Classification of 3-dimensional Landsbergian \((\alpha, \beta)\)-metrics

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Dedicated to the memory of Professor Lajos Tamássy

Abstract. In this paper we classify the class of 3-dimensional \((\alpha, \beta)\)-metrics with vanishing Landsberg curvature. More precisely, we show that these metrics belong to one of the following main classes: Berwald metrics which contain the Randers or Kropina metrics, or satisfy an ODE. Some of the well-known unicorns are special solutions of this ODE.

1. Introduction

Given an \(n\)-dimensional Finsler manifold \((M, F)\), a global vector field \(G\) is induced by \(F\) on \(TM_0\), which in a standard local coordinate system \((x^i, y^i)\) for \(TM_0\) is given by

\[
G_i = y^j \frac{\partial}{\partial x^i} - 2G^j_i \frac{\partial}{\partial y^i},
\]

where the functions

\[
G^i = \frac{1}{4} y^{ij} \left[ \frac{\partial^2 F^2}{\partial x^k \partial y^j} y^k - \frac{\partial F^2}{\partial x^i} \right].
\]

are the spray coefficients.

A Finsler metric is called a Berwald metric if its spray coefficients are quadratic in \(y \in T_x M\) for all \(x \in M\), i.e., there exist smooth functions \(\Gamma^i_{jk} = \Gamma^i_{kj}\) on \(M\) such that \(G^i = (\Gamma^i_{jk} \circ \pi) y^j y^k\), where \(\pi : TM \to M\) is the tangent bundle projection. For \(y \in T_x M_0\), define \(B_y : T_x M \times T_x M \times T_x M \to T_x M\) by

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The Finsler tensor field $\mathbf{B}$ is called the Berwald curvature of $F$. A Finsler metric is a Berwald metric if and only if $\mathbf{B} = 0$ [20]. An important characteristic of a Berwald space is that all of its tangent spaces are linearly isometric to a common Minkowski space [5]. Berwald metrics have been classified by Szabó in [18].

For $y \in T_x M$, define the Landsberg curvature $\mathbf{L}_y : T_x M \times T_x M \times T_x M \to \mathbb{R}$ by

$$\mathbf{L}_y (u, v, w) := -\frac{1}{2} g_y (\mathbf{B}_y (u, v, w), y),$$

where $g$ is the fundamental tensor of $(M, F)$. The Finsler tensor field $\mathbf{L}$ is called the Landsberg curvature of $F$. If $\mathbf{L} = 0$, then $F$ is called a Landsberg metric [15]. All of the tangent spaces of a Landsberg space are isometric to a common Minkowski space.

It is obvious from (1) that every Berwald metric is a Landsberg metric. One of the long standing open problems in Finsler geometry is whether there exists a Landsberg metric which is not a Berwald metric [21]. In 2005, Bao called such spaces unicorns in Finsler geometry. In 2006, Asanov found a special family of unicorns which consists of non-regular $(\alpha, \beta)$-metrics [1]. Shen found a more complicated family of unicorns in the class of non-regular $(\alpha, \beta)$-metrics which contains the Asanov's metrics. More precisely, Shen proved that every almost regular non-Riemannian $(\alpha, \beta)$-metric $F = \alpha \phi (\beta / \alpha)$ on a manifold $M$ of dimension $n \geq 3$ is a Landsberg metric if and only if it is Berwald metric, or the scalar function $\phi : (-r_0, r_0) \to \mathbb{R}$ is given by

$$\phi (t) = c_3 \exp \left[ \int_{0}^{t} \frac{c_1 \sqrt{1 - (u/b_0)^2} + c_2 u}{1 + u (c_1 \sqrt{1 - (u/b_0)^2} + c_2 u)} \, du \right],$$

and $\beta$ satisfies $b = b_0$, $s_{ij} = 0$ and $r_{ij} = k (b^2 a_{ij} - b_i b_j)$, where $c_1$, $c_2$, $c_3$ are constants with $c_1 \neq 0$, $1 + c_2 b_0 > 0$ and $c_3 > 0$, and $k$ is a scalar function on $M$. Moreover, $F$ is not a Berwald metric if and only if $k \neq 0$ (see [17], [25]).

In this paper, we are going to improve Shen's result for 3-dimensional $(\alpha, \beta)$-metrics. Our strategy is as follows. According to Matsumoto's theorem, every $(\alpha, \beta)$-metric is semi-C-reducible. First, we introduce a non-Riemannian quantity $\mathbf{R}$ defined by (32) that characterizes C-reducible Finsler metrics, namely the Randers and Kropina metrics, among the 3-dimensional Finsler metrics. It turns
out that the function $R$ takes its minimum if and only if $F$ is $C$-reducible. Then,
by a completely different approach from that used by Shen, employing the theory
of main scalars and using the fact that the function $R$ is constant along geodesics
in the Landsbergian case, we prove the following.

**Theorem 1.** Every 3-dimensional non-Riemannian almost regular Landsberg
$(\alpha, \beta)$-metric $F = \alpha \phi(\beta/\alpha)$ belongs to one of the following three classes of
Finsler metrics:

(i) $F$ is a Berwald metric. In this case, $F$ is a Randers metric or a Kropina
metric.

(ii) $\phi(t)$ is given by the ODE

$$\phi(t)^{4-4c}(\phi(t) - t\phi'(t))^{4-4c}\left[\phi(t) - t\phi'(t) + (b^2 - t^2)\phi''(t)\right]^{-c} = e^{k_0},$$

where $c$ is a nonzero real constant, $k_0$ is a real number and $b := ||\beta||_\alpha$. In this
case, $F$ is a Berwald metric (regular case) or an almost regular unicorn.

The ODE (3) might be solvable in general, but we have not been able to
prove this yet. Here, we mention two examples of Finsler metrics satisfying (3).

**Example 1.** If $c = 4/3$, we get

$$\phi(t) = c_3 \exp\int_0^t \frac{\sqrt{b^2 - u^2} + c_1(\kappa - 1)u}{c_1b^2 + u[\sqrt{b^2 - u^2} + c_1(\kappa - 1)u]} \, du,$$

where $c_1$ and $c_2$ are real constants and $c_3 > 0$. Suppose that $\beta$ satisfies $r_{ij} \neq 0$.
Then $F$ is a Landsberg metric which is not Berwaldian. In this case, $F$ is a uni-
corn [26]. If $r_{ij} = 0$ and $s_{ij} = 0$, then $F$ reduces to a Berwald metric. If $\kappa = 1$
and $b = 1$, then

$$\phi(t) = c_3 \exp\left[\int_0^t \sqrt{1 - u^2} du\right].$$

The family of unicorns (5) was found by Asanov in [1].

**Example 2.** If $c = 1$, by putting $\psi(t) := \phi(t) - t\phi'(t)$, we get

$$t \psi^3(t) = [t \psi(t) - (b^2 - t^2)\psi'(t)]e^{k_0}.\] Dividing it by $\psi^3(t)$ and putting $v(t) := \psi^{-2}(t)$, we get

$$2t = [2tv(t) + (b^2 - t^2)v'(t)]e^{k_0}$$
in terms of $v$. Its solution is given by

$$v(t) = c(b^2 - t^2) + e^{-k_0}.\] Consequently, we conclude that the solution of (3) is
given by the formula

$$\phi(t) = c_1t + \frac{\sqrt{c_2(b^2 - t^2) + 1}}{c_2b^2 + 1},$$

where $c_1, c_2$ are real constants.
where $c_1$ and $c_2$ are real constants. Then $F$ is a Randers-type metric. It is well-known that every Randers-type metric with vanishing Landsberg curvature is a Berwald metric [7].

**Remark 1.** Throughout the paper, we use the Cartan connection on Finsler manifolds. The $h$- and $v$-covariant derivatives of a Finsler tensor field are denoted by “$|$” and “$\nabla$,” respectively.

## 2. Preliminaries

Let $(M, F)$ be an $n$-dimensional Finsler manifold. The fundamental tensor $g$ of $F$ is defined by

$$g_y(u, v) := \left. \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \left[ F^2(y + su + tv) \right] \right|_{s, t = 0}, \quad u, v \in T_y M.$$  

Let $x \in M$ and $F_x := F|_{T_x M}$. To measure the non-Euclidean feature of $F_x$, define $C_y : T_x M \times T_x M \times T_x M \to \mathbb{R}$ by

$$C_y(u, v, w) := \left. \frac{1}{2} \frac{d}{dt} \left[ g_{y+tw}(u, v) \right] \right|_{t=0}, \quad u, v, w \in T_x M.$$  

Then $C_y$ is a symmetric trilinear form on $T_x M$. The Cartan torsion of $(M, F)$ is the Finsler tensor $C : y \in TM_0 \to C_y \in T^0_3(T\pi(y)M)$. It is well-known that $C = 0$ if and only if $F$ is Riemannian.

For $y \in T_x M_0$, define $I_y : T_x M \to \mathbb{R}$ by

$$I_y(u) = \sum_{i=1}^n g^{ij}(y) C_y(u, \partial_i, \partial_j),$$  

where $\{\partial_i\}$ is a basis for $T_x M$ at $x \in M$. The Finsler 1-form $I : y \in TM_0 \to I_y \in T^*\pi(y)M$ is called the mean Cartan torsion of $(M, F)$. By Diecke’s theorem, a positive-definite Finsler metric $F$ is Riemannian if and only if $I = 0$.

We need the Euclidean norms of the tensors $C$ and $I$, given locally by

$$||C|| := \sqrt{\sum_{p,q,r} g^{ij} g^{kl} g^{qr} C_{ij} C_{lk} C_{pq}}, \quad ||I|| := \sqrt{\sum_{i,j} I_i I_j}.$$  

(7)  

(8)
For a vector \( y \in TM_0 \), define the Matsumoto torsion \( M : y \in TM_0 \to M_y \in T^0_3(T\pi(y)M) \) by

\[
M_y(u, v, w) := C_y(u, v, w) - \frac{1}{n+1} \left\{ I_y(u)h_y(v, w) + I_y(v)h_y(u, w) + I_y(w)h_y(u, v) \right\},
\]

where \( h_y(u, v) := g_y(u, v) - F^{-2}(y)g_y(y, u)g_y(y, v) \) is the angular metric. A Finsler metric \( F \) is said to be \( C \)-reducible if \( M = 0 \). In [12], MATSUMOTO and Hōjō proved the following.

**Lemma 1.** A Finsler metric on a manifold of dimension \( n \geq 3 \) is a Randers metric or a Kropina metric if and only if \( M = 0 \).

A Finsler metric is called semi-\( C \)-reducible if its Cartan tensor is given by

\[
C_y(u, v, w) = \frac{p}{n+1} \left\{ I_y(u)h_y(v, w) + I_y(v)h_y(u, w) + I_y(w)h_y(u, v) \right\} + \frac{q}{\|I_y\|^2} I_y(u)I_y(v)I_y(w),
\]

where \( p \) and \( q \) are scalar functions on \( TM \) such that \( p + q = 1 \). The function \( p \) is called the characteristic scalar of \((M, F)\). M. Matsumoto proved the following:

**Proposition 1** ([11]). Let \( F = \alpha \phi(s) \), \( s = \beta/\alpha \), be a non-Riemannian \((\alpha, \beta)\)-metric on a manifold \( M \) of dimension \( n \geq 3 \). Then \( F \) is semi-\( C \)-reducible.

We recall that the Landsberg curvature of \( F \) can also be defined as follows:

\[
L_y(u, v, w) := \frac{d}{dt} \left[ C_{\sigma(t)} \left( U(t), V(t), W(t) \right) \right]_{t=0},
\]

where \( y \in T_xM \), \( \sigma \) is the geodesic with \( \sigma(0) = x \), \( \dot{\sigma}(0) = y \), and \( U, V, W \) are linearly parallel vector fields along \( \sigma \) with \( U(0) = u \), \( V(0) = v \), \( W(0) = w \). Thus the Landsberg curvature \( L_y \) is the rate of change of \( C_y \) along geodesics for any \( y \in T_xM_0 \) (see [19], [15]).

For \( y \in T_xM \), define \( J_y : T^*_x(y)M \to \mathbb{R} \) by \( J_y(u) := \sum_{i=1}^n g^{ij}(y)L_y(u, \partial_i, \partial_j) \). The Finsler 1-form \( J : y \in TM_0 \to J_y \in T^*_x(y)M \) is called the mean Landsberg curvature of \((M, F)\). We say that \( F \) is a weakly Landsberg metric if \( J = 0 \). It is easy to see that the mean Landsberg curvature of \( F \) can also be given by

\[
J_y(u) := \frac{d}{dt} \left[ I_{\sigma(t)} \left( U(t) \right) \right]_{t=0},
\]

where \( y \in T_xM \), \( \sigma \) is the geodesic with \( \sigma(0) = x \), \( \dot{\sigma}(0) = y \) and \( U \) is a linearly parallel vector field along \( \sigma \) with \( U(0) = u \). Thus the mean Landsberg curvature at \( y \) is the rate of change of \( I_y \) along geodesics for any \( y \in T_xM \).
3. Proof of Theorem 1

In his paper [13], A. Moór introduced a (smooth, local) orthonormal frame field \((\ell, m, n)\) for a three dimensional Finsler manifold. Here

\[ \ell : v \in TM_0 \mapsto \ell(v) := \frac{1}{F(v)}(v, v) \in \pi^* TM_0 = TM_0 \times_M TM \]

is the normalized support element field. In local coordinates, \(\ell = \ell_i \partial / \partial x_i\), where \(\ell_i = y_i / F\). The second member of the frame arises from the mean Cartan torsion. If

\[ I_i := g^{jk} C^i_{jk}, \quad I^i := g_{ij} I_j, \]

then \(m = I^i \partial / \partial x_i\). Finally, the third member \(n = n^i \partial / \partial x_i\) is orthogonal to both \(\ell\) and \(m\). The frame \((\ell, m, n)\) is called a Moór frame for the Finsler manifold. We denote its dual by \((\ell^*, m^*, n^*)\). Then, e.g., \(\ell^* = \ell_i dx^i\), where \(\ell_i = g_{ij} \ell^j = \partial F / \partial y_i\).

**Lemma 2.** Let \((M, F)\) be a 3-dimensional non-Riemannian Finsler manifold. Relative to a Moór frame, the components of the Cartan torsion of \(F\) are

\[ C_{ijk} = \left\{ \lambda_i h_{jk} + \lambda_j h_{ki} + \lambda_k h_{ij} \right\} + \left\{ \mu_i I_j I_k + \mu_j I_k I_i + \mu_k I_i I_j \right\}, \quad (10) \]

where \(h_{ij} := FF^{y_i y_j}\) are the components of the angular tensor field \(h\),

\[ \lambda_i := \frac{1}{3F} \left[ 3I_i m_i + J n_i \right], \quad \mu_i := \frac{F}{3(H + I)^2} \left[ (H - 3I) m_i - 4J n_i \right]; \quad (11) \]

\(H, J\) and \(I\) stand for the main scalars of \((M, F)\).

**Proof.** For 3-dimensional Finsler manifolds, we have

\[ g_{ij} = \ell_i \ell_j + m_i m_j + n_i n_j. \]

Thus

\[ g^{ij} = \ell^i \ell^j + m^i m^j + n^i n^j. \]

If \(C_{ijk}\) are the components of \(C\) relative to a Moór frame, then the Cartan torsion of \(F\) can be written as follows:

\[ FC_{ijk} = \mathcal{H} m_i m_j m_k - J \left\{ m_i m_j n_k + m_j m_k n_i + m_k m_i n_j - n_i n_j n_k \right\} + I \left\{ n_i n_j m_k + n_j n_k m_i + n_k n_i m_j \right\}. \quad (12) \]

Contracting (12) with \(g^{ij}\), we get

\[ FI_k = (H + I) m_k. \quad (13) \]
Multiplying (13) with $g^{mk}$ yields

$$FI^k = (\mathcal{H} + \mathcal{I})m^k.$$  \hspace{1cm} (14)

From (13) and (14) we obtain

$$\mathcal{H} + \mathcal{I} = F||I||,$$  \hspace{1cm} (15)

therefore $\mathcal{H} + \mathcal{I} \neq 0$. 

The components of the angular metric can be expressed as
\[ h_{ij} = m_im_j + n_in_j. \] (16)

By considering (13) and (16), one can rewrite (12) as (10), where \( \lambda_i \) and \( \mu_i \) are given by (11). It is easy to see that \( \lambda_i y^i = 0 \) and \( \mu_i y^i = 0 \). This completes the proof. □

The following result was obtained by Matsumoto.

**Lemma 3** ([6]). Let \((M,F)\) be a 3-dimensional Finsler manifold. Then \(F\) is a \(C\)-reducible metric if and only if the main scalars satisfy the following:
\[ H = 3I, \quad J = 0. \] (17)

In this case, \( I = F\|I\|/4. \)

**Example 3.** Let \( F = \alpha + \beta \) be a Randers metric on a 3-dimensional Finsler manifold \( M \), where \( \alpha = \sqrt{(a_{ij} \circ \pi) y^i y^j} \) is a Riemannian metric and \( \beta = (b_i \circ \pi) y^i \) is a 1-form on \( M \). We have
\[ |I| = 2F \sqrt{\frac{b^2 \alpha^2 - \beta^2}{\alpha F}}, \]
where \( b := |\beta|_\alpha = \sqrt{a^{ij} b_i b_j} \) (see [2]). The main scalars are
\[ I = \frac{1}{2} \sqrt{\frac{b^2 \alpha^2 - \beta^2}{\alpha F}}, \quad J = 0, \quad H = \frac{3}{2} \sqrt{\frac{b^2 \alpha^2 - \beta^2}{\alpha F}}. \]

**Example 4.** Let \( F = \alpha^2/\beta \) be a Kropina metric on a 3-dimensional Finsler manifold \( M \). We have
\[ I_i = \frac{\partial}{\partial y^i} \ln \sqrt{\det(g_{ij})} = 4 \left[ \frac{y_i}{\alpha^2} - \frac{b_i}{\beta} \right]. \] (18)
Thus
\[ g^{ij} I_i I_j = \frac{16 \alpha^2}{2} \left[ \left( \frac{2 \beta^2 - b^2 \alpha^2}{b^2 \alpha^2} \right) y^i - \frac{\beta}{b^2 \alpha^2} b_j \right] \left[ \frac{y_j}{\alpha^2} - \frac{b_j}{\beta} \right]. \]

Then
\[ |I| = \frac{2\sqrt{2}}{bF \alpha} \sqrt{b^2 \alpha^2 - \beta^2}, \]
for more details, see [24]. The main scalars in this case are
\[ I = \frac{\sqrt{2}}{2b\alpha} \sqrt{b^2 \alpha^2 - \beta^2}, \quad J = 0, \quad H = \frac{3\sqrt{2}}{2b\alpha} \sqrt{b^2 \alpha^2 - \beta^2}. \]
Now we recall a characterization of the 3-dimensional Berwald metrics. To do this, we remark that the horizontal covariant derivatives of the members of the dual Moör frame are given by the formulas
\[ \ell_{ij} = 0, \quad m_{ij} = h_j n_i, \quad n_{ij} = -h_j m_i, \]  
where the \( h_i \)s are called the \( h \)-connection vectors. The following result was obtained also by Matsumoto as Theorem 3.7.4.3 in [6]. Here we give a new proof for it.

**Lemma 4** ([6]). Let \((M, F)\) be a 3-dimensional Finsler manifold with nonzero mean Cartan torsion. Then \(F\) is a Berwald metric if and only if it has horizontally constant main scalars with vanishing \( h \)-connection vectors.

**Proof.** Since
\[ h_{ij|s} = g_{ij|s} = 0, \]
by taking a horizontal covariant derivative of (10), we get
\[ C_{ijk|s} = \left\{ \lambda_{ij|s} h_{jk} + \lambda_{jk|s} h_{ki} + \lambda_{ki|s} h_{ij} \right\} + \left\{ \mu_{ij|s} I_j I_k + \mu_{jk|s} I_i I_k + \mu_{ki|s} I_i I_j \right\} + \left\{ \mu_i I_{ij|s} I_k + I_j I_{k|s} + I_i I_{k|s} \right\}. \]  
(20)
Applying (19), we get
\[ \lambda_{ij|s} = \frac{1}{3F} \left[ (3I_{ij|s} - J h_s) m_i + (3I h_s + J f_s) n_i \right], \]  
(21)
\[ \mu_{ij|s} = \frac{-2I m I_{mn|s}}{3F ||I||^2} \left[ (H - 3I) m_i - 4J n_i \right] + \frac{1}{3F ||I||^2} \left[ (H_{ij|s} - 3I_{ij|s} + 4J h_s) m_i \right] + \left[ (H - 3I) h_s - 4J f_s \right] n_i. \]  
(22)
Let \(F\) be a Berwald metric. Then
\[ I_{k|s} = (g^{ij} C_{ijk})|_{s} = g^{ij} C_{ijk}|_{s} = 0, \]
so (20), (21) and (22) reduce to the following:
\[ \left\{ \lambda_{ij|s} h_{jk} + \lambda_{jk|s} h_{ki} + \lambda_{ki|s} h_{ij} \right\} + \left\{ \mu_{ij|s} I_j I_k + \mu_{jk|s} I_i I_k + \mu_{ki|s} I_i I_j \right\} = 0, \]  
(23)
\[ \lambda_{ij|s} = \frac{1}{3F} \left[ (3I_{ij|s} - J h_s) m_i + (3I h_s + J f_s) n_i \right], \]  
(24)
\[ \mu_{ij|s} = \frac{1}{3F ||I||^2} \left[ (H_{ij|s} - 3I_{ij|s} + 4J h_s) m_i + \left[ (H - 3I) h_s - 4J f_s \right] n_i \right]. \]  
(25)
Plugging (24) and (25) into (23), we get
\begin{align*}
3I_s(m_i h_{jk} + m_j h_{ki} + m_k h_{ij}) + J_s(n_i h_{jk} + n_j h_{ki} + n_k h_{ij}) \\
+ 3(H_s - 3I_s) m_i m_j m_k - 4J_s (n_i m_j m_k + n_j m_k m_i + n_k m_i m_j) = 0. \quad (26)
\end{align*}
By contracting (26) with $m^i m^j m^k$ and $n^i m^j m^k$, respectively, and using $h_{ij} m^j = m_i$ and $h_{ij} m^j m^i = 1$, we get $H_s = 0$ and $J_s = 0$, respectively. Thus (26) reduces to
\begin{align*}
I_s (m_i h_{jk} + m_j h_{ki} + m_k h_{ij} - 3m_i m_j m_k) = 0. \quad (27)
\end{align*}
Multiplying (27) with $g^{ij} m^k$ gives $I_s = 0$. By definition, we have $m_k = (1/\|I\|) I_k$. Thus $m_{kij} = 0$, and consequently $h_i = 0$.

Conversely, let $h_i = 0$ and $I_s = J_s = H_s = 0$. In this case, (13) implies that $I_{ijs} = 0$. Then (21) and (22) infer $\lambda_{ijs} = 0$ and $\mu_{ijs} = 0$, respectively. Putting these into (20), we get $C_{ijk|s} = 0$. Thus $F$ is a Berwald metric. $\square$

In [24], TAYEBI and SADEGHI proved the following.

**Theorem 2.** Let $F = \alpha \phi(\beta/\alpha)$ be a non-Riemannian $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$. Then the following holds:
\begin{align*}
\|C\| = \sqrt{\frac{(n-2)p^2 - 2(n-2)p + n + 1}{n + 1} \|I\|}, \quad (28)
\end{align*}
where $p(x, y)$ is given by (9).

It has been proved that $p = P \circ s$, where the scalar function $P : (-r_0, r_0) \rightarrow \mathbb{R}$ is given by
\begin{align*}
P(t) := \frac{n + 1}{a(t) A(t)} \left[ t (\phi(t) \phi''(t) + \phi'(t) \phi'(t)) - \phi(t) \phi'(t) \right], \quad (29)
\end{align*}
\begin{align*}
a(t) := \phi(t) (\phi(t) - t \phi'(t)), \quad (30)
\end{align*}
\begin{align*}
A(t) := (n - 2) \frac{t \phi''(t)}{\phi(t) - t \phi'(t)} - (n + 1) \frac{\phi'(t)(t - \phi'(t))}{\phi(t) \phi(t) - t \phi'(t)} + 3t \phi''(t) + (b^2 - t^2) \phi''(t), \quad (31)
\end{align*}
and $s : TM_0 \rightarrow \mathbb{R}$ is given by $s(v) := \beta(v)/\alpha(v)$ (see [14]).

In [9], MATSUMOTO found a necessary and sufficient condition under which a 3-dimensional Finsler manifold is semi-$C$-reducible.

**Lemma 5 ([9]).** Let $(M, F)$ be a 3-dimensional Finsler manifold. Then $F$ is semi-$C$-reducible if and only if $J = 0$. 
In [16], Shen proved that the Cartan torsion of a Randers metric satisfies \( ||C|| \leq 3/\sqrt{2} \). In [24], Tayebi and Sadeghi completed Shen’s theorem and proved that the Cartan torsion of a Kropina metric satisfies \( ||C|| \leq 3/\sqrt{2} \). It turns out that every \( C \)-reducible Finsler metric has bounded Cartan torsion.

Let \((M,F)\) be a 3-dimensional Finsler manifold. Define

\[
\mathcal{R} := \frac{4}{||I||^2F^2} \left[ 3I^2 + \mathcal{H}^2 \right],
\]

where \( \mathcal{H} \) and \( I \) are the main scalars of \( F \).

**Proposition 2.** Let \( F = \alpha \phi(\beta/\alpha) \) be a non-Riemannian \((\alpha,\beta)\)-metric on a 3-dimensional manifold \( M \). Then \( F \) is \( C \)-reducible if and only if \( \mathcal{R} = 3 \).

**Proof.** By (10) we get

\[
C_{ijk} = \left\{ \lambda^i h^{jk} + \lambda^j h^{ki} + \lambda^k h^{ij} \right\} + \left\{ \mu^i I^j I^k + I^i \mu^j I^k + I^i I^j \mu^k \right\}.
\]

Thus

\[
||C||^2 := C_{ijk} C^{ijk} = \left\{ \lambda^i h^{jk} + \lambda^j h^{ki} + \lambda^k h^{ij} \right\} \times \left\{ \lambda^i h^{jk} + \lambda^j h^{ki} + \lambda^k h^{ij} + \mu^i I^j I^k + I^i \mu^j I^k + I^i I^j \mu^k \right\}
\]

\[
= 12\lambda^2 + 6\mu \lambda^m ||I||^2 + 12\lambda \mu \lambda^m I^m + 3\mu^2 ||I||^4 + 6(\mu \lambda^m I^m)^2||I||^2.
\]

(34)

where \( \lambda^2 := \lambda_i \lambda^i \) and \( \mu^2 := \mu_i \mu^i \). By Lemma 5, it follows that every \((\alpha,\beta)\)-metric, on a 3-dimensional manifold \( M \) satisfies \( J = 0 \). Taking into account (11), we get

\[
\lambda^2 = \frac{1}{F^2} I^2, \quad \mu \lambda^i = \frac{1}{3F^2 ||I||^2} T(\mathcal{H} - 3I), \quad I^i \lambda^i = \frac{1}{F^2} T(\mathcal{H} + I),
\]

\[
\mu I^i = \frac{1}{3F^2 ||I||^2} (\mathcal{H} - 3I), \quad \mu^2 = \frac{1}{9F^2 ||I||^2} (\mathcal{H} - 3I)^2.
\]

So we obtain \( ||C||^2 = \frac{1}{F^2} \left( 3I^2 + \mathcal{H}^2 \right) \), whence

\[
||C|| = \frac{1}{F} \sqrt{3I^2 + \mathcal{H}^2}.
\]

(35)

Observe that replacing \( p = 4I/(\mathcal{H} + I) \) and \( q = (I - 3I)/(\mathcal{H} + I) \) into (28), we get (35).
Now we are going to prove the second part of the proposition. By (28) and (35) we get
\[ R = p^2 - 2p + 4. \] (36)

Let \( R = 3 \). Then \( p = 1 \), and \( F \) is \( C \)-reducible.

Conversely, let \( F \) be a \( C \)-reducible metric. Then we have
\[
\frac{1}{1 + n} \left\{ h_{ij} I_k + h_{jk} I_i + h_{ki} I_j \right\} = \frac{p}{1 + n} \left\{ h_{ij} I_k + h_{jk} I_i + h_{ki} I_j \right\} + \frac{1 - p}{\| I \|^2} I_j I_j. \] (37)

Contracting (37) with \( I^I J^j \) yields
\[(p - 1)\| I \|^2 I_k = 0.\]

Since \( F \) is non-Riemannian, \( p = 1 \). Putting it into (36), we find that \( R = 3 \). This completes the proof. \( \square \)

**Remark 2.** By (36), we get \( R \geq 3 \), so it follows from Proposition 2 that \( R \) takes its minimum if and only if \( F \) is \( C \)-reducible.

Given an \((\alpha, \beta)\)-metric \( F := \alpha \phi (\beta / \alpha) \), where as above, \( \alpha = \sqrt{(a_{ij} \circ \pi) y^i y^j} \) is a Riemannian metric and \( \beta = (b_i \circ \pi) y^i \), we define \( b_{ij} \) by
\[ b_{ij} \omega^j := db_i - b_j \omega^j, \]
where \( \omega^i := dx^i \) and \( \omega^j := \gamma^j_{ik} dx^k \) denote the Levi-Civita connection forms of \( \alpha \) (see [22] and [27]).

In the next calculation, we need the chain rule
\[ G \cdot (f \circ H) = (f' \circ H) G \cdot H, \quad f \in C^\infty(\mathbb{R}), \quad H \in C^\infty(TM_0). \] (38)

Put
\[
\begin{align*}
r_{ij} := & \frac{1}{2} (b_{ij} + b_{ji}), \\
s_{ij} := & \frac{1}{2} (b_{lij} - b_{lij}), \\
s_j := & b^i s_{ij}, \\
s_0 := & s_j y^j, \\
r_{00} := & r_{ij} y^i y^j,
\end{align*}
\]
and taking into account that \( p = \mathcal{P} \circ s \), for the horizontal derivation of characteristic scalar of \((M, F)\) along geodesics, Najafi and Tayebi obtained
\[
\nabla_0 p := G_p \overset{(38)}{=} \mathcal{P}' \circ s s_{ij} y^j
\]
\[=
\frac{1}{\alpha^2} \mathcal{P}' \circ s \left[ 1 - 2(b^2 - t^2) \Psi(t) \right] \circ s (r_{00} - 2 \alpha s_0 \mathcal{Q} \circ s), \] (39)
where
\[ Q(t) := \frac{\phi'(t)}{\phi(t) - t\phi'(t)}, \quad \Psi(t) := \frac{\phi''(t)}{2[(\phi(t) - t\phi'(t)) + (b^2 - t^2)\phi''(t)]}. \]

For details, see [14], where the authors gave a characterization of the constancy of \( p \) along any Finslerian geodesics in the case of non-Riemannian regular \((\alpha, \beta)\)-metrics. Their proof is based on Lemmas 3.2 and 3.6 of [14], which are valid also for almost regular \((\alpha, \beta)\)-metrics. Therefore, we have the following.

**Lemma 6.** Let \( F \) be an almost regular non-Riemannian \((\alpha, \beta)\)-metric on a manifold \( M \) of dimension \( n \geq 3 \). Then \( p \) is a first integral of \( G \), i.e., \( p \circ \gamma : I \rightarrow \mathbb{R} \) is a constant function for any geodesic \( \gamma : I \rightarrow M \) of \((M,F)\) if and only if one of the following holds:

(i) \( \beta \) satisfies
\[ r_{ij} = 0, \quad s_i = 0; \quad (40) \]

(ii) \( \phi \) satisfies the ODE
\[ (n + 1) \left[ t(\phi(t)\phi''(t) + \phi'(t)) - \phi(t)\phi'(t) \right] = ca(t)A(t), \quad (41) \]

where \( c \) is a real constant.

In order to prove Theorem 1, we find a necessary and sufficient condition under which a 3-dimensional Finsler metric becomes a Landsberg metric.

**Lemma 7.** \( \nabla_{0j} = 0 \) if and only if
\[ \left[ (\nabla_0 H - 3\nabla_0 I) + 4J h_0 \right] ||I||^2 = 2I_m J^m \left[ H - 3I \right], \quad (42) \]

and
\[ \left[ (H - 3I) h_0 - 4\nabla_0 J \right] ||I||^2 = -8I_m J^m J. \quad (43) \]

**Proof.**
\[ \nabla_{0j} = \frac{1}{3F||I||^2} \left\{ (\nabla_0 H - 3\nabla_0 I) m_i + (H - 3I) h_0 n_i 
- 4(\nabla_0 J n_i - J h_0 m_i) ||I||^2 - 2I_m J^m \left[ (H - 3I) m_i - 4J n_i \right] \right\}, \quad (44) \]

therefore \( \nabla_{0j} = 0 \) if and only if
\[ 2I_m J^m \left[ H - 3I \right] - (\nabla_0 H - 3\nabla_0 I + 4J h_0) ||I||^2 \] \[ = \left[ (H h_0 - 3I h_0 - 4\nabla_0 J)||I||^2 + 8I_m J^m J \right] n_i. \quad (45) \]

Multiplying (45) with \( m^i \) and \( n^i \) yields (42) and (43), respectively. \qed
Lemma 8. If \((M,F)\) is a 3-dimensional Finsler manifold, then the components of the Landsberg curvature of \(F\) relative to a Mo"{o}b frame are

\[
L_{ijk} = \frac{1}{4} \left\{ J_i h_{jk} + J_j h_{ki} + J_k h_{ij} \right\} - \frac{1}{2} \left\{ J^m \mu_m + \nabla_0 \mu_m I^m \right\} \left\{ I_i h_{jk} + I_j h_{ki} + I_k h_{ij} \right\} \\
- \frac{1}{4} \left\{ I_m J^m + J_m I^m \right\} \left\{ \mu_i h_{jk} + \mu_j h_{ki} + \mu_k h_{ij} \right\} \\
+ \left\{ \nabla_0 \mu_i I_j I_k + \nabla_0 \mu_j I_i I_k + \nabla_0 \mu_k I_i I_j \right\} - \frac{1}{2} \left\{ J_i h_{jk} + J_j h_{ki} + J_k h_{ij} \right\} \mu_m I^m \\
- \frac{1}{4} \left\{ \nabla_0 \mu_i h_{jk} + \nabla_0 \mu_j h_{ki} + \nabla_0 \mu_k h_{ij} \right\} ||I||^2 \\
+ \left\{ \mu_i (J_i I_k + I_i J_k) + \mu_j (J_j I_k + I_j J_k) + \mu_k (J_j I_j + I_j J_j) \right\}.
\]

(46)

Proof. Multiplying (10) with \(g^{ij}\) implies that

\[
a_i = \frac{1}{4} \left( 1 - 2I^m b_m \right) I_i - ||I||^2 b_i.
\]

(47)

Putting (47) into (10) yields

\[
C_{ijk} = \frac{1}{4} \left\{ I_i h_{jk} + I_j h_{ki} + I_k h_{ij} \right\} - \frac{b_m I^m}{2} \left\{ I_i h_{jk} + I_j h_{ki} + I_k h_{ij} \right\} \\
- \frac{||I||^2}{4} \left\{ b_i h_{jk} + b_j h_{ki} + b_k h_{ij} \right\} + \left\{ b_i I_j I_k + I_j b_j I_k + I_k b_i I_j \right\}
\]

(48)

Relation (48) can be written as

\[
M_{ijk} = -\frac{\mu_m I^m}{2} \left\{ I_i h_{jk} + I_j h_{ki} + I_k h_{ij} \right\} - \frac{||I||^2}{4} \left\{ \mu_i h_{jk} + \mu_j h_{ki} + \mu_k h_{ij} \right\} \\
+ \left\{ \mu_i I_j I_k + I_i \mu_j I_k + I_j \mu_k I_i \right\}
\]

(49)

By taking a horizontal derivation of (49), we get

\[
M_{ijk} g^s = -\frac{1}{2} \left\{ \mu_m J^m + \nabla_0 \mu_m I^m \right\} \left\{ I_i h_{jk} + I_j h_{ki} + I_k h_{ij} \right\} \\
- \frac{\mu_m I^m}{2} \left\{ J_i h_{jk} + J_j h_{ki} + J_k h_{ij} \right\} \\
- \frac{||I||^2}{4} \left\{ \nabla_0 \mu_i h_{jk} + \nabla_0 \mu_j h_{ki} + \nabla_0 \mu_k h_{ij} \right\} \\
- \frac{1}{4} \left\{ I_m J^m + J_m I^m \right\} \left\{ \mu_i h_{jk} + \mu_j h_{ki} + \mu_k h_{ij} \right\} \\
+ \left\{ \mu_i (J_i I_k + I_i J_k) + \mu_j (J_j I_k + I_j J_k) + \mu_k (J_j I_j + I_j J_j) \right\} \\
+ \left\{ \nabla_0 \mu_i I_j I_k + \nabla_0 \mu_j I_i I_k + \nabla_0 \mu_k I_i I_j \right\}
\]

(50)
On the other hand, we have
\[ M_{ijk} | s y s = L_{ijk} - \frac{1}{4} \left\{ J_i h_{jk} + J_j h_{ki} + J_k h_{ij} \right\}. \] (51)

By (50) and (51) we get the proof. \( \square \)

**Lemma 9.** Every 3-dimensional non-Riemannian Landsberg metric satisfies \( \nabla_0 \mu_i = 0 \).

**Proof.** Indeed, in this case (46) reduces to
\[ 2\nabla_0 \mu_m I^m \left\{ I_i h_{jk} + I_j h_{ki} + I_k h_{ij} \right\} - 4 \left\{ \nabla_0 \mu_i I_j I_k + \nabla_0 \mu_j I_i I_k + \nabla_0 \mu_k I_i I_j \right\} + ||I||^2 \left\{ \nabla_0 \mu_i h_{jk} + \nabla_0 \mu_j h_{ki} + \nabla_0 \mu_k h_{ij} \right\} = 0. \] (52)

Contracting (52) with \( I_i I_j \) implies that \( \nabla_0 \mu_k ||I||^4 = 0. \) Since \( ||I|| \neq 0 \), we get \( \nabla_0 \mu_k = 0. \) \( \square \)

In his paper [8], MATSUMOTO characterized the 3-dimensional Landsberg metrics as follows.

**Lemma 10.** Let \((M, F)\) be a 3-dimensional Finsler manifold. Then \(F\) is a Landsberg metric if and only if the main scalars are first integrals of \( G \) and \( h_0 := h_i y^i = 0 \).

**Proof of Theorem 1.** By assumption, \( F \) is a Landsberg metric, so by Lemma 10, we get \( \nabla_0 R = 0 \). On the other hand, by taking a horizontal derivation of (36) along geodesics, we get
\[ \nabla_0 R = 2(p - 1) \nabla_0 p. \] (53)

Thus we have the following two main cases:

**Case (i).** If \( p = 1 \), then \( F \) is \( C \)-reducible. In this case, by Lemma 1, \( F \) reduces to a Randers metric or Kropina metric. In [7], MATSUMOTO proved that a Randers metric \( F = \alpha + \beta \) has vanishing Landsberg curvature if and only if \( \beta \) is parallel with respect to \( \alpha \). Also, MATSUMOTO proved in [10] that every Kropina metric with vanishing Landsberg curvature is a Berwald metric on a manifold of dimension \( n \geq 3 \). Thus every \( C \)-reducible metric with vanishing Landsberg curvature is a Berwald metric on a manifold of dimension \( n \geq 3 \).

**Case (ii).** If \( \nabla_0 p = 0 \), then by Lemma 6, \( \beta \) satisfies (40) or (41). Thus, we have the following two subcases:
Case (iia). If (40) holds, then by [3, Theorem 1.2] we have $S = 0$, which implies that $E = 0$. In [4], it is proved that every Landsberg metric with vanishing mean Berwald curvature is a Berwald metric, as wanted.

Case (iib). Suppose that (41) holds. Then, we get

$$4 \left[ t(\phi(t)\phi''(t) + \phi'(t)\phi'(t)) - \phi(t)\phi'(t) \right] = ca(t)A(t).$$

(54)

Thus

$$4 \left[ t(\phi(t)\phi''(t) + \phi'(t)\phi'(t)) - \phi(t)\phi'(t) \right]$$

$$= c\phi(t) \left( \phi(t) - t\phi'(t) \right) \left[ \frac{t\phi''(t)}{\phi(t) - t\phi'(t)} - 4 \frac{\phi'(t)}{\phi(t) - t\phi'(t)} + 3\frac{t\phi''(t) - (b^2 - t^2)\phi'''(t)}{\phi(t) - t\phi'(t) + (b^2 - t^2)\phi''(t)} \right],$$

(55)

or equivalently,

$$4 \frac{t(\phi(t)\phi''(t) + \phi'(t)\phi'(t)) - \phi(t)\phi'(t)}{\phi(t)(\phi(t) - t\phi'(t))}$$

$$= c \left[ \frac{t\phi''(t)}{\phi(t) - t\phi'(t)} - 4 \frac{\phi'(t)}{\phi(t) - t\phi'(t)} + 3\frac{t\phi''(t) - (b^2 - t^2)\phi'''(t)}{\phi(t) - t\phi'(t) + (b^2 - t^2)\phi''(t)} \right].$$

(56)

If $c = 0$, we get

$$t(\phi(t)\phi''(t) + \phi'(t)\phi'(t)) - \phi(t)\phi'(t) = 0.$$  

(57)

By solving (57), we find $\phi(t) = \sqrt{c_1t^2 + c_2}$, where $c_1$ and $c_2$ are two real constants. In this case, $F$ is a Riemannian metric which contradicts with our assumptions. Thus $c$ is a nonzero constant.

Now we rewrite (56) as follows:

$$4 \frac{d}{dt} \ln \left( \phi(t)(\phi(t) - t\phi'(t)) \right)$$

$$= c \frac{d}{dt} \left[ \ln(\phi(t) - t\phi'(t)) + 4\ln(\phi(t)) + \ln(\phi(t) - t\phi'(t) + (b^2 - t^2)\phi''(t)) \right].$$

(58)

Then (56) implies that for some real number $k_0$ we have (3). This completes the proof. □

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