \), but the reverse curve \( T_{\tau}(p)^{-1} \) may be neither an axis of \( \tau^{-1} \) nor a straight line (cf. [16]).

2. Let \( \tau, \sigma \in \Phi \), \( \tau, \sigma \neq id \), \( \tau \neq \sigma \). Assume that an axis \( \gamma : (-\infty, \infty) \rightarrow N \) of \( \tau \) intersects an axis \( \alpha : (-\infty, \infty) \rightarrow N \) of \( \sigma \) at \( p = \gamma(0) = \alpha(0) \). Then \( \gamma((0, \infty)) \cap \alpha((0, \infty)) \neq \emptyset \) and \( \gamma((-\infty, 0)) \cap \alpha((-\infty, 0)) = \emptyset \) are true. However, \( \gamma((0, \infty)) \cap \alpha((-\infty, 0)) \neq \emptyset \) and \( \gamma((-\infty, 0)) \cap \alpha((0, \infty)) \neq \emptyset \) may happen (cf. [16]).

A straight line \( \gamma : (-\infty, \infty) \rightarrow N \) divides \( N \) into two connected components. We call them the right side \( E(\gamma) \) and the left side \( W(\gamma) \) of \( \gamma \).

In conjunction with Proposition 3.1, we have the following Proposition 3.7, using the same argument as in [5].

**Proposition 3.7.** Let \( \gamma : (-\infty, \infty) \rightarrow N \) be a straight line in \( N \). If \( \gamma \) is positively invariant under \( \tau \in \Phi \), i.e., \( \tau(\gamma(t)) = \gamma(t+c) \) for some \( c > 0 \), then \( c = \min d_{\tau} \) and \( \gamma \) is an axis of \( \tau \). Hence all points \( p \in \gamma((-\infty, \infty)) \) are minimum points of \( d_{\tau} \) and \( \gamma((-\infty, \infty)) = T_{\tau}(p) \). Moreover, there exists \( \tau_0 \in \Phi \) such that, if \( \tau \in \Phi \) leaves \( \gamma \) invariant, then \( \tau = \tau_0^k \) for some \( k \in \mathbb{Z} \). If \( \tau_0 = \varphi^m \circ \psi^n \), then \( m \) and \( n \) are relatively prime, where \( \varphi \) and \( \psi \) are the generators of \( \Phi \).

**Proof.** Let \( p = \gamma(t) \) for a number \( t \in (-\infty, \infty) \) and \( q \in N \). From the assumption, we then have \( c = d_{\tau}(p) \) and

\[
nd(p, \tau(p)) = d(p, \tau^n(p)) \leq d(p, q) + \sum_{k=1}^{n} d(\tau^{k-1}(q), \tau^k(q)) + d(\tau^n(q), \tau^n(p)) = d(p, q) + nd(q, \tau(q)) + d(q, p).
\]

Hence, we have

\[
d(p, \tau(p)) \leq d(q, \tau(q)) + \frac{d(p, q) + d(q, p)}{n}.
\]

As \( n \rightarrow \infty \), we conclude that \( c = d_{\tau}(p) \leq d_{\tau}(q) \), meaning that \( p \) is a minimum point of \( d_{\tau} \).

Let \( \Phi_1 = \{ \tau \in \Phi | \tau(\gamma(t)) = \gamma(t + \min d_{\tau}) \text{ for all } t \in (-\infty, \infty) \} \) and \( c = \inf \{ \min d_{\tau} | \tau \in \Phi_1, \tau \neq id \} \). Since \( \Phi \) is properly discontinuous, there exists \( \tau_0 \in \Phi_1 \) such that \( \min d_{\tau_0} = c > 0 \). Let \( \tau \in \Phi_1 \) and \( d = \min d_{\tau} \). If \( d = c \), then
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\(\tau = \tau_0\). Let \(d = kc + e\) for some \(k \in \mathbb{Z}\) with \(k \geq 0\) and some number \(e\) with \(0 \leq e < c\). We prove \(e = 0\). In fact, \(\tau_1 = \tau_0^{-k} \circ \tau\) satisfies that \(\tau_1 \in \Phi_1\) and \(e = \min d_{\tau_1}\), contradicting the choice of \(e\) if \(e \neq 0\). Since \(e = 0\), we have \(\tau_1 = id\). Hence \(\tau = \tau_0^k\) with \(k > 0\). This implies that if \(\gamma\) is positively invariant under \(\tau \in \Phi\), we then have \(\tau = \tau_0^k\) for some \(k > 0\).

In case there exists a number \(c > 0\) such that \(\tau(\gamma(t)) = \gamma(t - c)\) for all \(t \in (-\infty, \infty)\), we have \(\tau^{-1} = \tau_0^k\) for some \(k > 0\), since \(\tau^{-1}(t) = \gamma(t + c)\) for all \(t \in (-\infty, \infty)\). Then \(\tau = \tau_0^{-k}\).

Suppose for an indirect proof that \(m\) and \(n\) is not relatively prime, i.e., \(m = km_1\) and \(n = kn_1\) for some integers \(k > 1\), \(m_1\) and \(n_1\). Let \(\tau_1 = \varphi^{m_1} \circ \psi^{n_1}\). From the choice of \(c\), we have \(\tau_1(\gamma(\infty, \infty)) \cap \gamma((\infty, \infty)) = 0\), because both \(\tau_1 \circ \gamma\) and \(\gamma\) are axes of \(\tau_0\) (see Remark 3.6). Since \(\tau_1\) preserves the orientation of \(N\), if \(\tau_1 \circ \gamma\) is contained in \(E(\gamma)\) (resp., \(W(\gamma)\)), then \(\tau_1^k \circ \gamma\) is also contained in \(E(\gamma)\) (resp., \(W(\gamma)\)). This contradicts that \(\tau_1^k \circ \gamma(\infty, \infty)) = \tau_0 \circ \gamma((\infty, \infty)) = \gamma((\infty, \infty))\).

\[\square\]

4. Straight lines and slopes

4.1. Busemann functions and limit circles. Let \(\gamma : (-\infty, \infty) \to N\) be a straight line. We define the Busemann function \(B_\gamma : N \to \mathbb{R}\) of \(\gamma\) by

\[B_\gamma(p) = \lim_{t \to -\infty} d(\gamma(t), p) + t\]

for all \(p \in N\). It follows that

\[-d(p, q) \leq B_\gamma(p) - B_\gamma(q) \leq d(q, p)\]

for all \(p, q \in N\). Hence, \(B_\gamma\) is differentiable on a full measure set in \(N\). The structure of the level sets of a Busemann function has been studied in [18] and [27]. We say that a ray \(\alpha : (-\infty, 0] \to N\) is a co-ray to \(\gamma^- : (\infty, 0] \to N\), \(\gamma^- (t) = \gamma(t)\), ending at \(p = \alpha(0)\) if there exist a sequence of numbers \(t_j \to \infty\) and a sequence of points \(p_j \in N\) such that a sequence of minimal geodesics \(\alpha_j : [-d(\gamma(t_j), p_j), 0] \to N\) converges to \(\alpha\) as \(j \to \infty\), where \(\alpha_j(-d(\gamma(t_j), p_j)) = \gamma(t_j)\) and \(p_j = \alpha_j(0)\).

From [5], we see that a curve \(\alpha : (-\infty, 0] \to N\) is a co-ray to \(\gamma^-\) ending at \(\alpha(0)\) if and only if \(B_{\gamma}(\alpha(t)) = t + B_{\gamma}(\alpha(0))\) for all \(t \leq 0\). We call the end point of a maximal co-ray to \(\gamma^-\) a co-point to \(\gamma^-\). Let \(C(\gamma^-)\) denote the set of all co-points to \(\gamma^-\). Then \(B_{\gamma}\) is of class \(C^1\) on \(N \setminus C(\gamma^-)\) and the gradient vector of \(B_{\gamma}\) at \(p \not\in C(\gamma^-)\) is \(\dot{\alpha}(0)\) where \(\alpha : (-\infty, 0] \to N\) is a unique co-ray to \(\gamma^-\) ending at
Let \( B_N \) be the limit circle with central ray \( \gamma \). We say that a straight line \( \alpha : (-\infty, \infty) \rightarrow N \) is an asymptote to \( \gamma \) if \( B_\gamma(\alpha(t)) = t + B_\gamma(\alpha(0)) \) for all \( t \in (-\infty, \infty) \). In addition, if a restriction \( \alpha : [a, \infty) \rightarrow N \) is a co-ray to \( \gamma \), i.e., there exists a sequence of minimal geodesics \( \alpha_j \) from \( p_j = \alpha_j(a) \) to \( \gamma(t_j) = \alpha_j(d(p_j, \gamma(t_j))) \) such that \( \alpha_j \) converges to \( \alpha \) and \( t_j \to \infty \) as \( j \to \infty \), we call \( \alpha \) a parallel to \( \gamma \). The Busemann functions on the universal covering spaces of Finsler 2-tori are studied in [17] and [18].

For a function \( f \) on \( N \), let \([f = a] := \{ p \in N \mid f(p) = a \}\), \([f \leq a] := \{ p \in N \mid f(p) \leq a \}\), and so on. When \( \gamma \) is a straight line, it follows from (22.14) in [5, p. 133], that \([B_\gamma = a] = \lim_{t \to -\infty} S^d_N(\gamma(t), a-t)\) for all \( a \in R \). We call \([B_\gamma = a] \) a limit circle with central ray \( \gamma^- \).

**Lemma 4.1** (cf. [5, Theorem (32.4)]). Let \( \tau \in \Phi, \tau \neq id \). Then all axes of \( \tau \) are parallels to each other.

**Proof.** Let \( c = \min d_\tau \), and let \( \gamma \) and \( \alpha \) be two axes of \( \tau \). We prove that \( \alpha|_{(-\infty, s]} \) is a co-ray to \( \gamma^- \) for any \( s \in (-\infty, \infty) \). Since

\[
B_\gamma(\alpha(s-c)) = B_\gamma(\tau^{-1}(\alpha(s))) = \lim_{t \to -\infty} d(\gamma(t), \tau^{-1}(\alpha(s))) + t
\]

\[
= \lim_{t \to -\infty} d(\tau^{-1}(\gamma(t+c)), \tau^{-1}(\alpha(s))) + t
\]

\[
= \lim_{t \to -\infty} d(\gamma(t+c), \alpha(s)) + t = B_\gamma(\alpha(s)) - c,
\]

\( \alpha(s-c) \) is a foot of \( \alpha(s) \) on \([B_\gamma = B_\gamma(\alpha(s)) - c]\). From (22.17) and (22.18) in [5], we conclude that \( \alpha \) is an asymptote to \( \gamma^- \). The similar argument proves that \( \alpha|_{[s, \infty)} \) is a co-ray to \( \gamma \). \( \Box \)

**4.2. Fundamental domains over \( M_0 \) and slopes of straight lines.** Assume that \( \Phi \) is generated by two motions \( \{\varphi, \psi\} \). Let \( \mu : (-\infty, \infty) \rightarrow N \) be an axis of \( \varphi \). Then \( \psi \circ \mu \) is an axis of \( \varphi \) also. We may assume that \( \psi \circ \mu \in W(\mu) \).

We take a simple curve \( c : [0,1] \rightarrow N \) in the strip bounded by \( \mu((-\infty, \infty)) \) and \( \psi \circ \mu((-\infty, \infty)) \) such that \( c(0) \in \mu((-\infty, \infty)) \) and \( c(1) = \psi(c(0)) \). Let \( \nu : (-\infty, \infty) \rightarrow N \) be a parametrization of a curve \( \cup_{i=-\infty}^{\infty} \psi^i(c([0,1])) \) such that \( \nu(t) = \psi^i(c(s)) \) if \( t = i+s, 0 \leq s < 1 \), for some integer \( i \). We use this \( \nu \) instead of any axis of \( \psi \) because of the fact (2) in Remark 3.6. The domain bounded by \( \mu, \psi \circ \mu, \nu \) and \( \varphi \circ \nu \) is denoted by \( N(0,0) \). Obviously, \( N(0,0) \) covers \( M_0 \), i.e., \( \pi(N(0,0)) = M_0 \). If we set \( N(i,j) = \varphi^i \circ \psi^j(N(0,0)) \), then \( N = \cup_{(i,j) \in Z^2} N(i,j) \).

Using this notation, if \( \gamma : (-\infty, \infty) \rightarrow N \) is a straight line and \( \gamma(t) \in N(i(t),j(t)) \) for \( t \in (-\infty, \infty) \), we then have \( [i(t)] \to \infty \) or \( [j(t)] \to \infty \) as \( t \to \pm \infty \). Hereafter, we use the word “ray” in the following sense: a minimal geodesic \( \gamma : [0, \infty) \rightarrow N \) (resp., \( (-\infty, 0] \rightarrow N \)) such that \( \gamma(t) \in N(i(t),j(t)) \) for all \( t \) is
a ray if \(|i(t)|\) or \(|j(t)|\) goes to \(\infty\) as \(t \to \infty\) (resp., \(-\infty\)). The half parts of axes and their co-rays are rays.

Let \(\gamma: (-\infty, 0] \to N\) be a ray. We define the slope \(A(\gamma)\) of \(\gamma\) by

\[
A(\gamma) = \liminf_{t \to -\infty} \left\{ \frac{j(t)}{i(t)} \Big| \gamma(t) \in N(i(t), j(t)) \right\}.
\]

We prove that “\(\lim inf\)” is replaced by “\(\lim\)” in Lemma 4.3.

**Lemma 4.2.** If \(\gamma: (-\infty, \infty) \to N\) is an axis of \(\tau = \varphi^{m} \circ \psi^{n} \in \Phi\), \(\tau \neq id\), we then have \(A(\gamma) = n/m\) if \(m \neq 0\), and \(A(\gamma) = \infty\) if \(m = 0\).

**Proof.** Assume that \(p = \gamma(0) \in N(m_0, n_0)\) is a minimum point of \(d_\tau\) and \(c = \min d_\tau\). Let \(L\) be the maximum of those numbers \(|m_0 - i|\) and \(|n_0 - j|\) where \(N(i, j)\) intersects a \(c\)-ball with center \(p\) with respect to \(d(\cdot, p)\). If \(t = kc + r\) for some integer \(k\) and some number \(r\) with \(0 \leq r < c\), then \(\gamma(t) \in N(m_0 + km + m_1, n_0 + kn + n_1)\) for some \(m_1\) and \(n_1\) with \(0 \leq |m_1| < L\) and \(0 \leq |n_1| < L\). Hence we have

\[
A(\gamma) = \lim_{t \to -\infty} \frac{n_0 + kn + n_1}{m_0 + km + m_1} = \frac{n}{m}.
\]

All axes of \(\tau\) and \(\tau^{-1}\) have the same slope.

**Lemma 4.3.** Let \(\gamma: (-\infty, 0] \to N\) be a ray. We then have

\[
A(\gamma) = \lim_{t \to -\infty} \left\{ \frac{j(t)}{i(t)} \Big| \gamma(t) \in N(i, j) \right\}.
\]

Furthermore, for a straight line \(\gamma: (-\infty, \infty) \to N\), we have

\[
A(\gamma) = \lim_{t \to \pm\infty} \left\{ \frac{j(t)}{i(t)} \Big| \gamma(t) \in N(i, j) \right\}.
\]

**Proof.** Suppose for an indirect proof that there exists a rational number \(n/m\) such that

\[
\liminf_{t \to -\infty} \left\{ \frac{j}{i} \Big| \gamma(t) \in N(i, j) \right\} < \frac{n}{m} < \limsup_{t \to -\infty} \left\{ \frac{j}{i} \Big| \gamma(t) \in N(i, j) \right\}.
\]

Then there exists an axis of \(\tau = \varphi^{m} \circ \psi^{n}\) such that it intersects \(\gamma\) many times. Since the axis and ray \(\gamma\) are minimal, this is impossible, proving this lemma.

The second statement is proved in the same way. \(\square\)
Under a slightly different definition of slopes or rotation numbers, we see the complete structure of all sets of all straight lines with slope $h \in \mathbb{R}$ when $N$ is the universal covering plane of a 2-torus with Riemannian or reversible Finsler metric (cf. [3], [24]).

Instead of classifying the straight lines in $N$, we pay our attention to a restricted set of straight lines with slope $h \in \mathbb{R}$.

Let $X_h$ denote a set of straight lines for $h \in \mathbb{R}$:

1. If $h = n/m$ is a rational number, then $X_h$ is the set of all axes of $\tau = \varphi^m \circ \psi^n$ in $N$ for some $(m, n) \in \mathbb{Z}^2$ with $m > 0$.
2. If $h$ is an irrational number, then $X_h$ is the set of all straight lines such that there exists a sequence of axes in $X_\ell$ converging to $\alpha$ as $\ell \to h - 0$, where $\ell$ are rational numbers.

For two straight lines $\gamma$ and $\alpha$, we write $\gamma > \alpha$ when $\alpha$ is contained in $E(\gamma)$. The relation $>$ is a partial order on the set of all straight lines in $N$. Because all straight lines in $X_h$ are mutually disjoint, the following lemma is obvious.

**Lemma 4.4.** All geodesics in $X_h$ are straight lines with slope $h \in \mathbb{R}$ and $X_h$ is $\Phi$-invariant, i.e., $\tau \circ \gamma \in X_h$ for any $\gamma \in X_h$ and any $\tau \in \Phi$. The set $X_h$ is a totally ordered set. If $\alpha, \gamma \in X_h$ such that $\alpha < \gamma$, then $\alpha$ is an asymptote to $\gamma^-$.

### 5. Level sets of Busemann functions

Let $\gamma : (-\infty, \infty) \to N$ be a straight line. Note that the boundary of $[B_\gamma > a]$ possibly contains sub-arcs of the boundary of $N$, and that $[B_\gamma = a]$ may be divided by a removed point if $M$ is not complete.

**Lemma 5.1.** For all $a \in \mathbb{R}$, there exists the unique connected component of $[B_\gamma > a]$ whose boundary is unbounded in $N$.

**Proof.** Since $\gamma([a + 1, \infty)) \subset [B_\gamma > a]$, there exists at least one unbounded connected component $W_1$ of $[B_\gamma > a]$. Because of the topological structure of $N$ and Theorem 2.1 (Jordan curve theorem), the boundary of $W_1$ is unbounded. Suppose for an indirect proof that there exists another unbounded connected component $W_2$ of $[B_\gamma > a]$ such that the boundary of $W_2$ is unbounded. Then we have a compact set $K$ in $N$ such that $N \setminus W_1 \cup W_2 \cup K$ has at least two unbounded connected components one of which contains $\gamma((-\infty, a - 1))$. If $p_b$ is a boundary point of $W_2$ contained in another unbounded connected component of
If lines contained in $E^2$ because of the definition of $X$ cross the co-rays to $\gamma^-$ such that $\alpha_k(0) \in K$. As $p_k$ goes to $\infty$, choosing a subsequence of $\alpha_k$ converging a straight line $\alpha$, we have an asymptote $\alpha$ to $\gamma^-$. However, this is impossible, because $B_\gamma(\alpha(t))$ is bounded above by $a$. \hfill \Box

Let $Y_\gamma(a)$ denote the boundary of the unbounded connected component of $[B_\gamma > a]$ containing $\gamma([a + 1, \infty])$ for each $a \in \mathbb{R}$ in $N$. Obviously, $[B_\gamma = a] := Y_\gamma(a) \setminus \partial N \subset [B_\gamma = a]$. Furthermore, $Y_\gamma(a)$ divides $N$ into two connected components $N^+$ and $N^-$ such that $\gamma((a, \infty)) \subset N^+$ and $\gamma((a, \infty)) \subset N^-$. If $p \in N^+$, then $p \in [B_\gamma > a]$. If $p \in [B_\gamma < a]$, then $p \in N^-$. In general, it follows that $[B_\gamma > a] \cap N^-$ \neq \emptyset. The parameterized curve $Y_\gamma(a)(t)$, $t \in \mathbb{R}$, is assumed to cross the co-rays to $\gamma^-$ from left to right.

Let $\gamma \in X_h$, and let $X_h(\gamma)$ denote a subset of $X_h$ consisting of all straight lines contained in $E(\gamma)$. Then all straight lines $\alpha \in X_h(\gamma)$ are asymptotes to $\gamma^-$ because of the definition of $X_h$ (see Lemma 4.4). We use a parametrization of $\alpha \in X_h$ such that $B_\gamma(\alpha(t)) = t$ for all $t \in (-\infty, \infty)$ if $\alpha < \gamma$, and $B_\alpha(\gamma(t)) = t$ for all $t \in (-\infty, \infty)$ if $\alpha > \gamma$.

**Lemma 5.2.** If $\alpha \in X_h(\gamma)$, then $B_\alpha = B_\gamma$ on $E(\alpha)$.

**Proof.** If $\beta$ is a co-ray from $p \in E(\alpha)$ to $\gamma^-$, then $\beta$ is a co-ray to $\alpha^-$ as well, since $\alpha$ is an asymptote to $\gamma^-$. Conversely, a co-ray $\beta$ to $\alpha^-$ in $E(\alpha)$ is a co-ray to $\gamma^-$. Hence, $B_\gamma - B_\alpha$ is constant on $E(\alpha)$, because the distribution of co-rays of $\gamma^-$ and $\alpha^-$ in $E(\alpha)$ are identified. In particular, the gradient vectors of $B_\gamma$ and $B_\alpha$ are equal almost everywhere (see Subsection 4.1). We have $B_\gamma(p) - B_\alpha(p) = B_\gamma(\alpha(0)) - B_\alpha(\alpha(0)) = 0$. \hfill \Box

From this lemma, we can define a function $B_h : N \to \mathbb{R}$ by $B_h(p) = B_\alpha(p)$ for all $p \in N$ where $\alpha$ is a straight line in $X_h$ such that $p \in E(\alpha)$.

**Lemma 5.3.** Let $h, k \in \mathbb{R}$ with $h \neq k$. If $Y_h(a)(t_0) = Y_k(b)(t_1) =: p$, then $Y_h(a)((t_0, \infty)) \cap Y_k(b)((t_1, \infty)) \setminus \partial N = \emptyset$.

**Proof.** We may assume that $h < k$. Suppose for an indirect proof that there exist numbers $s_0 > t_0$ and $s_1 > t_1$ such that $Y_h(a)((t_0, s_0)) \cap [B_k = a] \cap Y_k(b)((t_1, s_1)) \cap [B_k = b] = \emptyset$ and $Y_h(a)(s_0) = Y_k(b)(s_1) =: q \in [B_k = a] \cap [B_k = b]$.

Let $\alpha : (-\infty, \infty) \to N$ (resp., $\beta : (-\infty, \infty) \to N$) be a straight line in $X_h$ (resp., $X_k$) such that $p, q \in E(\alpha)$ (resp., $p, q \in E(\beta)$). We may assume that the sequences of minimal geodesics $T(\alpha(t), p)$, $T(\beta(t), p)$, $T(\alpha(q), q)$ and $T(\beta(q), q)$ converge to $\alpha_1$, $\beta_1$, $\alpha_2$ and $\beta_2$, respectively. Then $\alpha_1$ and $\beta_1$ (resp., $\alpha_2$ and $\beta_2$) are co-rays from $p$ to $\alpha^-$ and $\beta^-$, respectively, (resp., from $q$ to $\alpha^-$ and $\beta^-$,
respectively. Since \( h < k \), the co-ray \( \beta_1 \) intersects the co-ray \( \alpha_2 \) at some point \( r \in N \). This means that

\[
\liminf_{t \to -\infty} (d(\alpha(t), q) + d(\beta(t), p) - d(\alpha(t), p) - d(\beta(t), q)) > 0,
\]

since there exists a number \( \delta > 0 \) such that

\[
d(\alpha(t), q) + d(\beta(t), p) = d(\alpha(t), r_t) + d(r_t, q) + d(\beta(t), r_t) + d(r_t, p)
\]

\[
> d(\alpha(t), p) + d(\beta(t), q) + \delta
\]

for any \( t < 0 \) with sufficiently large \(|t|\) and \( r_t \to r \) as \( t \to -\infty \) where \( r_t = T(\alpha(t), q) \cap T(\beta(t), p) \). This contradicts the following equality.

\[
0 = (B_h(q) - B_h(p)) - (B_k(q) - B_k(p))
\]

\[
= \lim_{t \to -\infty} (d(\alpha(t), q) + d(\beta(t), p) - d(\alpha(t), p) - d(\beta(t), q)).
\]

**Lemma 5.4.** Let \( \tau \in \Phi \). Then the function \( f_h(\tau) = B_h \circ \tau - B_h \) is constant on \( N \). Moreover, \( f_h : \Phi \to \mathbb{R} \) is a homomorphism, i.e., \( f_h(\tau) \circ \sigma) = f_h(\tau) + f_h(\sigma) \) for all \( \tau, \sigma \in \Phi \). In particular, if \( \tau = \varphi^m \circ \psi^n \in \Phi \), we then have \( f_h(\tau) = mf_h(\varphi) + nf_h(\psi) \).

**Proof.** For any points \( p, q \in N \), let \( \gamma \in X_h \) be a straight line such that \( p \) and \( q \) are in the right side of \( \gamma \) and \( \tau^{-1} \circ \gamma \), i.e., \( p, q \in E(\gamma) \cap E(\tau^{-1} \circ \gamma) \). We then have

\[
B_h(\tau(p)) = B_h(\tau(q)) = \lim_{t \to -\infty} d(\gamma(t), \tau(p)) - d(\gamma(t), \tau(q))
\]

\[
= \lim_{t \to -\infty} d(\tau^{-1} \circ \gamma(t), p) - d(\tau^{-1} \circ \gamma(t), q) = B_h(p) - B_h(q).
\]

From this we conclude that \( f_h(\tau) \) is constant on \( N \).

Since

\[
f_h(\tau \circ \sigma)(p) = B_h(\tau(\sigma(p))) - B_h(p)
\]

\[
= (B_h(\tau(\sigma(p))) - B_h(\sigma(p))) + (B_h(\sigma(p)) - B_h(p))
\]

\[
= f_h(\tau)(\sigma(p)) + f_h(\sigma)(p)
\]

for all \( p \in N \), we have \( f_h(\tau \circ \sigma) = f_h(\tau) + f_h(\sigma) \). \( \square \)
Let $\Phi_0(h) = \text{Ker}(f_h) = \{\tau \mid f_h(\tau) = 0\}$ for each slope $h \in \mathbb{R}$. If $\tau \in \Phi_0(h)$, then $\tau(Y_h(a)) = Y_h(a)$ for all $a \in \mathbb{R}$. There exists $\tau_0 \in \Phi_0(h)$ such that $\tau = \tau_0^k$ for any $\tau \in \Phi_0(h)$ and some $k \in \mathbb{Z}$, as was seen in the proof of Proposition 3.7. In fact, if $c(t)$, $t \in (-\infty, \infty)$, is a parametrization of $Y_0(0)$ such that $c(0) = \gamma(0)$ and $c((0, \infty))$ is in the right side of $\gamma$ in $N$ and $t(\tau)$ are the numbers such that $\gamma(c(\tau)) = c(t(\tau))$ for any $\tau \in \Phi_0(h)$, then $\tau_0$ or $\tau_0^{-1}$ satisfies that $t(\tau_0) = \min\{|t(\tau)| > 0 \mid \tau \in \Phi_0(h) \setminus \{id\}\}$.

**Lemma 5.5.** Let $\Phi_0(h)$ be generated by $\tau_0 = \varphi^{m_0} \circ \psi^{n_0} \neq id$. Then, $m_0$ and $n_0$ are relatively prime and $f_h(\psi)/f_h(\varphi) = -m_0/n_0$ if $f_h(\varphi) \neq 0$, and $f_h(\varphi)/f_h(\psi) = -n_0/m_0$ if $f_h(\psi) \neq 0$.

**Proof.** Suppose for an indirect proof that $m_0 = km_1$ and $n_0 = kn_1$ for some $k \in \mathbb{Z}$ with $k \neq 1$. Hence, if $\tau_1 = \varphi^{m_1} \circ \psi^{n_1}$, then $\tau_1(\gamma(0)) \notin Y_h(0)$, implying that $f_h(\tau_1) \neq 0$. Then we get a contradiction: $0 = f_h(\tau_0) = kf_h(\tau_1) \neq 0$.

The second part of the theorem immediately follows from $0 = f_h(\tau_0) = m_0f_h(\varphi) + n_0f_h(\psi)$.

**Lemma 5.6.** Let $\tau \in \Phi_0(h)$. If a straight line $\gamma \in \cup_{k \in \mathbb{R}} X_k$ is not any axis of $\tau$, then $\gamma((-\infty, \infty))$ intersects $Y_h(a)$ for all $a \in \mathbb{R}$.

**Proof.** We first assume that $\gamma \in X_k$ for some rational number $k \in \mathbb{R}$ such that it is an axis of $\tau_1 \in \Phi$ with $f_h(\tau_1) \neq 0$. Then $|f_h(\tau_1^n)| = |nf_h(\tau_1)|$ goes to $\infty$ as $n \to \pm \infty$. This implies that $|B_h(\gamma(t))|$ goes to $\infty$ as $t \to \pm \infty$.

If the slope $k$ of $\gamma$ is irrational, then there exist a sequence of rational numbers $k_j$ with $k_j < k$ converging to $k$ and a sequence of axes $\gamma_j$ with slopes $k_j$ converging to $\gamma$. Since all axes $\gamma_j$ intersect $Y_h(a)$, $\gamma_j$ intersects $Y_h(a)$ for all $a \in \mathbb{R}$.

Let $\ell_h = \inf\{f_h(\tau) \mid \tau \in \Phi \setminus \Phi_0(h)\}$ such that $f_h(\tau) > 0$. Since $\ell_h = \inf\{|mf_h(\varphi) + nf_h(\psi)|| \to \infty\}$ such that $f_h(\varphi^{m} \circ \psi^{n}) \neq 0$, if $f_h(\psi)/f_h(\varphi)$ or $f_h(\varphi)/f_h(\psi)$ is an irrational number, we then have $\ell_h = 0$ (cf. [1], [19]). Assume that $f_h(\psi)/f_h(\varphi) = i/j$, where $i$ and $j$ are relatively prime integers. Then we have

$$f_h(\varphi^{m} \circ \psi^{n}) = \frac{mj + ni}{j}f_h(\varphi).$$

Since $i$ and $j$ are relatively prime integers, there exist integers $m$ and $n$ such that $mi + nj = 1$. Therefore, we see that

$$\ell_h = \min\left\{\left|\frac{f_h(\varphi)}{j}\right|, \left|\frac{f_h(\psi)}{i}\right|\right\}.$$
Note that $|f_h(\varphi)| \leq \min d_\varphi$ and $|f_h(\psi)| \leq \min d_\psi$. If one of the denominators $i$ and $j$ in the above estimate of $\ell_h$ is greater than $Q := \max \{\min d_\varphi, \min d_\psi\}/\varepsilon$ for a number $\varepsilon > 0$, we then have $\ell_h < \varepsilon$

**Lemma 5.7.** For any $\varepsilon > 0$, the number of slopes $h \in \mathbb{R}$ such that $\ell_h > \varepsilon$ is finite.

**Proof.** Assume that $\ell_h > \varepsilon$. Then there exists a $\tau_1 \in \Phi$ such that $f_h(\tau_1) = \ell_h$. Here we write $\tau_1 = \varphi^{m_1} \circ \psi^{n_1}$. Since $f_h(\Phi)$ is a subgroup generated by $\ell_h$, there exists an integer $k_1$ such that $f_h(\varphi) = k_1f_h(\tau_1)$. Hence, we then have $(k_1m_1 - 1)f_h(\varphi) + k_1n_1f_h(\psi) = 0$. We assume that $k_1m_1 - 1 = km_0$ and $k_1n_1 = kn_0$ for some integer $k$ where the integers $m_0$ and $n_0$ are relatively prime. Set $\tau_0 = \varphi^{m_0} \circ \psi^{n_0}$. Then $\tau_0$ is a generator of $\Phi_0(h)$. It follows from the argument just before Lemmas 5.7 and 5.5 that both $j\varphi^{m_0}$ and $j\psi^{n_0}$ are less than $Q$. Thus we have at most finitely many $\tau_0 = \varphi^{m_0} \circ \psi^{n_0}$ such that $f_h(\tau_0) = 0$ even if there exist infinitely many $\tau_1 \in \Phi$ such that $f_h(\tau_1) = \ell_h$. Furthermore, how to choose $m_0$ and $n_0$ depends only on $Q$, which does not depend on the slope $h$. From Lemma 5.3, there exists at most one slope $h \in \mathbb{R}$ such that $f_h(\tau_0) = 0$ for each $\tau_0$. This implies that the number of the slopes $h$ with $\ell_h > \varepsilon$ is finite.

6. A domain consisting of slices covering $M_0$

Let $h \in \mathbb{R}$ be a slope and $\gamma : (-\infty, \infty) \to N$ a straight line in $N$ such that $\gamma \in X_h$. Take a $\tau \in \Phi$ such that $\tau \circ \gamma \neq \gamma$. Let $\square(i, j; u, v)$ denote the rectangle bounded by $Y_\gamma(-if_h(\tau)), Y_\gamma(-jf_h(\tau)), Y^u \circ \gamma$ and $Y^v \circ \gamma$.

**Lemma 6.1.** Under the notation above, we have

$$
\tau^s(\square(i, j; u, v)) = \square(i - s, j - s; u + s, v + s).
$$

**Proof.** This lemma follows from the fact that $\tau^s \circ Y_\gamma(a) = Y_\gamma(a + sf_h(\tau))$ and $\tau^s \circ Y^u = \tau^{s+u}$.

Let $\Phi(\tau)$ denote the infinite cyclic subgroup of $\Phi$ generated by $\tau$. Then $N_1 = N/\Phi(\tau)$ is topologically a cylinder with disks and points removed. If $\rho_1 : N \to N_1$ is the quotient map, then $\rho_1 \circ \gamma$ may not be a minimal geodesic in $N_1$. By the way, $\rho_1(Y_\gamma(0))$ is a curve like a helix contained in $N_1$ with pitch $|f_h(\tau)|$ if $|f_h(\tau)| \neq 0$. In particular, we note that $\rho_1(Y_\gamma(0))$ is not a level set of the Busemann function $B_{\rho_1 \circ \gamma}$ in $N_1$ even if $\rho_1 \circ \gamma$ is a straight line in $N_1$. 


We may assume that $\min \{ B_h(x) \mid x \in N(0,0) \} = 0$ (see Subsection 4.2 for the definition of $N(i,j)$). Let $b > \max \{ a \in \mathbb{R} \mid Y_\gamma(a) \cap N(0,0) \neq \emptyset \}$. Hence, $N(0,0)$ is contained in the strip bounded by $Y_\gamma(0)$ and $Y_\gamma(b)$. It does not imply that $B_h(x) \leq b$ for all $x \in N(0,0)$, although $0 \leq B_h(x)$ are true for all $x \in N(0,0)$. Further, when $f_h(\tau) < 0$, we may assume that the domain bounded by $Y(0)$, $Y(\gamma(\tau))$, and $Y(\gamma(\tau)) \cap N(0,0) \neq \emptyset$. Hence, $N(0,0)$ is contained in the strip bounded by $Y(0)$ and $Y(\gamma(\tau))$. It does not imply that $B_h(x)$ for all $x \in N(0,0)$, although $0 \leq B_h(x)$ are true for all $x \in N(0,0)$. In particular, $M_0 = \pi(\square(0,k;0,1))$, where $\pi : N \to M_0$ is the covering map.

**Lemma 6.2.** Assume that $f_h(\tau) < 0$ and $b > |f_h(\tau)|$. Let $k$ be an integer such that $k |f_h(\tau)| > b$. We then have $\square(0,k;0,1) \subset \bigcup_{i=0}^{k-1} \square(i,i+1;-i,k-i)$ and $\pi(\square(i,i+1;-i,k-i)) = M_0$ for each $i = 0,\ldots,k-1$.

**Proof.** The first part of the statement follows from the definition.

We prove the second part. Since $\tau^{-i}(\square(0,1;i,i+1)) = \square(i,i+1;0,1)$, we have

$$\square(0,k;0,1) \subset \bigcup_{i=0}^{k-1} \square(i,i+1;0,1) = \bigcup_{i=0}^{k-1} \tau^{-i}(\square(0,1;i,i+1)).$$

Therefore, we have

$$\pi(\square(0,1;k)) = \pi(\bigcup_{i=0}^{k-1} \square(0,1;i,i+1)) = \pi(\bigcup_{i=0}^{k-1} \tau^{-i}(\square(0,1;i,i+1))) = \pi(\square(0,k;0,1)) = M_0.$$

Since $\square(i,i+1;-i,k-i) = \tau^{-i}(\square(0,1;0,k))$, we have

$$\pi(\square(i,i+1;-i,k-i)) = \pi(\square(0,1;0,k)) = M_0.$$

---

**Figure 4.** Domain which covers $M_0$. 

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7. The asymptotic behavior of distance circles

We start preparing the notations which are used in the proof of Theorem 2.3. For any \( \varepsilon > 0 \), we choose a slope \( h \in \mathbb{R} \) and a straight line \( \gamma \in X_h \) such that \( N(0,0) \) lies between \( Y_\gamma(0) \) and \( Y_\gamma(b) \), \( \ell_b < \varepsilon \), and then choose an isometry \( \tau \in \Phi \) such that \( -\varepsilon < f_h(\tau) < 0 \). The integer \( k_1 \) is defined to satisfy \( k_1 f_h(\tau) \geq b \).

Let \( \gamma_1 \in X_h \) be a straight line, and let \( \alpha \) be a \( \psi \)-invariant curve such that \( \alpha \) intersects \( \gamma_1 \) and \( \psi \circ \gamma_1 \) exactly once, respectively. Then the domains \( Q(j) \) bounded by \( \varphi^j \circ \alpha \), \( \varphi^{j+1} \circ \alpha \), \( \gamma_1 \) and \( \psi \circ \gamma_1 \) cover \( M_0 \) for all integers \( j \in \mathbb{Z} \). Therefore, for any point \( p \in N \), there exists a sequence of points \( p_j \in Q(j) \) such that \( \pi(p_j) = \pi(p) \), i.e., \( \tau_j(p) = p_j \) for some \( \tau_j \in \Phi \). Since \( -\infty < h < \infty \) and the perimeters of \( \psi^j(Q(j)) \) equal for all \( i, j \in \mathbb{Z} \), there exists a number \( K_1 \) such that \( d(p_j, p_{j+1}) < K_1 \) for all \( j \in \mathbb{Z} \) (as was seen in the proof of Lemma 7.1 in [18, p. 356]). Let \( L \) be a number such that \( L > \max\{b, K_1\} \), and \( k, k_2 \) be integers such that \( k|f_h(\tau)| > L, k = k_1 + k_2 \). We change the parameterization of \( \gamma \) such that \( \tilde{\gamma}(s) = \gamma(s + (k_2 - 1)f_h(\tau)) \).

After those preparations, using \( \tilde{\gamma} \), we construct a domain \( D = \bigcup_{i=0}^{k-1} \square(i, i + 1; i+k_2, i-k) \) each of whose slices covers \( M_0 \), i.e., \( \pi(\square(i, i+1; i+k_2, i-k)) = M_0 \) for each \( i = 0, \ldots, k - 1 \). We may assume that \( \gamma_1 \in X_h \) satisfies \( D \subset E(\gamma_1) \).

**Lemma 7.1** (cf. [18, Assertion 7.2]). There exists an integer \( j_1 \) such that

\[
d(p_j, \gamma_1(0)) < d(p_{j+1}, \gamma_1(L))
\]

for all integers \( j < j_1 \).

**Proof.** The sequences of minimal geodesics \( T(p_j, \gamma_1(0)) \) and \( T(p_{j+1}, \gamma_1(L)) \) converge to sub-rays of \( \gamma_1 \) as \( j \to -\infty \), so there exists a sequence of points \( r_{j+1} \in T(p_{j+1}, \gamma_1(L)) \) converging to \( \gamma_1(0) \) as \( j \to -\infty \). Therefore, there exists an integer \( j_1 \) such that

\[
d(p_{j+1}, \gamma_1(L)) - d(p_j, \gamma_1(0)) = d(p_{j+1}, r_{j+1}) + d(r_{j+1}, \gamma_1(L)) - d(p_j, \gamma_1(0))
\]

\[
> -(d(p_j, p_{j+1}) + d(r_{j+1}, \gamma_1(0))) + d(r_{j+1}, \gamma_1(L))
\]

\[
> -K_1 + L + d(r_{j+1}, \gamma_1(0)) - d(\gamma_1(0), r_{j+1}) > 0
\]

for all \( j < j_1 \), since \( d(p_j, p_{j+1}) < K_1 < L \) and \( d(r_{j+1}, \gamma_1(0)) \to 0, d(\gamma_1(0), r_{j+1}) \to 0 \) as \( j \to -\infty \). \( \square \)

Let \( a_j = d(p_j, \gamma_1(0)) \) and \( b_j = d(p_j, \gamma_1(L)) \). Then, for any \( t \in [a_j, b_j] \), there exists a point \( x_t \in \gamma_1([0, L]) \) such that \( d(p_j, x_t) = t \). Since \( d(p_j, \gamma_1(0)) \to \infty \),
as $j \to -\infty$, there exists an integer $j_0$ with $j_0 < j_1$ such that $a_j < b_j$ and $d(p_{j_0}, \gamma_1(0)) \leq d(p_j, \gamma_1(0))$ for all integers $j < j_0$. Hence, when $R_1 := a_{j_0}$, we have $R_1 = \min\{a_j \mid j \leq j_0\}$.

**Lemma 7.2** (cf. [18, Assertion 7.3]). For any $t > R_1$, there exist a point $x_t \in \gamma_1([0, L])$ and an integer $j < j_0$ such that $d(p_j, x_t) = t$.

**Proof.** Let $K_j = \cup_{i=j}^{j_0}[a_i, b_i]$ for $j < j_0$. We prove that $K_j$ is connected for all $j \leq j_0$. Suppose for an indirect proof that $K_{i_0}$ is connected but not $K_{i_0-1}$. From the definition of $R_1$, we have $K_{i_0} = [R_1, b_{j_2}]$ for some $j_2 < j_0$.

Since $K_{i_0-1}$ is not connected and $R_1 \leq a_{i_0-1}$, we have $b_{j_2} < a_{i_0-1}$. On the other hand, we have $b_{i_0} > a_{i_0-1}$ because of Lemma 7.1. Since $b_{i_0} \leq b_{j_2}$, we have $a_{i_0-1} \leq b_{j_2}$, a contradiction. Since $d(p_j, \gamma_1(0)) \to \infty$ as $j \to -\infty$, we have $\cup_{i>j_0}[a_i, b_i] = [R_1, \infty]$. For any $t > R_1$, if we choose an integer $j$ such that $t \in [a_j, b_j]$, then there exists a point $x_t \in \gamma_1([0, L])$ such that $d(p_j, x_t) = t$. \qed

**Lemma 7.3** (cf. [18, Lemma 6.1]). Let $\epsilon > 0, \gamma_1, L, D, p \in N$ and $p_j \in \Phi(p)$ be as above. Then there exists an integer $j_0 = j_0(D, \epsilon) > 0$ such that

$$B_h^{-1}(B_h(x)) \cap D \subset B(S_N^p(p_j, d(p_j, x)), \epsilon)$$

for all points $x \in \gamma_1([0, L])$ and all integers $j < j_0$. In particular, for any point $q \in B_h^{-1}(B_h(x)) \cap D$, we have $B(q, \epsilon) \cap S(p_j, d(p_j, x)) \not= \emptyset$.

**Proof.** Since $g(z, t) = d(\gamma_1(t), z) + t$ is monotone increasing for $t < 0$ and converges to $B_h(z)$ uniformly on any compact set contained in $D$ as $t \to -\infty$, there exists a number $T < 0$ such that $0 \leq g(z, t) - B_h(z) < \epsilon/3$ for all $z \in D$ and $t < T$.

If $q \in B_h^{-1}(B_h(x)) \cap D$ for a point $x \in \gamma_1([0, L])$, we then have

$$0 \leq d(\gamma_1(t), q) - d(\gamma_1(t), x) < \epsilon/3$$

for any number $t < T$, because

$$0 \leq d(\gamma_1(t), q) - d(\gamma_1(t), x) = (d(\gamma_1(t), q) + t) - (d(\gamma_1(t), x) + t)$$

$$= g(q, t) - B_h(x) = g(q, t) - B_h(q) < \frac{\epsilon}{3}.$$  

Set $A = (B_h^{-1}(B_h(x)) \cap D) \setminus B(x, \epsilon/2)$. Since $\gamma_1$ is an asymptote to $(\psi \circ \gamma_1)^-$, there exists a positive integer $j_0 = j_0(D, \epsilon)$ such that, for all integers $j < j_0$, a minimal geodesic segment $T(p_j, x)$ from $p_j$ to the point $x \in \gamma_1([0, L])$ (resp.,
We prove Theorem 2.3, which is sufficient for Theorem 1.1. Let $p$, $q$ be any point $q \in A$) passes through $B(\gamma_1(T + 1), \varepsilon/3)$ (resp., intersects $\gamma_1$ at $\gamma_1(t_j)$ with some $t_j < T$).

If $p' \in T(p_j, x)$ satisfies $\max\{d(p', \gamma_1(T + 1)), d(\gamma_1(T + 1), p')\} < \varepsilon/3$, we then have, from (1) (see Figure 5),

$$
0 < d(\gamma_1(t_j), q) - d(\gamma_1(t_j), x) = d(p_j, q) - d(p_j, \gamma_1(t_j)) - d(\gamma_1(t_j), x)
$$

$$
\leq d(p_j, q) - d(p_j, x) \leq d(p', q) + d(p_j, p') - d(p_j, x) = d(p', q) - d(p', x)
$$

$$
< (d(\gamma_1(T + 1), q) + \varepsilon/3) - (d(\gamma_1(T + 1), x) - \varepsilon/3) < \varepsilon
$$

for all $q \in A$. Therefore, we have

$$
d(p_j, x) < d(p_j, q) < d(p_j, x) + \varepsilon.
$$

![Figure 5. The asymptotic behavior of geodesics from $p_j$.](image)

If $y_j(q)$ is a point at which $T(p_j, q)$ and $S^N_X(p_j, d(p_j, x))$ intersect, then we have $q \in B(y_j(q), \varepsilon)$, and therefore, $q \in B(S^N_X(p_j, d(p_j, x)), \varepsilon)$.

For $q \in (B_h^{-1}(B_h(x)) \cap D) \cap B(x, \varepsilon/2)$, we have $q \in B(S^N_X(p_j, d(p_j, x)), \varepsilon)$, since $x \in S^N_X(p_j, d(p_j, x))$ and $d(x, q) < \varepsilon/2$.

Lemma 7.3 states that we can find a distance sphere $S^N_X(p_j, d(p_j, x))$ meeting the $\varepsilon$-ball $B(q, \varepsilon)$ for any point $q \in D$ with $B_h(q) = B_h(x)$. From Lemma 7.2, any point $q \in (\cup_{x \in Y_{\gamma_1}(\partial N)} Y_{\gamma_1}(B_h(x)) \setminus \partial N) \cap D$ satisfies this condition. We have to treat another case, $q \not\in Y_{\gamma_1}(a) \setminus \partial N$ for any $a \in \mathbb{R}$, in order to complete the proof of Theorem 2.3.

**Proofs of Theorem 1.1 and 2.3.** We prove Theorem 2.3, which is sufficient for Theorem 1.1. Let $p$, $q$ and $\varepsilon$ be as in Theorem 2.3. If $q \in (Y_{\gamma_1}(0) \setminus \partial N) \cap D$ for a suitable parametrization of $\gamma_1$, then it follows from Lemmas 7.1, 7.2 and 7.3.
that for any \(t > R_1\), there exist sequences of points \(p_j \in \Phi(p)\) and \(q_j \in D \cap \Phi(q)\) such that \(S_N^d(p_j, t) \cap B(q_j, \varepsilon) \neq \emptyset\).

In case \(q \not\in Y_{\gamma_1}(a) \setminus \partial N\) for any \(a \in \mathbb{R}\), we find a point \(q_1 \in D \cap \Phi(q)\) in a strip bounded by \(Y_{\gamma_1}(0)\) and \(Y_{\gamma_1}([f_h(\tau)])\). Assume that a sequence of minimal geodesics from \(p_j\) to \(q_1\) converges to a co-ray \(\alpha : (-\infty, 0] \to N\) from \(q_1\) to \(\gamma_1^{-}\) and \(r_1 = \alpha(-d(Y_{\gamma_1}(0), q_1))\). Then the sequence of intersection points \(r_j = T(p_j, q_1) \cap Y_{\gamma_1}(0)\) converges to \(r_1 \in Y_{\gamma_1}(0)\) as \(j \to -\infty\). This implies that for any \(t > R_1 + d(r_1, q_1)\), we have \(S_N^d(p_j, t) \cap B(q_j, \varepsilon) \neq \emptyset\) for some \(p_j \in \Phi(p)\) and \(q_j \in D \cap \Phi(q_1)\). \(\square\)

**Remark 7.4.** In the above argument, if \(p_j = \tau_j(p)\) and \(q_j = \tau_j'(q)\) for \(\tau_j, \tau_j' \in \Phi\), we then have \(S_N^d(p_j, t) \cap B(\tau_j^{-1} \circ \tau_j'(q), \varepsilon) = S_N^d(p_j, t) \cap B(q_j, \varepsilon) \neq \emptyset\).

For any \(\varepsilon > 0\) and any points \(p, q \in M\), let \(\tilde{p}\) (resp., \(\tilde{q}_k \in \Phi(D)\)) be the lifts of \(p\) (resp., \(q\)). Then it follows from the above consequence that the geodesic circle with center \(\tilde{p}\) meets the union of \(B(\tilde{q}_k, \varepsilon)\)'s for any \(t > R\) on \(N\). Combining with Lemma 2.2, we can see the asymptotic behavior of the geodesic circles emanating from \(p\) in \(M\) (see Figure 6).

\[\text{Figure 6. The geodesic circle with center } \tilde{p} \text{ in } N.\]

**Proof of Corollary 1.5.** We work in \(M_0\) instead of \(M\). Let \(n > 0\) be an integer and \(\varepsilon > 0\). Let \(\tilde{p} \in N\) be chosen so that \(\pi(\tilde{p}) = p\). From Lemma 5.7, there exist slopes \(h_i, i = 1, \ldots, n\), such that \(h_i \neq h_k\) for \(i \neq k\) and \(\ell_{h_i} < \varepsilon\) for all \(i\). As was seen in the proof of Theorem 1.1, for each slope \(h_i\), we can find domains \(D_i\) and numbers \(R_i\) satisfying the following: for any \(t > R_i\), there exist sequences of points \(p_{ij} \in N\) and \(q_{ij} \in D_i\) such that \(\pi(p_{ij}) = p, \pi(q_{ij}) = q\) and \(S_{M_0}(p_{ij}, t) \cap B(q_{ij}, \varepsilon) \neq \emptyset\). Let \(\tau_{ij} \in \Phi\) be such that \(\tau_{ij}(p_{ij}) = \tilde{p}\). The sequence of minimal geodesics \(T(\tilde{p}, \tau_{ij}(q_{ij}))\) from \(\tilde{p}\) to \(\tau_{ij}(q_{ij})\) converges to a ray with slope \(h_i\) as \(j \to -\infty\) for each \(i = 1, \ldots, n\). \(\square\)
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References


