Sincov’s inequalities on topological spaces

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Abstract. Assume that $X$ is a non-empty set, and $T$ and $S$ are real or complex mappings defined on the product $X \times X$. Additive and multiplicative Sincov’s equations are:

$T(x, z) = T(x, y) + T(y, z), \quad x, y, z \in X$

and

$S(x, z) = S(x, y) \cdot S(y, z), \quad x, y, z \in X,$

respectively. In the present paper, we study three related inequalities. We begin with functional inequality

$G(x, z) \leq G(x, y) \cdot G(y, z), \quad x, y, z \in X,$

and assume that $X$ is a topological space and $G : X \times X \to \mathbb{R}$ is a continuous mapping. In some our statements a considerably weaker regularity than continuity of $G$ is needed. Next, we study the reverse inequality:

$F(x, z) \geq F(x, y) \cdot F(y, z), \quad x, y, z \in X,$

as well as the additive inequality:

$H(x, z) \leq H(x, y) + H(y, z), \quad x, y, z \in X.$

A corollary for generalized metric is derived.

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1. Introduction

Throughout the paper it is assumed that \( \mathbb{R} \) denotes the set of real numbers, \( \mathbb{Q} \) is the set of rationals, and \( \mathbb{N} \) stands for the set of positive integers. Moreover, for \( c, d \) from \( \mathbb{R} \) or from \( \mathbb{R} \cup \{-\infty, +\infty\} \) such that \( c < d \), open, closed and half-open intervals with endpoints \( c \) and \( d \) are denoted by \((c, d), [c, d], [c, d)\) and \((c, d]\), respectively. For an arbitrary set \( A \), by \( \chi_A \) we denote its characteristic function.

Assume that \( X \) is a non-void set and \( S: X \times X \to \mathbb{R} \) is an arbitrary mapping. By multiplicative Sincov’s equation we mean

\[ S(x, z) = S(x, y) \cdot S(y, z), \quad x, y, z \in X. \quad (1) \]

The general solution of (1) is the following: either \( S = 0 \) on \( X \times X \) or there exists a function \( f: X \to \mathbb{R} \setminus \{0\} \) such that

\[ S(a, b) = \frac{f(a)}{f(b)}, \quad a, b \in X \quad (2) \]

(see D. Gronau [4, Theorem]). Equation (1) is of significant importance and its history goes back to the 19th century; for more information we refer the reader to works by D. Gronau [4], [5]. A connection of Hyers–Ulam stability of equation (1) with some generalizations of the Cauchy–Schwarz inequality was recently observed in our paper [2].

Remark 1.1. Directly from the representation (2) of solutions of Sincov’s equation one can easily observe that if two given solutions of (1) defined on the same set \( X \) are comparable, then they are equal. In particular, it makes no sense to search for maximal or minimal solutions of Sincov’s equation. This observation is important in the light of our subsequent results (see Corollaries 2.2, 3.1, 4.1 below), in which we provide representations of solutions of inequalities under study as a pointwise supremum or infimum of a certain family of functions of the form (2).

2. Multiplicative Sincov’s inequality

In this section, we study the following functional inequality:

\[ G(x, z) \leq G(x, y) \cdot G(y, z), \quad x, y, z \in X, \quad (3) \]

which will be called multiplicative Sincov’s inequality. In our main results we assume that \( X \) is a topological space and \( G: X \times X \to \mathbb{R} \) is continuous, or it satisfies
Sincov’s inequalities

a weaker regularity condition. We prove that either $G$ is in a sense trivial solution, or there is a map which lies below $G$ on $X \times X$, is equal to $G$ at a given point $(x_0, y_0) \in X \times X$ and solves Sincov’s equation (1). From this, we derive a representation of solutions of (3) as supremum of mappings of the form (2).

We begin with sorting out two classes of solutions, namely non-positive mappings and mappings whose image is contained in a compact interval of positive real numbers.

Example 2.1. For an arbitrary non-void set $X$, every map $G : X \times X \to (-\infty, 0]$ and every map $G : X \times X \to [c, c^2]$ with some $c \geq 1$ yield solutions of (3).

Proposition 2.1. Assume that $X$ is a connected topological space and $G : X \times X \to \mathbb{R}$ is a continuous solution of (3). If $G$ attains a non-positive value, then $G$ is non-positive on $X \times X$.

Proof. By assumption, there exists a point $(a_0, x_0) \in X \times X$ such that $G(a_0, x_0) \leq 0$. Suppose, for the contrary, that for some $(a_1, x_1) \in X \times X$, we have $G(a_1, x_1) > 0$. We can find another point $(a_2, x_2) \in X \times X$ such that $G(a_2, x_2) = 0$ (consider the sign of $G(a_0, x_1)$ and apply the continuity of one of the mappings $G(a_0, \cdot)$ or $G(\cdot, x_1)$). From (3) we derive

$$G(a_1, x_2) \leq G(a_1, a_2)G(a_2, x_2) = 0.$$ 

Next, by continuity used once more, we obtain the existence of some $x_3 \in X$ such that $G(a_1, x_3) = 0$. Consequently,

$$0 < G(a_1, x_1) \leq G(a_1, x_3)G(x_3, x_1) = 0;$$

a contradiction. □

Let us denote by (Z) the following property of a function $f : X \to \mathbb{R}$ defined on a non-void set:

(Z) if there exist $x, y \in X$ such that $f(x) \leq 0 \leq f(y)$, then there exists $z \in X$ such that $f(z) = 0$.

Clearly, every real-valued continuous mapping acting on a connected topological space has property (Z).

In Proposition 2.1 it is enough to assume that each section of $G$ has property (Z); in particular, no topology on $X$ is needed and the proof remains unchanged. Below we provide some examples illustrating the situation.
Example 2.2. Let \( A : \mathbb{R} \to \mathbb{R} \) be a discontinuous additive function with connected graph. Such functions do exist, see, e.g., L. Székelyhidi [9], and necessarily have the Darboux (intermediate value) property. Take \( X = \mathbb{R} \) and define

\[
G_1(a, b) = \exp(A(a) - A(b)), \quad a, b \in \mathbb{R}.
\]

Then \( G_1 \) is a discontinuous solution of (3) (in fact, it is a solution of (1)) with all sections having the Darboux property.

Let us modify the above mapping a bit. Let \( X = \{(x, A(x)) : x \in \mathbb{R}\} \) and

\[
G_2((a, A(a)), (b, A(b))) = \exp(A(a) - A(b)), \quad a, b \in \mathbb{R}.
\]

Note that this time \( G_2 \) is continuous and \( X \) is a connected space.

Finally, let \( X \) be a disconnected topological space. Define \( G_3 : X \times X \to \mathbb{R} \) as follows: \( G_3(a, b) = 1 \) whenever \( a, b \) lies in the same connected component of \( X \), and \( G_3(a, b) = -1 \) elsewhere. It is easy to check that \( G_3 \) is a continuous solution of (3). In particular, the assumption that \( X \) is connected cannot be dropped in the statement of Proposition 2.1.

Next, we make an easy observation that a special case of Proposition 2.1 with \( G \) attaining non-negative values only remains valid without any topological assumptions.

**Proposition 2.2.** Assume that \( X \) is a non-void set, and \( G : X \times X \to [0, +\infty) \) is a solution of (3). If \( G \) has a zero, then \( G = 0 \) on \( X \times X \).

**Proof.** Let \((a_0, x_0) \in X \times X\) be such that \( G(a_0, x_0) = 0\). Then for arbitrary \( a, b \in X \), we have

\[
0 \leq G(a, b) \leq G(a, a_0)G(a_0, b) \leq G(a, a_0)G(a_0, x_0)G(x_0, b) = 0.
\]

In view of Propositions 2.1 and 2.2, from now on we focus on positive solutions of (3). First, we show that in the case of positive and bounded solutions there is an estimate from below by a positive number.

**Proposition 2.3.** Assume that \( X \) is a non-void set, and \( G : X \times X \to (0, +\infty) \) is a bounded solution of (3). Then there exists a constant \( c \geq 1 \) such that \( G(X \times X) \subseteq [1/c, c] \).

**Proof.** Put \( c = \sup\{G(a, b) : a, b \in X\} \in \mathbb{R} \). Next, observe that directly from (3) we get

\[
G(a, a) \leq G(a, a)G(a, a), \quad a \in X.
\]
Therefore, since $G$ is positive, $G(a,a) \geq 1$ for every $a \in X$. Utilizing this and using (3), we obtain

$$1 \leq G(a,a) \leq G(a,b)G(b,a) \leq c \cdot G(a,b), \quad a,b \in X.$$ 

Thus, $\inf\{G(a,b) : a,b \in X\} \geq 1/c$. \hfill $\Box$

**Proposition 2.4.** Assume that $X$ is a non-void set, and $G : X \times X \rightarrow (0, +\infty)$ is a solution of (3). Then the following estimate holds true:

$$1 \leq G(x,y) \leq G(y,a)G(a,x) \leq c \cdot G(y,x), \quad a,x,y \in X. \quad (4)$$

**Proof.** Estimate (4) follows immediately from (3) applied twice. \hfill $\Box$

Let us associate with $G : X \times X \rightarrow \mathbb{R}$ a map $G^* : X \times X \rightarrow \mathbb{R}$ given by

$$G^*(x,y) = G(y,x), \quad x,y \in X. \quad (5)$$

It is clear that $G$ solves (3) if and only if $G^*$ solves (3).

**Remark 2.1.** An analogue of Proposition 2.4 with the roles of variables reversed is true, as well. More precisely, we have

$$1 \leq G^*(y,x) \leq G^*(y,a)G(a,x) \leq c \cdot G^*(y,x), \quad a,x,y \in X. \quad (6)$$

It is enough to consider map $G^*$ defined by (5) and apply Proposition 2.4. On the other hand, an easy example shows that it is possible that a solution $G$ of (3) has every left section bounded and at the same time every right section unbounded, or conversely. Indeed, consider $X = (1, +\infty)$ and take $G(x,y) = xy^{-1}$ for $x,y \in (1, +\infty)$ (see [2, Example 2]).

Let us denote by $\Delta$ the diagonal of the product $X \times X$, i.e.

$$\Delta = \{(x,x) : x \in X\}.$$

**Lemma 2.1.** Assume that $X$ is a non-void set, and $G : X \times X \rightarrow (0, +\infty)$ is a solution of (3). Then there exists a map $\tilde{G} : X \times X \rightarrow (0, +\infty)$ which enjoys the following properties:

(i) $1/G^* \leq \tilde{G} \leq G$ on $X \times X$;
(ii) $\tilde{G} = 1$ on $\Delta$;
(iii) if $G = 1$ on $\Delta$, then $G = \tilde{G}$;
(iv) $\tilde{G}$ solves inequality (3);

(v) if $X$ is a topological space and $G$ is continuous, then $\tilde{G}$ is lower semi-continuous;

(vi) if $X$ is a topological space and the family $\{G(x, \cdot) : x \in X\}$ is pointwise equi-continuous, then $\tilde{G}$ is continuous at every point of $\Delta$.

**Proof.** Define $\tilde{G} : X \times X \to (0, +\infty)$ by

$$\tilde{G}(x,y) = \sup \left\{ \frac{G(a,y)}{G(a,x)} : a \in X \right\}, \quad x, y \in X.$$ 

Due to estimate (4), the definition is correct and property (i) is fulfilled.

Part (ii) is obvious.

To prove (iii) note that if $G = 1$ on $\Delta$, then by (4)

$$\tilde{G}(x,y) \geq \frac{G(x,y)}{G(x,x)} = \frac{G(x,y)}{G(x,x)}, \quad x, y \in X.$$ 

The converse inequality is a consequence of (i).

To justify (iv), fix arbitrary $x, y, z, a \in X$. Directly from the definition of $\tilde{G}$ one has

$$\tilde{G}(x,y)\tilde{G}(y,z) \geq \frac{G(a,y)}{G(a,x)} \frac{G(a,z)}{G(a,y)} = \frac{G(a,z)}{G(a,x)},$$

and (iv) follows by passing to the supremum with $a$ on the right-hand side.

Point (v) is clear by the definition of $\tilde{G}$.

To prove (vi), fix arbitrary $y \in X$ and $\varepsilon > 0$. Let $U \subset X$ be a neighbourhood of $y$ such that for every $y_1, y_2 \in U$ and for all $x \in X$, one has

$$\left| \frac{G(x,y_1)}{G(x,y_2)} - 1 \right| < \frac{\varepsilon}{2}.$$ 

For fixed $y_1, y_2 \in U$ there exists some $x_0 \in X$ such that

$$\frac{G(x_0,y_1)}{G(x_0,y_2)} > \sup \left\{ \frac{G(x,y_1)}{G(x,y_2)} : x \in X \right\} - \frac{\varepsilon}{2} = \tilde{G}(y_2, y_1) - \frac{\varepsilon}{2}.$$ 

Apply both estimates to get

$$|\tilde{G}(y_2, y_1) - \tilde{G}(y, y)| = |\tilde{G}(y_2, y_1) - 1|$$

$$\leq \left| \tilde{G}(y_2, y_1) - \frac{G(x_0,y_1)}{G(x_0,y_2)} \right| + \left| \frac{G(x_0,y_1)}{G(x_0,y_2)} - 1 \right| < \varepsilon.$$  

$\square$
Lemma 2.2. Assume that $X$ is a countable set, $(a_n) \subset X$ is an arbitrary sequence, and $G : X \times X \to (0, +\infty)$ is a solution of (3). Then there exists a sequence $(\alpha_n) \subset X$ such that $(\alpha_n)$ is a subsequence of $(a_n)$, the following limit exists:
\[
S(b, a) = \lim_{n \to \infty} \frac{G(\alpha_n, a)}{G(\alpha_n, b)}
\]
for every $a, b \in X$ and map $S : X \times X \to (0, +\infty)$ defined by (7) solves (1).

Proof. Let $\{q_k : k \in \mathbb{N}\}$ be an arrangement of $X \times X$ into a sequence. We will construct an auxiliary family of sequences associated to each $q_k$. Put $a_0 := a_n$ for $n \in \mathbb{N}$. Next, fix a $k \in \mathbb{N}$, assume that sequence $(a_{n-1})$ is already defined and denote $(a_k, b_k) = q_k$. From Proposition 2.4 we know that the values $G(a_{n-1}, a_k)/G(a_{n-1}, b_k)$ for all $n \in \mathbb{N}$ lie in a compact interval. Consequently, there exists a sub-sequence $(a_{kn})$ of $(a_{n-1})$ such that the sequence $(G(a_{kn}, a_k)/G(a_{kn}, b_k))$ is convergent.

We have constructed inductively a countable family of sequences $\{(a_k^n)_k : k \in \mathbb{N}\}$ with the property that every sequence $(a_{k+1})$ is a sub-sequence of $(a_k^n)$ and the sequences $(G(a_k^n, a_k)/G(a_k^n, b_k))$ are convergent, where $(a_k, b_k) = q_k$. Now, define the sequence $(\alpha_n)$ by $\alpha_n = a_k^n$ for $n \in \mathbb{N}$. To see that formula (7) holds true for every $a, b \in X$, consider $q_k = (a, b)$ and note that the sequence $(\alpha_n)$ is from a certain moment a subsequence of $(a_k^n)$. Having (7) proved, it is straightforward to check that (1) holds true. □

Corollary 2.1. Assume that $X$ is a countable set, $G : X \times X \to (0, +\infty)$ is a solution of (3), and $(x_0, y_0) \in X \times X$ is an arbitrary point. Then there exists a function $S : X \times X \to (0, +\infty)$ such that $S$ is a solution of (1), $S(x_0, y_0) = \tilde{G}(x_0, y_0)$, where $\tilde{G}$ is postulated by Lemma 2.1, and
\[
1 \leq \frac{G}{G^*} \leq S \quad \text{on } X \times X.
\]

Proof. Take as $(\alpha_n)$ a sequence such that
\[
\lim_{n \to \infty} \frac{G(a_n, y_0)}{G(a_n, x_0)} = \sup \left\{ \frac{G(a, y_0)}{G(a, x_0)} : a \in X \right\}.
\]
To finish the proof, it is enough to observe that the assertion follows from Lemma 2.2 and Proposition 2.4. □

With the aid of results of this section, now we are able to prove our first main result.
Theorem 2.1. Assume that $X$ is a separable topological space, $G : X \times X \to (0, +\infty)$ is a solution of (3) such that $G$ is continuous and equal to 1 at every point of $\Delta$, and $(x_0, y_0) \in X \times X$ is arbitrarily fixed. Then there exists a function $S : X \times X \to (0, +\infty)$ such that $S$ is a solution of (1), $S(x_0, y_0) = G(x_0, y_0)$, and estimate (8) is satisfied. Moreover, $S$ is given by formula (7) on $X \times X$ with some sequence $(\alpha_n) \subset X$.

Proof. Equality $G = \tilde{G}$ follows from part (iii) of Lemma 2.1. Let $X_0$ be a countable dense subset of $X$ such that $x_0, y_0 \in X_0$. Corollary 2.1 applied for $X_0$ and $G$ give us a sequence $(\alpha_n) \subset X_0$ and a map $S_0$ defined on $X_0 \times X_0$ which satisfies

\[ S_0(b, a) = \lim_{n \to \infty} \frac{G(\alpha_n, a)}{G(\alpha_n, b)} \] (9)

and $S_0(x_0, y_0) = G(x_0, y_0)$. In the next step, we will justify that the definition of mapping $S : X \times X \to (0, +\infty)$ via formula (7) is correct for all $(b, a) \in X \times X$ (i.e., the limit always exists). Fix some $a, b \in X$ and take $a', b' \in X_0$ sufficiently close to $a$ and $b$. From (4) we obtain

\[ \frac{1}{G(a', a)} \leq \frac{G(x, a')}{G(x, a)} \leq G(a, a'), \quad x \in X. \]

This means that the middle term is as close to 1 as desired, since $G$ is continuous and equal to 1 at $(a, a)$. An analogous estimate holds for $b$ and $b'$. On the other hand, we have

\[ \frac{G(\alpha_n, a)}{G(\alpha_n, b)} = \frac{G(\alpha_n, a)}{G(\alpha_n, a')} \cdot \frac{G(\alpha_n, a')}{G(\alpha_n, b')} \cdot \frac{G(\alpha_n, b')}{G(\alpha_n, b)}. \]

Note that the first and third fractions are close to 1, whereas by (9), the middle one tends to $S_0(b', a')$. This proves our claim. By (4) we derive that estimate (8) holds true on $X$. □

Remark 2.2. The second part of the above proof, showing that $S$ is well-defined on $X \times X$, proves a fact which is interesting on its own. Namely, if $X$ is a topological space and $G : X \times X \to (0, +\infty)$ is a solution of (3) such that $G$ is continuous and equal to 1 at every point of $\Delta$, then the family of functions \{\(G(x, \cdot) : x \in X\)\} is equi-continuous. This is the converse statement of part (vi) of Lemma 2.1.

Example 2.3. The assumption that $G = 1$ on $\Delta$ cannot be omitted. Take $X = [1, +\infty)$ and define $G : X \times X \to (0, +\infty)$ by

\[ G(a, b) = a + b, \quad a, b \in X. \]
In particular, for \((x_0, y_0) = (1, 1)\), there is no sequence \((a_n) \subset X\) such that function \(S\) defined by (7) satisfies \(S(1, 1) = G(1, 1)\).

What is more, there exist non-measurable solutions of (1) which satisfy (8) together with \(G\) as above. Let \(A \subset [1, +\infty)\) be a non-measurable set, and define \(S: X \times X \to (0, +\infty)\) by

\[
S(a, b) = \frac{b + \chi_A(b)}{a + \chi_A(a)}, \quad a, b \in X.
\]

Then \(S\) is a non-measurable solution of (1) and estimate (8) is satisfied by \(G\) and \(S\).

Let us introduce a class of functions of one variable associated with a map \(G: X \times X \to (0, +\infty)\):

\[
\mathcal{G}(G) = \left\{ f: X \to (0, +\infty) : \forall x, y \in X \frac{f(x)}{f(y)} \leq G(x, y) \right\}.
\]

**Corollary 2.2.** Assume that \(X\) is a separable topological space, and function \(G: X \times X \to (0, +\infty)\) is a solution of (3) such that \(G\) is continuous and equal to 1 at every point of \(\Delta\). Then

\[
G(a, b) = \sup \left\{ \frac{f(a)}{f(b)} : f \in \mathcal{G}(G) \right\}, \quad a, b \in X.
\] (10)

Conversely, for an arbitrary family \(\mathcal{G}\) of positive functions on \(X\), every mapping \(G: X \times X \to (0, +\infty)\) defined by (10) (with \(\mathcal{G}(G)\) replaced by \(\mathcal{G}\)) solves (3), it is equal to 1 on \(\Delta\) and \(\mathcal{G} \subseteq \mathcal{G}(G)\).

**Proof.** Apply Theorem 2.1 for points \((x_0, y_0)\) running through the whole space \(X \times X\), and use the form (2) of solutions of (1).

To prove the converse statement, fix arbitrary \(x, y, z \in X\) and \(\varepsilon > 0\). There exists some \(f \in \mathcal{G}\) such that \(G(x, y) < f(x)/f(y) + \varepsilon\). From this we have

\[
G(x, z) < \frac{f(x)}{f(y)} \cdot \frac{f(y)}{f(z)} + \varepsilon \leq G(x, y)G(y, z) + \varepsilon,
\]

and the assertion follows. \(\square\)

### 3. Second multiplicative Sincov’s inequality

The case of the reverse inequality to (3), i.e., the inequality

\[
F(x, z) \geq F(x, y) \cdot F(y, z), \quad x, y, z \in X
\] (11)

is not fully symmetric to (3), but in some situations it can be reduced to (3).

First, we list some examples.
Example 3.1. Functions $G_1$ and $G_2$ of Example 2.2 are solutions of (11), since they solve (1). If the topological space $X$ spoken of in the same example consists of precisely two connected components, then $G_3$ is a solution of equation (1), as well (and thus solves (11)).

Example 3.2. Function $F_1: \mathbb{R} \times \mathbb{R} \to [0, 1]$ given by

$$F_1(a, b) = \chi_Q(a - b), \quad a, b \in \mathbb{R},$$

function $F_2: X \times X \to [0, 1]$ defined as

$$F_2(a, b) = \chi_A(a) \cdot \chi_B(b), \quad a, b \in X,$$

where $X$ is a non-void set and $A, B \subset X$ are arbitrary subsets, and function $F_3: [1, +\infty) \times [1, +\infty) \to [0, +\infty)$ given by

$$F_3(a, b) = \frac{a - 1}{b}, \quad a, b \in [1, +\infty)$$

are all solutions of (11).

Note that if $F$ is positive and solves (11), then map $G = 1/F$ solves (3).

Next, introduce a class of functions associated with a map $F: X \times X \to (0, +\infty)$:

$$\mathcal{F}(F) = \left\{ f: X \to (0, +\infty) : \forall x, y \in X \frac{f(x)}{f(y)} \geq F(x, y) \right\}.$$  

From Corollary 2.2, we derive the following description of solutions of (11).

**Corollary 3.1.** Assume that $X$ is a separable topological space, and function $F: X \times X \to (0, +\infty)$ is a solution of (11) which is continuous and equal to 1 at every point of $\Delta$. Then

$$F(a, b) = \inf \left\{ \frac{f(a)}{f(b)} : f \in \mathcal{F}(F) \right\}, \quad a, b \in X. \quad (12)$$

Conversely, for an arbitrary family $\mathcal{F}$ of positive functions on $X$, every mapping $F: X \times X \to (0, +\infty)$ defined by (12) solves (11) (with $F(F)$ replaced by $\mathcal{F}$), it is equal to 1 on $\Delta$ and $\mathcal{F} \subseteq \mathcal{F}(F)$.

It remains to consider the case when $F$ attains a non-positive value.

**Proposition 3.1.** Assume that $X$ is a non-void set, and $F: X \times X \to \mathbb{R}$ is a solution of (11). If for every $x, y \in X$ at least one of the mappings $F(x, \cdot)$, $F(\cdot, y)$ has property (Z), then $F$ is non-negative on $X \times X$. 
Proof. Directly from (11) applied for \( y = z = x \), we obtain
\[
F(x, x) \in [0, 1], \quad x \in X.
\]
Now, suppose that \( F(a_1, x_1) < 0 \) for some \( a_1, x_1 \in X \). By (11) we have
\[
F(a_1, x_1) \geq F(a_1, a_1)F(a_1, x_1),
\]
thus \( F(a_1, a_1) = 1 \). Similarly, we show that \( F(x_1, x_1) = 1 \).

Apply property (Z) for sections of \( F \) crossing the point \((a_1, x_1)\) to deduce that, either there exists some \( x_2 \in X \) such that \( F(a_1, x_2) = 0 \), or there exists some \( a_2 \in X \) such that \( F(a_2, x_1) = 0 \). Utilizing this, we arrive at
\[
0 > F(a_1, x_1) \geq F(a_1, y_2)F(y_2, x_1) = 0,
\]
where \( y_2 \in \{a_2, x_2\} \); a contradiction. \( \square \)

Lemma 3.1. Assume that \( X \) is a topological space, and \( F : X \times X \to [0, +\infty) \) is a continuous solution of (11). Suppose that the set
\[
Z = \{(x, y) \in X \times X : F(x, y) = 0\}
\]
of zeros of \( F \) is non-empty and \((a, b) \in Z \) is arbitrary. Then \( \{a\} \times X \subseteq Z \) or \( X \times \{b\} \subseteq Z \) or there exist open non-void sets \( U_1, U_2 \subset X \) such that \( U_1 \times \{b\} \cup \{a\} \times U_2 \subseteq Z \).

Proof. Directly from (11), we have
\[
0 = F(a, b) \geq F(a, x)F(x, b), \quad x \in X.
\]
Therefore, taking into account the fact that we assume that \( F \) is non-negative, we obtain an alternative:
\[
\forall x \in X[(a, x) \in Z \text{ or } (x, b) \in Z]. \tag{13}
\]
Assume that none of the sets \( \{a\} \times X \) and \( X \times \{b\} \) is contained in \( Z \). Thus, there exist two points, say \( x_1, x_2 \in X \) such that for \( x_1 \) the first part of the alternative is not true, and for \( x_2 \), the second one is not valid, i.e., \( F(a, x_1) > 0 \) and \( F(x_2, b) > 0 \). Since sections of \( F \) are continuous, there exist two non-void open sets \( U_1, U_2 \subset X \) such that \( F(a, \cdot) > 0 \) on \( U_1 \), and \( F(\cdot, b) > 0 \) on \( U_2 \). Now, apply alternative (13) for all elements of \( U_1 \) and \( U_2 \) to derive the equality \( F = 0 \) on \( U_1 \times \{b\} \cup \{a\} \times U_2 \). \( \square \)
We will utilize Lemma 3.1 to show that the set $Z$ of zeros of $F$, if it is non-empty, is large in some sense. We will use the notion of set ideals. Recall that a family $I \subset 2^X$ is a set ideal if
(a) $A \in I$ and $B \subset A$ implies $B \in I$;
(b) $A, B \in I$ implies $A \cup B \in I$.

We call the elements of an ideal small sets, further, a set is large if it is not small. Two model examples of set ideals are: the family of all subsets of the first category of a topological space, and the family of sets of Lebesgue measure zero of $\mathbb{R}^n$. Given a set ideal $I$ of subsets of a set $X$, we define the product ideal $I \otimes I$ of subsets of $X \times X$ as the family of all sets $A \subseteq X \times X$ such that
$$\{x \in X : A[x] \notin I\} \in I,$$
where
$$A[x] = \{y \in X : (x, y) \in A\}.$$

For a comprehensive study of the notion of set ideals and several further examples, we refer the reader to the monograph of J. C. Oxtoby [8].

**Corollary 3.2.** Assume that $X$ is a topological space, $I \subset 2^X$ is a set ideal which does not contain open non-void sets, and $F : X \times X \to [0, +\infty)$ is a continuous solution of (11) such that the set $Z$ of zeros of $F$ is non-empty and $(a, b) \in Z$ is arbitrary. Then $\{a\} \times X \subseteq Z$ or $X \times \{b\} \subseteq Z$ or $Z$ is a large set with respect to the product ideal $I \otimes I$.

**Proof.** Let us pick some $(a, b) \in Z$ arbitrarily, and assume that $Z$ does not contain any of the sets $\{a\} \times X$ and $X \times \{b\}$. Apply Lemma 3.1 repeatedly to obtain
$$U_1 \times \{b\} \cup \{a\} \times U_2 \subseteq Z$$
for some open non-void sets $U_1, U_2 \subset X$. Then, use the same lemma for every point of this set to deduce from the definition of the product ideal that $Z \notin I \otimes I$. \(\square\)

With the same proof, one can deduce that in Lemma 3.1 and Corollary 3.2 it is enough to assume that both sections of $F$ are continuous.

Function $F_3$ of Example 3.2 is a solution of inequality (11) for which the set of zeros $Z$ contain both a horizontal and a vertical line, and is small with respect to the product ideal $I \otimes I$ on $X \times X$ (for every ideal $I$ satisfying the assumptions of Corollary 3.2).

We terminate this section with an open problem related to the last statement.
Problem. Is it true that, under the assumptions of Corollary 3.2, if the set $Z$ does not contain a set of the form $\{a\} \times X$ or $X \times \{b\}$, then it has a non-void interior with respect to the product topology on $X \times X$?

4. Additive Sincov’s inequality

Generalized metric space or Lawvere space (see F. W. Lawvere [7]) is a non-
void set $X$ together with a function $H : X \times X \to \mathbb{R}$, called a generalized metric, which is non-negative, vanishes on $\Delta$ and satisfies the triangle inequality:

$$H(x, z) \leq H(x, y) + H(y, z), \quad x, y, z \in X. \tag{14}$$

One can find several different names for this notion in the literature. In [3], J. Goubault-Larrecq called it hemi-metric, whereas in [6], H.-P. A. Künzi used the term quasi-metric. Let us also note that M. J. Campión, E. Induráin, G. Ochoa and O. Valero [1] studied weightable quasi-metric in connection with several functional equations, in particular with additive Sincov’s equation.

We can apply our results of Section 2 to obtain a characterization of solutions of (14). Our settings are fairly general in comparison to the definition of a generalized metric, but instead we assume that we already have a topology on the set $X$.

For an arbitrary function $H : X \times X \to \mathbb{R}$, let us define

$$\mathcal{H}(H) = \{\varphi : X \to \mathbb{R} : \forall x, y \in X \varphi(x) - \varphi(y) \leq H(x, y)\}.$$

Corollary 4.1. Assume that $X$ is a separable topological space and function $H : X \times X \to \mathbb{R}$ is a solution of (14) which is continuous and equal to 0 at every point of $\Delta$. Then

$$H(a, b) = \sup \{\varphi(a) - \varphi(b) : \varphi \in \mathcal{H}(H)\}, \quad a, b \in X. \tag{15}$$

Conversely, for an arbitrary family $\mathcal{H}$ of real functions on $X$, every mapping $H : X \times X \to \mathbb{R}$ defined by (15) (with $\mathcal{H}(H)$ replaced by $\mathcal{H}$) solves (14), it is equal to 0 on $\Delta$ and $\mathcal{H} \subseteq \mathcal{H}(H)$.

Proof. Apply Corollary 2.2 for $G := \exp \circ H$, and define $\varphi := \log \circ f$ for all $f \in \mathcal{G}(G)$. □

Corollary 4.2. Under the assumptions of Corollary 4.1, there exists a quotient subspace $X_0$ of $X$ such that:
(a) the family $\mathcal{H}(H)$ separates points of $X_0$;
(b) every $\varphi \in \mathcal{H}(H)$ satisfies a Lipschitz-type condition:
\[ |\varphi(a) - \varphi(b)| \leq \frac{1}{2} (H(a, b) + H(b, a)), \quad a, b \in X; \]
(c) $H|_{X_0 \times X_0}(a, b) = 0$ if and only if $a = b$.

Proof. Introduce a relation on $X$ as follows. We will write $a \sim b$ whenever $\varphi(a) = \varphi(b)$ for every $\varphi \in \mathcal{H}(H)$. Clearly $\sim$ is an equivalence relation. Let $X_0$ be the quotient space of $X$ with respect to $\sim$. We can embed $X_0$ in $X$ by choosing any representative of each class of abstraction, and thus (a) follows. Point (b) is a direct consequence of (15). By Corollary 4.1, we get $H(a, b) = 0$ whenever $a \sim b$, which proves (c). 

References